

# 伴有无穷时滞效应的不可压缩非 Newton 微极流方程组的确定模估计\*

袁艳杰<sup>1</sup>, 王永威<sup>2</sup>

(1. 河南科技职业大学 公共基础教学部, 河南 周口 466000;  
2. 长沙理工大学 数学与统计学院, 长沙 410114)

**摘要:** 本文在二维有界区域上估计一类伴有无穷时滞效应的不可压缩非 Newton 微极流方程组的确定模个数. 结果表明, 伴有无穷时滞效应的非 Newton 微极流方程组任意弱解的渐近行为可以完全由其前有限个 Fourier 模的渐近行为所决定.

**关键词:** 不可压缩非 Newton 微极流; 无穷时滞; 确定模; 渐近行为

**中图分类号:** O175.29      **文献标志码:** A      **DOI:** 10.21656/1000-0887.460032

## Estimate of Determining Modes for Incompressible Non-Newtonian Micropolar Fluid Equations With Infinite Delay

YUAN Yanjie<sup>1</sup>, WANG Yongwei<sup>2</sup>

(1. *Department of Public Basic Education, Henan Vocational University of Science and Technology, Zhoukou, Henan 466000, P.R.China;*

2. *School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha 410114, P.R.China*)

**Abstract:** The number of determining modes was estimated for the incompressible non-Newtonian micropolar fluid with infinite delay in 2D bounded domains. The results show that, the asymptotic behavior of any weak solution to the non-Newtonian micropolar fluid equations with infinite delay depends completely on the asymptotic behavior of their 1st finite number of Fourier modes.

**Key words:** incompressible non-Newtonian micropolar fluid; infinite delay; determining mode; asymptotic behavior

## 0 引 言

时滞微分方程是应用数学的一个重要研究领域, 由于物理原因或非即时传输, 使得时滞现象随处可见, 在许多情况下其影响是不容忽视的. 方程中包含某种时滞项的模型得到了广泛的研究, 如文献[1-7].

\* 收稿日期: 2025-02-25; 修订日期: 2025-04-17

基金项目: 国家自然科学基金(12071484)

作者简介: 袁艳杰(1989—), 女, 讲师, 硕士(通信作者. E-mail: yjy19890224@163.com);

王永威(1990—), 男, 讲师, 博士(E-mail: yongweiwang2025@163.com).

引用格式: 袁艳杰, 王永威. 伴有无穷时滞效应的不可压缩非 Newton 微极流方程组的确定模估计[J]. 应用数学和力学, 2026, 47(4): 516-528.

本文将在二维有界区域  $\Omega \subset \mathbb{R}^2$  上研究一类含无穷时滞项的流体动力学模型.在流体动力学理论中,若流体的应力张量与应变速率张量呈线性关系,则称该流体为 Newton 流,否则称为非 Newton 流<sup>[8]</sup>.例如,熔融的塑料、合成的纤维、油漆等均属于非 Newton 流.由于非 Newton 流在现实生活中的广泛应用,受到了许多学者的研究与关注并取得了丰富的研究成果<sup>[9-12]</sup>.当考虑流体流动时的旋转速度(角速度),可以用微极流方程来刻画.微极流方程是 Navier-Stokes 方程的重要推广,可以很好地表征一些经典的 Navier-Stokes 方程无法描述的伴有非均匀应力张量的流体的运动(如动物血液、液晶和稀释水溶性聚合物溶液的运动等)<sup>[13]</sup>.更多相关研究成果可见文献[14-20].

本文考虑一类伴有无穷时滞效应的不可压缩非 Newton 微极流模型,其方程组可表示如下

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla \cdot \mathbf{T}(\mathbf{e}(\mathbf{u})) + \nabla \mathbf{p} = 2\nu_r \nabla \times \boldsymbol{\omega} + \mathbf{f}(t, x) + \mathbf{g}(t, \mathbf{u}_t), & x \in \Omega, t > \tau, \\ \nabla \cdot \mathbf{u} = 0, & x \in \Omega, t > \tau, \\ \frac{\partial \boldsymbol{\omega}}{\partial t} - \nu \Delta \boldsymbol{\omega} + (\mathbf{u} \cdot \nabla) \boldsymbol{\omega} + 4\nu_r \boldsymbol{\omega} = 2\nu_r \nabla \times \mathbf{u} + \tilde{\mathbf{f}}(t, x) + \tilde{\mathbf{g}}(t, \boldsymbol{\omega}_t), & x \in \Omega, t > \tau, \end{cases} \quad (1)$$

其中  $\mathbf{u} := \mathbf{u}(t, x)$  表示速度,  $\boldsymbol{\omega} := \boldsymbol{\omega}(t, x)$  表示角速度,  $\mathbf{p} := \mathbf{p}(t, x)$  为流体的压力,  $\mathbf{f} := \mathbf{f}(t, x)$  和  $\tilde{\mathbf{f}}(t, x)$  分别表示外力 and 力矩,  $\mathbf{g} := \mathbf{g}(t, \mathbf{u}_t)$  和  $\tilde{\mathbf{g}}(t, \boldsymbol{\omega}_t)$  表示含有时滞特征的附加外力, 且

$$\mathbf{u}_t := \mathbf{u}_t(\cdot) = \mathbf{u}(t + \cdot), \quad \boldsymbol{\omega}_t := \boldsymbol{\omega}_t(\cdot) = \boldsymbol{\omega}(t + \cdot).$$

$\mathbf{T}(\mathbf{e}(\mathbf{u})) := (2\mu_0(\epsilon + |\mathbf{e}(\mathbf{u})|^2)^{-\alpha/2} \mathbf{e}(\mathbf{u}) - 2\mu_1 \Delta \mathbf{e}(\mathbf{u})), \mathbf{e}(\mathbf{u}) = (e_{jk}(\mathbf{u}))_{2 \times 2}$  是应变速率张量,

$$|\mathbf{e}(\mathbf{u})|^2 = \sum_{j, k=1}^2 e_{jk}^2(\mathbf{u}), \quad e_{jk}(\mathbf{u}) = \frac{1}{2} \left( \frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right), \quad j, k = 1, 2.$$

$\epsilon, \mu_0, \mu_1$  和  $\alpha \in (0, 1)$  为正常数,  $\nu$  和  $\nu_r$  为正的黏性系数.另外,

$$\nabla \cdot \mathbf{u} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, \quad \nabla \times \boldsymbol{\omega} = \begin{pmatrix} \frac{\partial \omega}{\partial x_2} & -\frac{\partial \omega}{\partial x_1} \end{pmatrix}.$$

对于非线性发展方程,其研究的一个基本、核心问题是方程解的存在性及其动力学行为<sup>[21-33]</sup>.鉴于不可压缩 Newton 微极流方程组有着重要的理论价值和现实指导意义,其受到了相关学者的广泛关注与研究.例如,在二维区域上,de Araújo 等<sup>[34]</sup>证明了一类非 Newton 微极流方程组拉回吸引子的存在性及其关于黏性系数  $\nu_r$  的上半连续性, Ai 和 Tan 研究了一类非 Newton 微极流方程组全局吸引子和指数吸引子的存在性<sup>[35-36]</sup>.最近, Zhao 等<sup>[37]</sup>得到了非 Newton 微极流方程组统计解的存在性及其退化正则性.随后, Chen 等<sup>[38]</sup>在三维有界区域上证明了该非 Newton 微极流方程组轨道统计解的存在性及其退化正则性.更多相关结果见文献[39-40].

研究方程组的动力学行为(吸引子)固然重要,而进一步揭示吸引子的内部结构特性也有着同等重要的地位.确定模、自由度与确定结点描述的是系统的渐近行为可以被某些有限量所确定,其概念是由数学家 Foias 等<sup>[41-42]</sup>提出,并阐明了不可压缩 Navier-Stokes 方程组具有有限的确定模与确定结点数.随后一些相关的研究工作相继出现<sup>[43-54]</sup>.粗略地说,演化系统的全局吸引子是相空间中的紧集,演化系统的解可以在相空间中展开成 Fourier 级数(无穷级数),该 Fourier 级数的前有限(比如说  $m$ )项的和也是相空间中的函数.如果由吸引子中任意两个解的前  $m$  项的和具有相同的渐近行为可以推出这两个解就具有相同的渐近行为,则称前  $m$  个 Fourier 模为这个吸引子的确定模,自然数  $m$  称为该吸引子的自由度.由此可见,演化系统的确定模可以深刻描述系统的动力学行为,自由度则刻画了吸引子的一个本质特征.确定结点则是指:如果吸引子内的两个解在有限个物理空间中的点具有相同的渐近行为,则这两个解在整个空间区域具有相同的渐近行为.因此,关于确定结点的个数从一定程度上刻画了吸引子的物理空间自由度.

本文的主要目的是证明伴有无穷时滞效应的不可压缩非 Newton 微极流方程组具有有限的确定模个数.具体地,我们将证明若系统的任意两个解的若干 Fourier 模具有相同的渐近性质,则其余无穷多个模也具有相同的渐近性质.为此,考虑如下初边值条件:

$$\begin{cases} \mathbf{u}_\tau(s, x) = \boldsymbol{\phi}(s, x), \quad \boldsymbol{\omega}_\tau(s, x) = \boldsymbol{\psi}(s, x), & s \in (-\infty, 0], \tau \in \mathbb{R}, x \in \Omega, \\ \mathbf{u}(t, x) \big|_{\partial\Omega} = \mathbf{0}, \quad \boldsymbol{\omega}(t, x) \big|_{\partial\Omega} = \mathbf{0}, \quad 2\mu_1 \frac{\partial e_{jk}}{\partial x_m} r_k r_m \big|_{\partial\Omega} = 0, & j, k, m = 1, 2, \forall t \geq \tau, \end{cases} \quad (2)$$

其中  $\mathbf{r} = (r_1, r_2)$  为垂直于边界的单位外法向量.

本文的主要思想来源于文献[42, 48]. 文献[42]和[48]分别估计了 Navier-Stokes 方程组和微极流方程组的确定模个数. 与 Navier-Stokes 方程组和微极流方程组相比, 无穷时滞不可压缩非 Newton 微极流方程组包含非线性项  $\nabla \cdot (\mu_0(\epsilon + |\mathbf{e}(\mathbf{u})|^2)^{-\alpha/2} \mathbf{e}(\mathbf{u}))$  和高阶项  $\nabla \cdot (-2\mu_1 \Delta \mathbf{e}(\mathbf{u}))$ . 鉴于此, 如何构建解满足微分不等式、如何证明解具有仿压挤性质等并不是容易的问题, 这也是证明系统具有有限的确定模个数的关键. 另外, 无穷时滞项的出现, 使得估计问题(1)–(2)的确定模更具困难性. 确切地说, 考虑到时滞效应可以无界增长, 相空间的选择是微妙的(见文献[50]). 为此本文引入了空间  $\mathcal{C}_\gamma$ , 并给出了关于时滞项的假设条件.

本文剩余部分安排如下: 第1节, 做一些准备工作; 第2节, 估计问题(1)–(2)的确定模个数.

## 1 预备知识

在这一节中, 我们主要做一些准备工作, 包括引入一些记号、定义及相关的一些结果和预备知识.

在本文中,  $\mathbb{N}, \mathbb{R}$  和  $\mathbb{R}_+$  分别表示自然数集、实数集和正实数集.  $L^p(\Omega)$  和  $W^{m,p}(\Omega)$  分别表示 Lebesgue 空间和 Sobolev 空间, 相应的范数分别记为  $\|\cdot\|_{L^p}$  和  $\|\cdot\|_{m,p}$ . 特别地,  $H^m(\Omega) := W^{m,2}(\Omega)$ .

$$\mathcal{V} := \{ \boldsymbol{\varphi} \in \mathcal{C}_0^\infty(\Omega) \times \mathcal{C}_0^\infty(\Omega) \mid \boldsymbol{\varphi} = (\varphi_1, \varphi_2), \nabla \cdot \boldsymbol{\varphi} = \mathbf{0} \};$$

$$H := \mathcal{V} \text{ 在 } L^2(\Omega) \times L^2(\Omega) \text{ 范数下的完备化空间, 范数为 } \|\cdot\|_H, \text{ 对偶空间记为 } H^*;$$

$$V := \mathcal{V} \text{ 在 } H^2(\Omega) \times H^2(\Omega) \text{ 范数下的完备化空间, 范数为 } \|\cdot\|_V, \text{ 对偶空间记为 } V^*;$$

$$\hat{H} := H \times L^2(\Omega), \text{ 范数为 } \|\cdot\|_{\hat{H}}, \text{ 对偶空间记为 } \hat{H}^*;$$

$$\hat{V} := V \times H_0^1(\Omega), \text{ 范数为 } \|\cdot\|_{\hat{V}}, \text{ 对偶空间记为 } \hat{V}^*.$$

令  $(\cdot, \cdot)$  和  $\langle \cdot, \cdot \rangle$  分别表示空间的内积和对偶积. 在不引起混淆的情况下, 范数  $\|\cdot\|_{L^2}, \|\cdot\|_H$  和  $\|\cdot\|_{\hat{H}}$  均简记为  $\|\cdot\|$ . 另外:

$L^p(I; \hat{H}) :=$  取值于  $\hat{H}$  空间中, 在区间  $I$  上  $p$  次可积的函数全体, 其范数为

$$\|\boldsymbol{\varphi}\|_{L^p(I; \hat{H})} = \left( \int_I \|\boldsymbol{\varphi}(t)\|^p dt \right)^{1/p}, \quad \forall \boldsymbol{\varphi} \in L^p(I; \hat{H}), 1 \leq p < \infty;$$

$\mathcal{C}(I; \hat{H}) :=$  取值于  $\hat{H}$  空间中, 在区间  $I$  上连续的函数全体, 其范数为

$$\|\boldsymbol{\varphi}\|_{\mathcal{C}(I; \hat{H})} = \max_{t \in I} \|\boldsymbol{\varphi}(t, x)\|_X;$$

$L_{\text{loc}}^2(I; \hat{H}) :=$  取值于  $\hat{H}$  空间中, 在区间  $I$  上局部可积的函数全体; “ $\hookrightarrow$ ” 表示两个空间之间紧嵌入关系.

为将问题式(1)–(2)写成抽象的形式, 我们引入几个重要的算子. 首先, 定义算子  $A_1$  和  $A_2$  如下:

$$\langle A_1 \mathbf{u}, \mathbf{v} \rangle := \sum_{i,j,k=1}^2 \int_{\Omega} \frac{\partial e_{ij}(\mathbf{u})}{\partial x_k} \frac{\partial e_{ij}(\mathbf{v})}{\partial x_k} dx, \quad \forall \mathbf{u}, \mathbf{v} \in V,$$

$$\langle A_2 \boldsymbol{\omega}, \boldsymbol{\xi} \rangle := (\nabla \boldsymbol{\omega}, \nabla \boldsymbol{\xi}), \quad \forall \boldsymbol{\omega}, \boldsymbol{\xi} \in H_0^1(\Omega).$$

**注 1** 由以上定义可知,  $A_1, A_2$  为椭圆算子. 根据椭圆算子的谱理论, 存在由  $A_1$  的特征值构成的序列  $\{\lambda_n\}_{n=1}^\infty$  和  $H$  空间的标准正交基构成的序列  $\{\mathbf{t}_n\}_{n=1}^\infty \subset D(A_1) := (H^1(\Omega))^2 \cap V$ , 满足  $\text{span}\{\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_n, \dots\}$  在  $V$  空间中稠密. 存在由  $A_2$  的特征值构成的序列  $\{\lambda_n^*\}_{n=1}^\infty$  和  $L^2(\Omega)$  空间的标准正交基构成的序列  $\{\mathbf{t}_n^*\}_{n=1}^\infty \subset D(A_2) := H^2(\Omega) \cap H_0^1(\Omega)$ , 满足  $\text{span}\{\mathbf{t}_1^*, \mathbf{t}_2^*, \dots, \mathbf{t}_n^*, \dots\}$  在  $H_0^1(\Omega)$  空间中稠密. 且对任意的  $n \in \mathbb{N}$ , 成立:

$$A_1 \mathbf{t}_n = \lambda_n \mathbf{t}_n, \quad 0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots, \text{ 且当 } n \rightarrow \infty \text{ 时, } \lambda_n \rightarrow \infty;$$

$$A_2 \mathbf{t}_n^* = \lambda_n^* \mathbf{t}_n^*, \quad 0 < \lambda_1^* \leq \lambda_2^* \leq \dots \leq \lambda_n^* \leq \dots, \text{ 且当 } n \rightarrow \infty \text{ 时, } \lambda_n^* \rightarrow \infty.$$

此外, 定义算子

$$b_1(\mathbf{u}, \mathbf{v}, \boldsymbol{\varphi}) := \sum_{j,k=1}^2 \int_{\Omega} u_j \frac{\partial v_k}{\partial x_j} \boldsymbol{\varphi}_k dx, \quad \forall \mathbf{u}, \mathbf{v}, \boldsymbol{\varphi} \in (H_0^1(\Omega))^2,$$

$$b_2(\mathbf{u}, \boldsymbol{\omega}, \boldsymbol{\xi}) := \sum_{i=1}^2 \int_{\Omega} u_i \frac{\partial \boldsymbol{\omega}}{\partial x_i} \boldsymbol{\xi} dx, \quad \forall \mathbf{u} \in (H_0^1(\Omega))^2, \boldsymbol{\omega}, \boldsymbol{\xi} \in H_0^1(\Omega).$$

不难验证,算子  $b_1(\cdot, \cdot, \cdot)$  和  $b_2(\cdot, \cdot, \cdot)$  分别在  $V \times V \times V$  和  $V \times H_0^1(\Omega) \times H_0^1(\Omega)$  上连续.记

$$\begin{aligned} \langle B_1(\mathbf{u}, \mathbf{v}), \boldsymbol{\varphi} \rangle &:= b_1(\mathbf{u}, \mathbf{v}, \boldsymbol{\varphi}), \quad \forall \mathbf{u}, \mathbf{v}, \boldsymbol{\varphi} \in V, \\ \langle B_2(\mathbf{u}, \boldsymbol{\omega}), \boldsymbol{\xi} \rangle &:= b_2(\mathbf{u}, \boldsymbol{\omega}, \boldsymbol{\xi}), \quad \forall \mathbf{u} \in V, \boldsymbol{\omega}, \boldsymbol{\xi} \in H_0^1(\Omega). \end{aligned}$$

令  $\mu(\mathbf{u}) := 2\mu_0(\epsilon + |\mathbf{e}(\mathbf{u})|^2)^{-\alpha/2}$ , 定义算子  $N$  如下:

$$\langle N(\mathbf{u}), \mathbf{v} \rangle := \sum_{j,k=1}^2 \int_{\Omega} \mu(\mathbf{u}) e_{jk}(\mathbf{u}) e_{jk}(\mathbf{v}) dx, \quad \forall \mathbf{u}, \mathbf{v} \in V.$$

可以验证算子  $N(\cdot)$  是从  $V$  到  $V^*$  的连续函数.

最后,定义算子:

$$\mathcal{K}_1(\boldsymbol{\omega}) := -2\nu_r \nabla \times \boldsymbol{\omega}, \quad \mathcal{K}_2(\mathbf{u}, \boldsymbol{\omega}) := -2\nu_r \nabla \times \mathbf{u} + 4\nu_r \boldsymbol{\omega}, \quad \forall \boldsymbol{\omega} \in H_0^1(\Omega), \mathbf{u} \in V.$$

接下来,给出相关算子的估计如下(见文献[51]).

**引理 1** ① 存在正常数  $c_1$  和  $c_2$ , 使得

$$\begin{cases} c_1 \|\mathbf{u}\|_V^2 \leq \langle A_1 \mathbf{u}, \mathbf{u} \rangle \leq c_2 \|\mathbf{u}\|_V^2, & \forall \mathbf{u} \in V, \\ c_1 \|\boldsymbol{\omega}\|_{1,2}^2 \leq \langle A_2 \boldsymbol{\omega}, \boldsymbol{\omega} \rangle \leq c_2 \|\boldsymbol{\omega}\|_{1,2}^2, & \forall \boldsymbol{\omega} \in H_0^1(\Omega). \end{cases}$$

② 存在仅依赖于区域  $\Omega$  的正常数  $c_3$  和  $c_4$ , 使得

$$\begin{aligned} b_1(\mathbf{u}, \mathbf{v}, \boldsymbol{\varphi}) &= -b_1(\mathbf{u}, \boldsymbol{\varphi}, \mathbf{v}), \quad b_1(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0, \quad \forall \mathbf{u}, \mathbf{v}, \boldsymbol{\varphi} \in V, \\ b_2(\mathbf{u}, \boldsymbol{\omega}, \boldsymbol{\xi}) &= -b_2(\mathbf{u}, \boldsymbol{\xi}, \boldsymbol{\omega}), \quad b_2(\mathbf{u}, \boldsymbol{\omega}, \boldsymbol{\omega}) = 0, \quad \forall \mathbf{u} \in V, \boldsymbol{\omega}, \boldsymbol{\xi} \in H_0^1(\Omega), \\ |b_1(\mathbf{u}, \mathbf{v}, \boldsymbol{\varphi})| &\leq c_3 \|\mathbf{u}\|^{1/2} \|\nabla \mathbf{u}\|^{1/2} \|\mathbf{v}\|^{1/2} \|\nabla \mathbf{v}\|^{1/2} \|\nabla \boldsymbol{\varphi}\|, \quad \forall \mathbf{u}, \mathbf{v}, \boldsymbol{\varphi} \in V, \\ |b_2(\mathbf{u}, \boldsymbol{\omega}, \boldsymbol{\xi})| &\leq c_3 \|\mathbf{u}\|^{1/2} \|\nabla \mathbf{u}\|^{1/2} \|\nabla \boldsymbol{\omega}\| \|\boldsymbol{\xi}\|^{1/2} \|\nabla \boldsymbol{\xi}\|^{1/2}, \quad \forall \mathbf{u} \in V, \boldsymbol{\omega}, \boldsymbol{\xi} \in H_0^1(\Omega). \end{aligned}$$

根据上面算子的定义,问题式(1)–(2)在分布  $\mathcal{D}'(\tau, +\infty; \hat{V})$  意义下可以写成如下抽象形式:

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + 2\mu_1 A_1 \mathbf{u} + B_1(\mathbf{u}, \mathbf{u}) + \mathcal{K}_1(\boldsymbol{\omega}) + N(\mathbf{u}) = \mathbf{f}(t, x) + \mathbf{g}(t, \mathbf{u}_t), & x \in \Omega, t > \tau, \\ \frac{\partial \boldsymbol{\omega}}{\partial t} + \nu A_2 \boldsymbol{\omega} + B_2(\mathbf{u}, \boldsymbol{\omega}) + \mathcal{K}_2(\mathbf{u}, \boldsymbol{\omega}) = \tilde{\mathbf{f}}(t, x) + \tilde{\mathbf{g}}(t, \boldsymbol{\omega}_t), & x \in \Omega, t > \tau, \\ \mathbf{u}_\tau(s, x) = \boldsymbol{\phi}(s, x), \quad \boldsymbol{\omega}_\tau(s, x) = \boldsymbol{\psi}(s, x), & s \in (-\infty, 0], \tau \in \mathbb{R}, x \in \Omega, \\ \mathbf{u}(t, x)|_{\partial\Omega} = \mathbf{0}, \quad \boldsymbol{\omega}(t, x)|_{\partial\Omega} = \mathbf{0}, \quad 2\mu_1 \frac{\partial e_{jk}}{\partial x_m} r_k r_m |_{\partial\Omega} = 0, & j, k, m = 1, 2, \forall t \geq \tau. \end{cases} \quad (3)$$

为了处理无穷时滞项,给定 Banach 空间  $X$ , 对于适当的  $\gamma$ , 引入空间  $\mathcal{C}_\gamma(X)$  如下:

$$\mathcal{C}_\gamma(X) := \{ \boldsymbol{\Phi} \in \mathcal{C}((-\infty, 0]; X) \mid \exists \lim_{s \rightarrow -\infty} e^{\gamma s} \boldsymbol{\Phi}(s) \in X \},$$

此空间为 Banach 空间,其范数为

$$\|\boldsymbol{\Phi}\|_\gamma := \sup_{s \in (-\infty, 0]} e^{\gamma s} \|\boldsymbol{\Phi}(s)\|_X.$$

问题(3)的弱解定义如下<sup>[20]</sup>.

**定义 1** 给定任意的初始条件  $(\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathcal{C}_\gamma(\hat{H})$ , 若存在函数对

$$(\mathbf{u}, \boldsymbol{\omega}) \in \mathcal{C}([\tau, +\infty); \hat{H}) \cap L^2(\tau, +\infty; \hat{V}) \text{ 且 } (\mathbf{u}_t, \boldsymbol{\omega}_t) \in \mathcal{C}_\gamma(\hat{H}), \quad \forall t \geq \tau,$$

使得问题(3)在分布  $\mathcal{D}'(\tau, +\infty; \hat{V}^*)$  意义下成立,则称  $(\mathbf{u}, \boldsymbol{\omega})$  是问题(3)的弱解.

给定如下假设条件.

**假设(A)** 假设  $(\mathbf{g}(t, \mathbf{u}_t), \tilde{\mathbf{g}}(t, \boldsymbol{\omega}_t)): [\tau, +\infty) \times \mathcal{C}_\gamma(\hat{H}) \rightarrow (L^2(\Omega))^3$  满足:

- ① 对于任意  $(\boldsymbol{\varphi}, \boldsymbol{\xi}) \in \mathcal{C}_\gamma(\hat{H})$ , 映射  $[\tau, +\infty) \ni t \mapsto (\mathbf{g}(t, \boldsymbol{\varphi}), \tilde{\mathbf{g}}(t, \boldsymbol{\xi})) \in (L^2(\Omega))^3$  是可测的;
- ②  $(\mathbf{g}(t, 0), \tilde{\mathbf{g}}(t, 0)) = (0, 0, 0)$ ;
- ③ 存在常数  $L > 0$ , 使得对于任意  $(\varphi_1, \xi_1), (\varphi_2, \xi_2) \in \mathcal{C}_\gamma(\hat{H})$ , 有

$$\begin{cases} \| \mathbf{g}(t, \varphi_1) - \mathbf{g}(t, \varphi_2) \| \leq L \| \varphi_1 - \varphi_2 \|_\gamma, \\ \| \tilde{\mathbf{g}}(t, \xi_1) - \tilde{\mathbf{g}}(t, \xi_2) \| \leq L \| \xi_1 - \xi_2 \|_\gamma, \end{cases} \quad \forall t \geq \tau.$$

在上面的假设条件下, 如下关于问题(3)弱解的适定性结论可由 Galerkin 方法证明, 详细证明过程可参阅文献[20].

**引理 2** 假设(A)成立且  $2L < \delta < 2\gamma$ , 给定  $(\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathcal{C}_\gamma(\hat{H})$ ,  $(\mathbf{f}, \tilde{\mathbf{f}}) \in L^2(\tau, +\infty; \hat{V}^*)$ , 则问题(3)存在唯一的弱解  $(\mathbf{u}, \boldsymbol{\omega})$ , 满足

$$\begin{aligned} (\mathbf{u}, \boldsymbol{\omega}) &\in \mathcal{C}([\tau, +\infty); \hat{H}) \cap L^2(\tau, +\infty; \hat{V}), \quad (\mathbf{u}_t, \boldsymbol{\omega}_t) \in \mathcal{C}_\gamma(\hat{H}), \\ \| (\mathbf{u}, \boldsymbol{\omega})(t) \|^2 &\leq \| (\mathbf{u}_t, \boldsymbol{\omega}_t) \|^2_\gamma \leq \\ &e^{-(\delta-2L)(t-\tau)} \| (\boldsymbol{\phi}, \boldsymbol{\psi}) \|^2_\gamma + \frac{1}{\min\{c_1\mu_1, c_1\nu\}} \int_\tau^t e^{-(\delta-2L)(t-\theta)} \| (\mathbf{f}, \tilde{\mathbf{f}})(\theta) \|^2_{\hat{V}^*} d\theta, \end{aligned} \quad (4)$$

其中  $\delta := \min\{2c_1\mu_1\lambda_1, c_1\nu\lambda_1^* + 8\nu_r\}$ .

最后, 我们引入如下引理<sup>[47,52]</sup>.

**引理 3** 设  $\vartheta(t)$  和  $\varrho(t)$  为  $[\tau, +\infty)$  上的实局部可积函数, 且存在  $T_0 > 0$ , 使得

$$\liminf_{t \rightarrow +\infty} \frac{1}{T_0} \int_t^{t+T_0} \vartheta(\theta) d\theta > 0, \quad \limsup_{t \rightarrow +\infty} \frac{1}{T_0} \int_t^{t+T_0} \vartheta^-(\theta) d\theta < +\infty, \quad (5)$$

$$\lim_{t \rightarrow +\infty} \frac{1}{T_0} \int_t^{t+T_0} \varrho^+(\theta) d\theta = 0, \quad (6)$$

其中  $\vartheta^-(t) = \max\{-\vartheta(t), 0\}$ ,  $\varrho^+(t) = \max\{\varrho(t), 0\}$ . 若  $\zeta(t)$  是  $[\tau, +\infty)$  上的非负且绝对连续函数, 并且在  $[\tau, +\infty)$  上  $d\zeta(t)/dt + \vartheta(t)\zeta(t) \leq \varrho(t)$  几乎处处成立, 则当  $t \rightarrow \infty$  时,  $\zeta(t) \rightarrow 0$ .

## 2 确定模个数估计

本节的主要目的是估计问题(3)的确定模个数.

首先, 由注 1 可知, 对于任意  $\mathbf{u} \in H, \boldsymbol{\omega} \in L^2(\Omega)$ ,

$$\mathbf{u}(t, x) = \sum_{i=1}^{\infty} (\mathbf{u}, \mathbf{e}_i) \mathbf{e}_i := \sum_{i=1}^{\infty} \hat{\mathbf{u}}_i(t) \mathbf{e}_i(x), \quad \boldsymbol{\omega}(t, x) = \sum_{i=1}^{\infty} (\boldsymbol{\omega}, \mathbf{e}_i^*) \mathbf{e}_i^* := \sum_{i=1}^{\infty} \hat{\boldsymbol{\omega}}_i(t) \mathbf{e}_i^*(x).$$

令  $\boldsymbol{\varepsilon}_n := \text{span}\{(\mathbf{e}_1, \mathbf{e}_1^*), (\mathbf{e}_2, \mathbf{e}_2^*), \dots, (\mathbf{e}_n, \mathbf{e}_n^*)\}$ , 定义 Galerkin 投影算子  $P_n$  如下

$$P_n \mathbf{u}(t, x) = \sum_{i=1}^n \hat{\mathbf{u}}_i(t) \mathbf{e}_i(x), \quad P_n \boldsymbol{\omega}(t, x) = \sum_{i=1}^n \hat{\boldsymbol{\omega}}_i(t) \mathbf{e}_i^*(x).$$

接下来, 我们给出确定模的定义如下(见文献[48,52]).

**定义 2** 设  $(\mathbf{u}_1(t, x), \boldsymbol{\omega}_1(t, x))$  和  $(\mathbf{u}_2(t, x), \boldsymbol{\omega}_2(t, x))$  为问题(3)分别对应于外力  $(\mathbf{f}_1, \tilde{\mathbf{f}}_1)$  和  $(\mathbf{f}_2, \tilde{\mathbf{f}}_2)$  的两个弱解, 如果由

$$\begin{cases} \lim_{t \rightarrow +\infty} (\| P_n \mathbf{u}_1(t, x) - P_n \mathbf{u}_2(t, x) \| + \| P_n \boldsymbol{\omega}_1(t, x) - P_n \boldsymbol{\omega}_2(t, x) \|) = 0, \\ \lim_{t \rightarrow +\infty} (\| \mathbf{f}_1(t, x) - \mathbf{f}_2(t, x) \|_{V^*} + \| \tilde{\mathbf{f}}_1(t, x) - \tilde{\mathbf{f}}_2(t, x) \|_{H^{-1}}) = 0 \end{cases} \quad (7)$$

可推得  $\lim_{t \rightarrow +\infty} (\| \mathbf{u}_1(t, x) - \mathbf{u}_2(t, x) \| + \| \boldsymbol{\omega}_1(t, x) - \boldsymbol{\omega}_2(t, x) \|) = 0$ , 则称与  $P_n$  相关的前  $n$  个 Fourier 模为问题(3)的确定模.

本部分的主要结论如下.

**定理 1** 设  $(\mathbf{u}_1, \boldsymbol{\omega}_1)$  和  $(\mathbf{u}_2, \boldsymbol{\omega}_2)$  为问题(3)分别对应于外力  $(\mathbf{f}_1, \tilde{\mathbf{f}}_1)$  和  $(\mathbf{f}_2, \tilde{\mathbf{f}}_2)$  的两个弱解, 若

$$c_2^2 \mu_1 \lambda_1 \min\{\mu_1, \nu\} \geq 4 \max\{8c_3^4/(c_1^3 \mu_1^3), \tilde{c}\} \| (\mathbf{f}_1, \tilde{\mathbf{f}}_1) \|^4_{L^2(\tau, +\infty; \hat{V}^*)}, \quad (8)$$

取  $\lambda_{n+1}$  (见注 1) 满足

$$\begin{cases} \lambda_{n+1} \geq \frac{4\nu_r^2}{c_1^2 \nu \mu_1} + \frac{c_4^2}{2c_1 \mu_1} + \frac{c_5}{2c_1 \mu_1} \sum_{i=1}^2 \| (\mathbf{f}_i, \tilde{\mathbf{f}}_i) \|^2_{L^2(\tau, +\infty; \hat{V}^*)}, \\ \lambda_{n+1}^* \geq \frac{4\nu_r^2}{c_1^2 \mu_1 \nu} - \frac{8\nu_r}{c_1 \nu} + \frac{c_3^2}{2c_1^3 \nu \sqrt{\mu_1 \nu} \min\{\mu_1, \nu\}} \| (\mathbf{f}_2, \tilde{\mathbf{f}}_2) \|^2_{L^2(\tau, +\infty; \hat{V}^*)}, \end{cases} \quad (9)$$

则在定义2的意义下,与 Galerkin 投影算子  $P_n$  相关的前  $n$  个 Fourier 模为问题(3)的确定模,其中  $c_4, c_5$  和  $\tilde{c}$  分别来自式(20)、(25)和(34).

**证明** 设  $(\mathbf{u}_1(t, x), \boldsymbol{\omega}_1(t, x))$  和  $(\mathbf{u}_2(t, x), \boldsymbol{\omega}_2(t, x))$  为问题(3)分别对应于外力  $(f_1, \tilde{f}_1)$  和  $(f_2, \tilde{f}_2)$  的两个弱解,其初值分别为  $(\phi_1, \psi_1) \in \mathcal{L}_\gamma(\hat{H})$  和  $(\phi_2, \psi_2) \in \mathcal{L}_\gamma(\hat{H})$ . 记  $\mathbf{U} = \mathbf{u}_1 - \mathbf{u}_2, \mathbf{W} = \boldsymbol{\omega}_1 - \boldsymbol{\omega}_2, \mathbf{F} = f_1 - f_2, \tilde{\mathbf{F}} = \tilde{f}_1 - \tilde{f}_2$ , 则在分布  $\mathcal{D}'(\tau, +\infty; \hat{H})$  意义下,成立:

$$\begin{cases} \frac{\partial \mathbf{U}}{\partial t} + 2\mu_1 A_1 \mathbf{U} + B_1(\mathbf{u}_1, \mathbf{u}_1) - B_1(\mathbf{u}_2, \mathbf{u}_2) + N(\mathbf{u}_1) - N(\mathbf{u}_2) + \mathcal{K}_1(\mathbf{W}) = \\ \mathbf{F} + \mathbf{g}(t, \mathbf{u}_{1,t}) - \mathbf{g}(t, \mathbf{u}_{2,t}), \\ \frac{\partial \mathbf{W}}{\partial t} + \nu A_2 \mathbf{W} + B_2(\mathbf{u}_1, \boldsymbol{\omega}_1) - B_2(\mathbf{u}_2, \boldsymbol{\omega}_2) + \mathcal{K}_2(\mathbf{U}, \mathbf{W}) = \tilde{\mathbf{F}} + \tilde{\mathbf{g}}(t, \boldsymbol{\omega}_{1,t}) - \tilde{\mathbf{g}}(t, \boldsymbol{\omega}_{2,t}). \end{cases} \quad (10)$$

令  $Q_n = I - P_n$ , 其中  $I$  为单位算子. 根据定义2, 只需证明存在  $n \in \mathbb{N}$ , 使得在条件(7)下, 下式成立:

$$\lim_{t \rightarrow +\infty} (\|Q_n \mathbf{U}(t, x)\| + \|Q_n \mathbf{W}(t, x)\|) = 0. \quad (11)$$

接下来, 我们将证明过程分为三步.

第一步: 能量估计. 首先, 用  $Q_n \mathbf{U}$  和  $Q_n \mathbf{W}$  分别与式(10)的第一式和第二式作对偶积, 可得

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|Q_n \mathbf{U}\|^2 + 2\mu_1 \langle A_1 \mathbf{U}, Q_n \mathbf{U} \rangle + b_1(\mathbf{U}, \mathbf{u}_1, Q_n \mathbf{U}) + b_1(\mathbf{u}_2, \mathbf{U}, Q_n \mathbf{U}) + \langle \mathcal{K}_1(\mathbf{W}), Q_n \mathbf{U} \rangle + \\ \langle N(\mathbf{u}_1) - N(\mathbf{u}_2), Q_n \mathbf{U} \rangle = \langle \mathbf{F}, Q_n \mathbf{U} \rangle + \langle \mathbf{g}(t, \mathbf{u}_{1,t}) - \mathbf{g}(t, \mathbf{u}_{2,t}), Q_n \mathbf{U} \rangle, \end{aligned} \quad (12)$$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|Q_n \mathbf{W}\|^2 + \nu \langle A_2 \mathbf{W}, Q_n \mathbf{W} \rangle + b_2(\mathbf{u}_1, \mathbf{W}, Q_n \mathbf{W}) + b_2(\mathbf{U}, \boldsymbol{\omega}_2, Q_n \mathbf{W}) + \langle \mathcal{K}_2(\mathbf{U}, \mathbf{W}), Q_n \mathbf{W} \rangle = \\ \langle \tilde{\mathbf{F}}, Q_n \mathbf{W} \rangle + \langle \tilde{\mathbf{g}}(t, \boldsymbol{\omega}_{1,t}) - \tilde{\mathbf{g}}(t, \boldsymbol{\omega}_{2,t}), Q_n \mathbf{W} \rangle. \end{aligned} \quad (13)$$

由引理1, 可知

$$2c_1 \mu_1 \|Q_n \mathbf{U}\|_V^2 \leq 2\mu_1 \langle A_1 \mathbf{U}, Q_n \mathbf{U} \rangle = 2\mu_1 \langle A_1 Q_n \mathbf{U}, Q_n \mathbf{U} \rangle, \quad (14)$$

$$c_1 \nu \|Q_n \mathbf{W}\|_{1,2}^2 \leq \nu \langle A_2 \mathbf{W}, Q_n \mathbf{W} \rangle = \nu \langle A_2 Q_n \mathbf{W}, Q_n \mathbf{W} \rangle. \quad (15)$$

由引理1, 可知

$$\begin{aligned} |b_1(\mathbf{U}, \mathbf{u}_1, Q_n \mathbf{U}) + b_1(\mathbf{u}_2, \mathbf{U}, Q_n \mathbf{U})| \leq \\ |b_1(P_n \mathbf{U}, \mathbf{u}_1, Q_n \mathbf{U}) + b_1(Q_n \mathbf{U}, \mathbf{u}_1, Q_n \mathbf{U})| + |b_1(\mathbf{u}_2, P_n \mathbf{U}, Q_n \mathbf{U}) + b_1(\mathbf{u}_2, Q_n \mathbf{U}, Q_n \mathbf{U})| \leq \\ c_3 \|P_n \mathbf{U}\|^{1/2} \|P_n \mathbf{U}\|_V^{1/2} \|\mathbf{u}_1\|^{1/2} \|\mathbf{u}_1\|_V^{1/2} \|Q_n \mathbf{U}\|_V + c_3 \|Q_n \mathbf{U}\| \|Q_n \mathbf{U}\|_V \|\mathbf{u}_1\|_V + \\ c_3 \|\mathbf{u}_2\|^{1/2} \|\mathbf{u}_2\|_V^{1/2} \|P_n \mathbf{U}\|^{1/2} \|P_n \mathbf{U}\|_V^{1/2} \|Q_n \mathbf{U}\|_V \leq \\ c_3 \|P_n \mathbf{U}\|^{1/2} \|P_n \mathbf{U}\|_V^{1/2} \|Q_n \mathbf{U}\|_V (\|\mathbf{u}_1\|^{1/2} \|\mathbf{u}_1\|_V^{1/2} + \|\mathbf{u}_2\|^{1/2} \|\mathbf{u}_2\|_V^{1/2}) + \\ \frac{c_3^2}{c_1 \mu_1} \|Q_n \mathbf{U}\|^2 \|\mathbf{u}_1\|_V^2 + \frac{c_1 \mu_1}{4} \|Q_n \mathbf{U}\|_V^2, \end{aligned} \quad (16)$$

$$\begin{aligned} |b_2(\mathbf{u}_1, \mathbf{W}, Q_n \mathbf{W}) + b_2(\mathbf{U}, \boldsymbol{\omega}_2, Q_n \mathbf{W})| \leq \\ |b_2(\mathbf{u}_1, P_n \mathbf{W}, Q_n \mathbf{W}) + b_2(\mathbf{u}_1, Q_n \mathbf{W}, Q_n \mathbf{W})| + |b_2(P_n \mathbf{U}, \boldsymbol{\omega}_2, Q_n \mathbf{W}) + b_2(Q_n \mathbf{U}, \boldsymbol{\omega}_2, Q_n \mathbf{W})| \leq \\ c_3 \|\mathbf{u}_1\|^{1/2} \|\mathbf{u}_1\|_V^{1/2} \|P_n \mathbf{W}\|^{1/2} \|P_n \mathbf{W}\|_{1,2}^{1/2} \|Q_n \mathbf{W}\|_{1,2} + \\ c_3 \|P_n \mathbf{U}\|^{1/2} \|P_n \mathbf{U}\|_V^{1/2} \|\boldsymbol{\omega}_2\|_{1,2} \|Q_n \mathbf{W}\|^{1/2} \|Q_n \mathbf{W}\|_{1,2}^{1/2} + \\ c_3 \|Q_n \mathbf{U}\|^{1/2} \|Q_n \mathbf{U}\|_V^{1/2} \|\mathbf{W}\|_{1,2} \|Q_n \mathbf{W}\|^{1/2} \|Q_n \mathbf{W}\|_{1,2}^{1/2} \leq \\ c_3 \|\mathbf{u}_1\|^{1/2} \|\mathbf{u}_1\|_V^{1/2} \|P_n \mathbf{W}\|^{1/2} \|P_n \mathbf{W}\|_{1,2}^{1/2} \|Q_n \mathbf{W}\|_{1,2} + \\ c_3 \|P_n \mathbf{U}\|^{1/2} \|P_n \mathbf{U}\|_V^{1/2} \|\boldsymbol{\omega}_2\|_{1,2} \|Q_n \mathbf{W}\|^{1/2} \|Q_n \mathbf{W}\|_{1,2}^{1/2} + \\ \frac{c_3^2}{4c_1 \sqrt{\mu_1 \nu}} \|\boldsymbol{\omega}_2\|_{1,2}^2 (\|Q_n \mathbf{U}\|^2 + \|Q_n \mathbf{W}\|^2) + \frac{c_1 \mu_1}{4} \|Q_n \mathbf{U}\|_V^2 + \frac{c_1 \nu}{4} \|Q_n \mathbf{W}\|_{1,2}^2. \end{aligned} \quad (17)$$

由算子  $\mathcal{K}_1$  和  $\mathcal{K}_2$  的定义, 不难推得

$$|\langle \mathcal{K}_1(\mathbf{W}), Q_n \mathbf{U} \rangle| \leq 2\nu_r \|P_n \mathbf{W}\| \|Q_n \mathbf{U}\|_V + 2\nu_r \|Q_n \mathbf{W}\| \|Q_n \mathbf{U}\|_V \leq$$

$$2\nu_r \|P_n \mathbf{W}\| \|Q_n \mathbf{U}\|_V + \frac{2\nu_r^2}{c_1 \mu_1} \|Q_n \mathbf{W}\|^2 + \frac{c_1 \mu_1}{2} \|Q_n \mathbf{U}\|_V^2, \quad (18)$$

$$\begin{aligned} \langle \mathcal{K}_2(\mathbf{U}, \mathbf{W}), Q_n \mathbf{W} \rangle &\geq -2\nu_r \|P_n \mathbf{U}\| \|Q_n \mathbf{W}\|_{1,2} - 2\nu_r \|Q_n \mathbf{U}\| \|Q_n \mathbf{W}\|_{1,2} + 4\nu_r \|Q_n \mathbf{W}\|^2 \geq \\ &-2\nu_r \|P_n \mathbf{U}\| \|Q_n \mathbf{W}\|_{1,2} - \frac{4\nu_r^2}{c_1 \nu} \|Q_n \mathbf{U}\|^2 - \frac{c_1 \nu}{4} \|Q_n \mathbf{W}\|_{1,2}^2 + 4\nu_r \|Q_n \mathbf{W}\|^2. \end{aligned} \quad (19)$$

通过直接计算, 可知存在与  $\mu_0$ ,  $\epsilon$  和  $\alpha$  有关的常数  $c_4 := c_4(\mu_0, \epsilon, \alpha)$ , 使得

$$\begin{aligned} -(N(\mathbf{u}_1) - N(\mathbf{u}_2), Q_n \mathbf{U}) &= -\sum_{j,k=1}^2 \int_{\Omega} (\mu(\mathbf{u}_1) e_{jk}(\mathbf{u}_1) - \mu(\mathbf{u}_2) e_{jk}(\mathbf{u}_2)) e_{jk}(Q_n \mathbf{U}) dx = \\ &- \sum_{j,k=1}^2 \int_{\Omega} [(\mu(\mathbf{u}_1) - \mu(\mathbf{u}_2)) e_{jk}(\mathbf{u}_1) + \mu(\mathbf{u}_2) (e_{jk}(\mathbf{u}_1) - e_{jk}(\mathbf{u}_2))] e_{jk}(Q_n \mathbf{U}) dx = \\ &- \sum_{j,k=1}^2 \int_{\Omega} (\mu(\mathbf{u}_1) - \mu(\mathbf{u}_2)) e_{jk}(\mathbf{u}_1) e_{jk}(Q_n \mathbf{U}) dx - \sum_{j,k=1}^2 \int_{\Omega} \mu(\mathbf{u}_2) e_{jk}(P_n \mathbf{U} + Q_n \mathbf{U}) e_{jk}(Q_n \mathbf{U}) dx \leq \\ &c_4 (\|\mathbf{u}_1\|_V + \|\mathbf{u}_1\|_V^2 + \|\mathbf{u}_2\|_V^2) \|Q_n \mathbf{U}\| + c_4 (\|\mathbf{u}_2\|_V + 1) \|P_n \mathbf{U}\| \|Q_n \mathbf{U}\|_V. \end{aligned} \quad (20)$$

另外, 易知

$$\langle \mathbf{F}, Q_n \mathbf{U} \rangle \leq \|\mathbf{F}\|_{V^*} \|Q_n \mathbf{U}\|_V, \quad \langle \tilde{\mathbf{f}}, Q_n \mathbf{W} \rangle \leq \|\tilde{\mathbf{F}}\|_{H^{-1}} \|Q_n \mathbf{W}\|_{1,2}. \quad (21)$$

由假设条件(A), 可得

$$\begin{cases} \langle \mathbf{g}(t, \mathbf{u}_{1,t}) - \mathbf{g}(t, \mathbf{u}_{2,t}), Q_n \mathbf{U} \rangle \leq \|\mathbf{g}(t, \mathbf{u}_{1,t}) - \mathbf{g}(t, \mathbf{u}_{2,t})\| \|Q_n \mathbf{U}\| \leq L \|\mathbf{u}_t\|_{\gamma} \|Q_n \mathbf{U}\|, \\ \langle \tilde{\mathbf{g}}(t, \boldsymbol{\omega}_{1,t}) - \tilde{\mathbf{g}}(t, \boldsymbol{\omega}_{2,t}), Q_n \mathbf{W} \rangle \leq L \|\mathbf{W}_t\|_{\gamma} \|Q_n \mathbf{W}\|. \end{cases} \quad (22)$$

将式(14)–(22) 代入式(12)和(13), 并结合如下 Poincaré 型不等式

$$\lambda_{n+1} \|Q_n \mathbf{U}\|^2 \leq \|Q_n \mathbf{U}\|_V^2, \quad \lambda_{n+1}^* \|Q_n \mathbf{W}\|^2 \leq \|Q_n \mathbf{W}\|_{1,2}^2,$$

可得

$$\begin{aligned} \frac{d}{dt} (\|Q_n \mathbf{U}\|^2 + \|Q_n \mathbf{W}\|^2) &+ \left( c_1 \nu \lambda_{n+1}^* + 8\nu_r - \frac{c_3^2}{2c_1 \sqrt{\mu_1 \nu}} \|\boldsymbol{\omega}_2\|_{1,2}^2 - \frac{4\nu_r^2}{c_1 \mu_1} \right) \|Q_n \mathbf{W}\|^2 + \\ &2 \left( c_1 \mu_1 \lambda_{n+1} - \frac{c_3^2}{c_1 \mu_1} \|\mathbf{u}_1\|_V^2 - \frac{c_3^2}{4c_1 \sqrt{\mu_1 \nu}} \|\boldsymbol{\omega}_2\|_{1,2}^2 - \right. \\ &\left. \frac{4\nu_r^2}{c_1 \nu} - c_4 (\|\mathbf{u}_1\|_V + \|\mathbf{u}_1\|_V^2 + \|\mathbf{u}_2\|_V^2) \right) \|Q_n \mathbf{U}\|^2 \leq \\ &2c_3 \|P_n \mathbf{U}\|^{1/2} \|P_n \mathbf{U}\|_V^{1/2} \|Q_n \mathbf{U}\|_V (\|\mathbf{u}_1\|^{1/2} \|\mathbf{u}_1\|_V^{1/2} + \|\mathbf{u}_2\|^{1/2} \|\mathbf{u}_2\|_V^{1/2}) + \\ &2c_3 \|\mathbf{u}_1\|^{1/2} \|\mathbf{u}_1\|_V^{1/2} \|P_n \mathbf{W}\|^{1/2} \|P_n \mathbf{W}\|_{1,2}^{1/2} \|Q_n \mathbf{W}\|_{1,2} + \\ &2c_3 \|P_n \mathbf{U}\|^{1/2} \|P_n \mathbf{U}\|_V^{1/2} \|\boldsymbol{\omega}_2\|_{1,2} \|Q_n \mathbf{W}\|^{1/2} \|Q_n \mathbf{W}\|_{1,2}^{1/2} + \\ &4\nu_r \|P_n \mathbf{W}\| \|Q_n \mathbf{U}\|_V + 4\nu_r \|P_n \mathbf{U}\| \|Q_n \mathbf{W}\|_{1,2} + \\ &2c_4 (1 + \|\mathbf{u}_2\|_V) \|P_n \mathbf{U}\| \|Q_n \mathbf{U}\|_V + 2 \|\mathbf{F}\|_{V^*} \|Q_n \mathbf{U}\|_V + \\ &2 \|\tilde{\mathbf{F}}\|_{H^{-1}} \|Q_n \mathbf{W}\|_{1,2} + 2L \|\mathbf{U}_t\|_{\gamma} \|Q_n \mathbf{U}\| + 2L \|\boldsymbol{\omega}_t\|_{\gamma} \|Q_n \mathbf{W}\|. \end{aligned} \quad (23)$$

令

$$\zeta(t) := \|Q_n \mathbf{U}(t)\|^2 + \|Q_n \mathbf{W}(t)\|^2,$$

$$\vartheta(t) := 2 \min \left\{ c_1 \mu_1 \lambda_{n+1} - \frac{c_3^2}{c_1 \mu_1} \|\mathbf{u}_1\|_V^2 - \frac{c_3^2}{4c_1 \sqrt{\mu_1 \nu}} \|\boldsymbol{\omega}_2\|_{1,2}^2 - \frac{4\nu_r^2}{c_1 \nu} - \right.$$

$$\left. c_4 (\|\mathbf{u}_1\|_V + \|\mathbf{u}_1\|_V^2 + \|\mathbf{u}_2\|_V^2), c_1 \nu \lambda_{n+1}^* + 8\nu_r - \frac{c_3^2}{2c_1 \sqrt{\mu_1 \nu}} \|\boldsymbol{\omega}_2\|_{1,2}^2 - \frac{4\nu_r^2}{c_1 \mu_1} \right\},$$

$$\begin{aligned} \varrho(t) &:= 2c_3 \|P_n \mathbf{U}\|^{1/2} \|P_n \mathbf{U}\|_V^{1/2} \|Q_n \mathbf{U}\|_V (\|\mathbf{u}_1\|^{1/2} \|\mathbf{u}_1\|_V^{1/2} + \|\mathbf{u}_2\|^{1/2} \|\mathbf{u}_2\|_V^{1/2}) + \\ &2c_3 \|\mathbf{u}_1\|^{1/2} \|\mathbf{u}_1\|_V^{1/2} \|P_n \mathbf{W}\|^{1/2} \|P_n \mathbf{W}\|_{1,2}^{1/2} \|Q_n \mathbf{W}\|_{1,2} + \\ &2c_4 (1 + \|\mathbf{u}_2\|_V) \|P_n \mathbf{U}\| \|Q_n \mathbf{U}\|_V + \end{aligned}$$

$$\begin{aligned}
 & 4\nu_r \| P_n \mathbf{U} \| \| Q_n \mathbf{W} \|_{1,2} + 4\nu_r \| P_n \mathbf{W} \| \| Q_n \mathbf{U} \|_V + \\
 & 2c_3 \| P_n \mathbf{U} \|^{1/2} \| P_n \mathbf{U} \|_V^{1/2} \| \boldsymbol{\omega}_2 \|_{1,2} \| Q_n \mathbf{W} \|^{1/2} \| Q_n \mathbf{W} \|_{1,2}^{1/2} + \\
 & 2 \| \mathbf{F} \|_{V^*} \| Q_n \mathbf{U} \|_V + 2 \| \tilde{\mathbf{F}} \|_{H^{-1}} \| Q_n \mathbf{W} \|_{1,2} + 2L \| \mathbf{U}_t \|_\gamma \| Q_n \mathbf{U} \| + 2L \| \mathbf{W}_t \|_\gamma \| Q_n \mathbf{W} \|,
 \end{aligned}$$

则式(23)可表示为

$$\frac{d\zeta(t)}{dt} + \vartheta(t)\zeta(t) \leq \varrho(t), \quad \forall t \geq \tau. \tag{24}$$

第二步: 验证  $\vartheta(t)$  和  $\varrho(t)$  分别满足引理 3 中条件(5)和(6).

(I) 验证  $\vartheta(t)$  满足条件(5). 为此, 用  $\mathbf{u}_i$  和  $\boldsymbol{\omega}_i$  分别与式(3)的第一式和第二式作对偶积并相加, 可得

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} (\| \mathbf{u}_i \|^2 + \| \boldsymbol{\omega}_i \|^2) + 2\mu_1 \langle A_1 \mathbf{u}_i, \mathbf{u}_i \rangle + \nu \langle A_2 \boldsymbol{\omega}_i, \boldsymbol{\omega}_i \rangle + \\
 & \langle B_1(\mathbf{u}_i, \mathbf{u}_i), \mathbf{u}_i \rangle + \langle B_2(\mathbf{u}_i, \boldsymbol{\omega}_i), \boldsymbol{\omega}_i \rangle + \\
 & \langle \mathcal{K}_1(\boldsymbol{\omega}_i), \mathbf{u}_i \rangle + \langle \mathcal{K}_2(\mathbf{u}_i, \boldsymbol{\omega}_i), \boldsymbol{\omega}_i \rangle + \langle N(\mathbf{u}_i), \mathbf{u}_i \rangle = \\
 & \langle \mathbf{f}_i(t, x), \mathbf{u}_i \rangle + \langle \tilde{\mathbf{f}}_i(t, x), \boldsymbol{\omega}_i \rangle + \langle \mathbf{g}(t, \mathbf{u}_{i,t}), \mathbf{u}_i \rangle + \langle \tilde{\mathbf{g}}(t, \boldsymbol{\omega}_{i,t}), \boldsymbol{\omega}_i \rangle, \quad i = 1, 2.
 \end{aligned}$$

结合算子  $\mathcal{K}_1, \mathcal{K}_2$  和  $N$  的定义, 以及引理 1 和引理 2, 可推得

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} (\| \mathbf{u}_i \|^2 + \| \boldsymbol{\omega}_i \|^2) + \frac{3}{2} c_1 \mu_1 \| \mathbf{u}_i \|_V^2 + \frac{1}{2} c_1 \nu \| \boldsymbol{\omega}_i \|_{1,2}^2 + 4\nu_r \| \boldsymbol{\omega}_i \|^2 \leq \\
 & \frac{1}{2c_1 \mu_1} \| \mathbf{f}_i \|_{V^*}^2 + \frac{1}{2c_1 \nu} \| \tilde{\mathbf{f}}_i \|_{H^{-1}}^2 + L \| \mathbf{u}_{i,t} \|_\gamma^2 + L \| \boldsymbol{\omega}_{i,t} \|_\gamma^2, \quad i = 1, 2.
 \end{aligned}$$

积分上式并结合式(4), 可得

$$\begin{aligned}
 & \| \mathbf{u}_i(t + T_0) \|^2 + \| \boldsymbol{\omega}_i(t + T_0) \|^2 + 3c_1 \mu_1 \int_t^{t+T_0} \| \mathbf{u}_i(\theta) \|_V^2 d\theta + c_1 \nu \int_t^{t+T_0} \| \boldsymbol{\omega}_i(\theta) \|_{1,2}^2 d\theta + \\
 & 8\nu_r \int_t^{t+T_0} \| \boldsymbol{\omega}_i(\theta) \|^2 d\theta \leq \\
 & \| \mathbf{u}_i(t) \|^2 + \| \boldsymbol{\omega}_i(t) \|^2 + \frac{1}{c_1 \mu_1} \int_t^{t+T_0} \| \mathbf{f}_i(\theta) \|_{V^*}^2 d\theta + \frac{1}{c_1 \nu} \int_t^{t+T_0} \| \tilde{\mathbf{f}}_i(\theta) \|_{H^{-1}}^2 d\theta + \\
 & 2L \int_t^{t+T_0} \| (\mathbf{u}_{i,\theta}, \boldsymbol{\omega}_{i,\theta}) \|_\gamma^2 d\theta \leq \\
 & 2e^{-(\delta-2L)(t-\tau)} \| (\boldsymbol{\phi}_i, \boldsymbol{\psi}_i) \|_\gamma^2 + \frac{2}{\min\{c_1 \mu_1, c_1 \nu\}} \| (\mathbf{f}_i, \tilde{\mathbf{f}}_i) \|_{L^2(\tau, +\infty; \hat{V}^*)}^2 + \\
 & 2L \int_t^{t+T_0} \| (\mathbf{u}_{i,\theta}, \boldsymbol{\omega}_{i,\theta}) \|_\gamma^2 d\theta, \quad i = 1, 2.
 \end{aligned}$$

由上式, 可推得

$$\begin{aligned}
 & \limsup_{t \rightarrow +\infty} \frac{1}{T_0} \int_t^{t+T_0} \vartheta(\theta) d\theta = \\
 & \limsup_{t \rightarrow +\infty} \frac{1}{T_0} \int_t^{t+T_0} \min \left\{ 2c_1 \mu_1 \lambda_{n+1} - \frac{2c_3^2}{c_1 \mu_1} \| \mathbf{u}_1 \|_V^2 - \frac{c_3^2}{2c_1 \sqrt{\mu_1 \nu}} \| \boldsymbol{\omega}_2 \|_{1,2}^2 - \frac{8\nu_r^2}{c_1 \nu} - \right. \\
 & \left. 2c_4 (\| \mathbf{u}_1 \|_V + \| \mathbf{u}_1 \|_V^2 + \| \mathbf{u}_2 \|_V^2), c_1 \nu \lambda_{n+1}^* + 8\nu_r - \frac{c_3^2}{2c_1 \sqrt{\mu_1 \nu}} \| \boldsymbol{\omega}_2 \|_{1,2}^2 - \frac{4\nu_r^2}{c_1 \mu_1} \right\} d\theta \leq \\
 & \min \left\{ 2c_1 \mu_1 \lambda_{n+1} + \frac{8\nu_r^2}{c_1 \nu} + \right. \\
 & \left. \limsup_{t \rightarrow +\infty} \frac{1}{T_0} \int_t^{t+T_0} \left[ \frac{2c_3^2}{c_1 \mu_1} \| \mathbf{u}_1 \|_V^2 + \frac{c_3^2}{2c_1 \sqrt{\mu_1 \nu}} \| \boldsymbol{\omega}_2 \|_{1,2}^2 + 2c_4 (\| \mathbf{u}_1 \|_V + \| \mathbf{u}_1 \|_V^2 + \| \mathbf{u}_2 \|_V^2) \right] d\theta, \right. \\
 & \left. c_1 \nu \lambda_{n+1}^* + 8\nu_r + \frac{4\nu_r^2}{c_1 \mu_1} + \frac{c_3^2}{2c_1 \sqrt{\mu_1 \nu}} \limsup_{t \rightarrow +\infty} \frac{1}{T_0} \int_t^{t+T_0} \| \boldsymbol{\omega}_2 \|_{1,2}^2 d\theta \right\} \leq
 \end{aligned}$$

$$\min \left\{ 2c_1\mu_1\lambda_{n+1} + \frac{8\nu_r^2}{c_1\nu} + c_4 + c_5 \sum_{i=1}^2 \| (f_i, \tilde{f}_i) \|_{L^2(\tau, +\infty; \hat{\nu}^*)}^2, \right. \\ \left. c_1\nu\lambda_{n+1}^* + 8\nu_r + \frac{4\nu_r^2}{c_1\mu_1} + \frac{c_3^2}{2c_1^2\sqrt{\mu_1\nu} \min\{\mu_1, \nu\}} \| (f_2, \tilde{f}_2) \|_{L^2(\tau, +\infty; \hat{\nu}^*)}^2 \right\} < +\infty, \quad (25)$$

其中  $c_5 := \frac{2T_0 + 2L}{T_0 \min\{c_1\mu_1, c_1\nu\}} \left( \frac{2c_3^2}{3c_1^2\mu_1^2} + \frac{c_4}{c_1\mu_1} + \frac{c_3^2}{2c_1^2\nu\sqrt{\mu_1\nu}} \right)$ . 类似地, 可推得

$$\liminf_{t \rightarrow +\infty} \frac{1}{T_0} \int_t^{t+T_0} \vartheta(\theta) d\theta \geq \\ \min \left\{ 2c_1\mu_1\lambda_{n+1} - \frac{8\nu_r^2}{c_1\nu} - c_4 - c_5 \sum_{i=1}^2 \| (f_i, \tilde{f}_i) \|_{L^2(\tau, +\infty; \hat{\nu}^*)}^2, \right. \\ \left. c_1\nu\lambda_{n+1}^* + 8\nu_r - \frac{4\nu_r^2}{c_1\mu_1} - \frac{c_3^2}{2c_1^2\sqrt{\mu_1\nu} \min\{\mu_1, \nu\}} \| (f_2, \tilde{f}_2) \|_{L^2(\tau, +\infty; \hat{\nu}^*)}^2 \right\}.$$

由于当  $n \rightarrow +\infty$  时,  $\lambda_n \rightarrow +\infty$  且  $\lambda_n^* \rightarrow +\infty$ , 故可取  $\lambda_n$  和  $\lambda_n^*$  满足式(9), 进而可得

$$\liminf_{t \rightarrow +\infty} \frac{1}{T_0} \int_t^{t+T_0} \vartheta(\theta) d\theta > 0, \quad T_0 > 0. \quad (26)$$

由式(25)和式(26), 可知  $\vartheta(t)$  满足条件(5).

(II) 验证  $\varrho(t)$  满足条件(6). 由  $\varrho(t)$  的表达式可知, 我们需要得到  $\|U_t\|_\gamma$  和  $\|W_t\|_\gamma$  的相关估计. 为此, 用  $U$  和  $W$  分别与式(10)的第一式和第二式作对偶积, 可得

$$\begin{cases} \frac{1}{2} \frac{d}{dt} \|U\|^2 + \langle 2\mu_1 A_1 U, U \rangle + \langle B_1(u_1, u_1) - B_1(u_2, u_2), U \rangle + \langle \mathcal{K}_1(W), U \rangle + \\ \langle N(u_1) - N(u_2), U \rangle = \langle F, U \rangle + \langle g(t, u_{1,t}) - g(t, u_{2,t}), U \rangle, \\ \frac{1}{2} \frac{d}{dt} \|W\|^2 + \langle \nu A_2 W, W \rangle + \langle B_2(u_1, \omega_1) - B_2(u_2, \omega_2), W \rangle + \langle \mathcal{K}_2(U, W), W \rangle = \\ \langle \tilde{F}, W \rangle + \langle \tilde{g}(t, \omega_{1,t}) - \tilde{g}(t, \omega_{2,t}), W \rangle. \end{cases} \quad (27)$$

由引理1, 易得

$$2c_1\mu_1 \|U\|_V^2 \leq \langle 2\mu_1 A_1 U, U \rangle, \quad c_1\nu \|W\|_{1,2}^2 \leq \langle \nu A_2 W, W \rangle. \quad (28)$$

根据文献[53]中的式(3.28), 可知

$$\langle N(u_1) - N(u_2), U \rangle \geq 0. \quad (29)$$

经直接计算, 可得

$$\langle \mathcal{K}_1(W), U \rangle + \langle \mathcal{K}_2(U, W), W \rangle = 4\nu_r \|W\|^2, \quad (30)$$

$$\langle F, U \rangle + \langle \tilde{F}, W \rangle \leq \|F\|_{V^*} \|U\|_V + \|\tilde{F}\|_{H^{-1}} \|W\|_{1,2} \leq \\ \frac{1}{c_1\mu_1} \|F\|_{V^*}^2 + \frac{1}{c_1\nu} \|\tilde{F}\|_{H^{-1}}^2 + \frac{c_1\mu_1}{4} \|U\|_V^2 + \frac{c_1\nu}{4} \|W\|_{1,2}^2, \quad (31)$$

$$\langle g(t, u_{1,t}) - g(t, u_{2,t}), U \rangle + \langle \tilde{g}(t, \omega_{1,t}) - \tilde{g}(t, \omega_{2,t}), W \rangle \leq L (\|U_t\|_\gamma^2 + \|W_t\|_\gamma^2). \quad (32)$$

另外, 由引理1及 Gagliardo-Nirenberg 不等式和 Young's 不等式, 可得

$$|\langle B_1(u_1, u_1) - B_1(u_2, u_2), U \rangle| = |\langle B_1(U, u_1) - B_1(u_2, U), U \rangle| \leq \\ c_3 \|U\|^{1/2} \|\nabla U\| \|u_1\| \|D^2 U\|^{1/2} \leq \frac{64c_3^4}{c_1^3\mu_1^3} \|u_1\|^4 \|U\|^2 + \frac{c_1\mu_1}{4} \|U\|_V^2, \quad (33)$$

$$|\langle B_2(u_1, \omega_1) - B_2(u_2, \omega_2), W \rangle| = \\ |\langle B_2(U, \omega_1) - B_2(u_2, W), W \rangle| \leq \|U\|_{L^\infty} \|\omega_1\| \|\nabla W\| \leq \\ c \|U\|^{1/2} \|D^2 U\|^{1/2} \|\omega_1\| \|\nabla W\| \leq \\ \tilde{c} \|\omega_1\|^4 \|U\|^2 + \frac{c_1\mu_1}{4} \|U\|_V^2 + \frac{c_1\nu}{4} \|W\|_{1,2}^2, \quad (34)$$

其中  $\tilde{c}$  为与  $c_1, \mu_1, \nu$  有关的常数. 将式(28)–(34)代入式(27), 得

$$\begin{aligned} & \frac{d}{dt} (\|U\|^2 + \|W\|^2) + 8\nu_r \|W\|^2 + c_1\nu \|W\|_{1,2}^2 + \\ & \frac{5c_1\mu_1}{2} \|U\|_{\dot{V}}^2 - \frac{16c_3^4}{c_1^3\mu_1^3} \|u_1\|^4 \|U\|^2 - 2\tilde{c} \|\omega_1\|^4 \|U\|^2 \leq \\ & 2L (\|U_t\|_{\dot{\gamma}}^2 + \|W_t\|_{\dot{\gamma}}^2) + \frac{2}{c_1\mu_1} \|F\|_{\dot{V}^*}^2 + \frac{2}{c_1\nu} \|\tilde{F}\|_{H^{-1}}^2. \end{aligned} \quad (35)$$

由式(4)可知, 对于  $\forall \varepsilon > 0$ , 存在  $T_1 > \tau$ , 使得

$$\|u_1(t)\|^4 + \|\omega_1(t)\|^4 < \varepsilon + \frac{1}{\min\{c_1\mu_1, c_1\nu\}} \|(f_1, \tilde{f}_1)\|_{L^2(\tau, +\infty; \dot{V}^*)}^4, \quad \forall t \geq T_1,$$

因此, 由式(8)和式(35), 可得

$$\begin{aligned} & \frac{d}{dt} (\|U\|^2 + \|W\|^2) + \delta (\|U\|^2 + \|W\|^2) \leq \\ & \frac{2}{c_1 \min\{\mu_1, \nu\}} \|(F, \tilde{F})\|_{\dot{V}^*}^2 + 2L (\|U_t\|_{\dot{\gamma}}^2 + \|W_t\|_{\dot{\gamma}}^2), \quad \forall t \geq T_1, \end{aligned}$$

其中  $\delta := \min\{2c_1\mu_1\lambda_1, c_1\nu\lambda_1^* + 8\nu_r\}$ . 运用与文献[54]中的式(2.34)相同的证明过程, 由上式可推得

$$\begin{aligned} & \|U_t\|_{\dot{\gamma}}^2 + \|W_t\|_{\dot{\gamma}}^2 \leq e^{-(\delta-2L)(t-\tau)} (\|\phi_1 - \phi_2\|_{\dot{\gamma}}^2 + \|\psi_1 - \psi_2\|_{\dot{\gamma}}^2) + \\ & \frac{2}{\min\{c_1\mu_1, c_1\nu\}} \int_{\tau}^t e^{-(\delta-2L)(t-\theta)} \|(F, \tilde{F})(\theta)\|_{\dot{V}^*}^2 d\theta, \quad \forall t \geq T_1 > \tau. \end{aligned} \quad (36)$$

接下来, 我们验证  $\varrho(t)$  满足条件式(6). 首先, 因为  $2L < \delta$ , 故由式(4)、(36)和 Hölder 不等式, 可得

$$\begin{aligned} & \lim_{t \rightarrow +\infty} \frac{1}{T_0} \int_t^{t+T_0} 2L (\|U_{\theta}\|_{\dot{\gamma}} \|Q_n U(\theta)\| + \|W_{\theta}\|_{\dot{\gamma}} \|Q_n W(\theta)\|) d\theta \leq \\ & \lim_{t \rightarrow +\infty} \frac{2L}{\min\{c_1\mu_1, c_1\nu\}^{1/2} \sqrt{T_0}} \|(F, \tilde{F})\|_{L^2(\tau, +\infty; \dot{V}^*)} \left[ \int_t^{t+T_0} (\|U_{\theta}\|_{\dot{\gamma}}^2 + \|W_{\theta}\|_{\dot{\gamma}}^2) d\theta \right]^{1/2} \leq \\ & \lim_{t \rightarrow +\infty} \frac{2L \|(F, \tilde{F})\|_{L^2(\tau, +\infty; \dot{V}^*)}}{\min\{c_1\mu_1, c_1\nu\}^{1/2} \sqrt{T_0}} \left[ \int_t^{t+T_0} e^{-(\delta-2L)(\theta-\tau)} (\|\phi_1 - \phi_2\|_{\dot{\gamma}}^2 + \|\psi_1 - \psi_2\|_{\dot{\gamma}}^2) + \right. \\ & \left. \frac{2}{\min\{c_1\mu_1, c_1\nu\}} \int_t^{t+T_0} \int_{\tau}^{\theta} e^{-(\delta-2L)(\theta-r)} \|(F, \tilde{F})\|_{\dot{V}^*}^2 dr d\theta \right]^{1/2} \leq \\ & \lim_{t \rightarrow +\infty} \frac{2\sqrt{2}L \|(F, \tilde{F})\|_{L^2(\tau, +\infty; \dot{V}^*)}}{\min\{c_1\mu_1, c_1\nu\} \sqrt{T_0}} \left[ \int_t^{t+T_0} \int_{\tau}^{T_2} e^{-(\delta-2L)(\theta-r)} \|(F, \tilde{F})\|_{\dot{V}^*}^2 dr d\theta + \right. \\ & \left. \int_t^{t+T_0} \int_{T_2}^{\theta} e^{-(\delta-2L)(\theta-r)} \|(F, \tilde{F})\|_{\dot{V}^*}^2 dr d\theta \right]^{1/2} \leq \\ & \lim_{t \rightarrow +\infty} \frac{2\sqrt{2}L \|(F, \tilde{F})\|_{L^2(\tau, +\infty; \dot{V}^*)}}{\min\{c_1\mu_1, c_1\nu\} \sqrt{T_0}} \left[ \frac{1}{\delta - 2L} \|(F, \tilde{F})\|_{L^2(\tau, +\infty; \dot{V}^*)}^2 \times \right. \\ & \left. (e^{-(\delta-2L)(t-T_2)} - e^{-(\delta-2L)(t+T_0-T_2)}) + \frac{T_0}{\delta - 2L} \|(F, \tilde{F})\|_{L^{\infty}(T_2, +\infty; \dot{V}^*)}^2 \right]^{1/2} \leq \\ & \lim_{t \rightarrow +\infty} \frac{2\sqrt{2}L \|(F, \tilde{F})\|_{L^2(\tau, +\infty; \dot{V}^*)}}{\min\{c_1\mu_1, c_1\nu\} (\delta - 2L)^{1/2}} \|(F, \tilde{F})\|_{L^{\infty}(T_2, +\infty; \dot{V}^*)}, \quad \forall T_2 \geq \tau. \end{aligned} \quad (37)$$

由式(7)可知, 当  $t \rightarrow +\infty$  时,  $\|(F, \tilde{F})(t)\|_{\dot{V}^*} \rightarrow 0$ . 所以, 对于  $\forall \varepsilon > 0$ , 可在式(37)中选取  $T_2$  (充分大), 使得  $T_2 > T_1 > \tau$  且

$$\|(F, \tilde{F})\|_{L^{\infty}(T_2, +\infty; \dot{V}^*)} < \varepsilon. \quad (38)$$

由  $\varepsilon$  的任意性, 并结合式(37)和式(38), 可知

$$\lim_{t \rightarrow +\infty} \frac{1}{T_0} \int_t^{t+T_0} 2L( \| \mathbf{U}_\theta \|_\gamma \| \mathbf{Q}_n \mathbf{U}(\theta) \| + \| \mathbf{W}_\theta \|_\gamma \| \mathbf{Q}_n \mathbf{W}(\theta) \| ) d\theta = 0. \quad (39)$$

另外, 在式(7)的条件下, 易得

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{1}{T_0} \int_t^{t+T_0} [ & 2c_3 \| \mathbf{P}_n \mathbf{U} \|^{1/2} \| \mathbf{P}_n \mathbf{U} \|^{1/2} \| \mathbf{Q}_n \mathbf{U} \|_\nu ( \| \mathbf{u}_1 \|^{1/2} \| \mathbf{u}_1 \|_\nu^{1/2} + \| \mathbf{u}_2 \|^{1/2} \| \mathbf{u}_2 \|_\nu^{1/2} ) + \\ & 2c_3 \| \mathbf{u}_1 \|^{1/2} \| \mathbf{u}_1 \|_\nu^{1/2} \| \mathbf{P}_n \mathbf{W} \|^{1/2} \| \mathbf{P}_n \mathbf{W} \|_{1,2}^{1/2} \| \mathbf{Q}_n \mathbf{W} \|_{1,2} + \\ & 2c_4 (1 + \| \mathbf{u}_2 \|_\nu) \| \mathbf{P}_n \mathbf{U} \| \| \mathbf{Q}_n \mathbf{U} \|_\nu + 2 \| \mathbf{F} \|_{\nu^*} \| \mathbf{Q}_n \mathbf{U} \|_\nu + 4\nu_r \| \mathbf{P}_n \mathbf{W} \| \| \mathbf{Q}_n \mathbf{U} \|_\nu + \\ & 2c_3 \| \mathbf{P}_n \mathbf{U} \|^{1/2} \| \mathbf{P}_n \mathbf{U} \|_\nu^{1/2} \| \boldsymbol{\omega}_2 \|_{1,2} \| \mathbf{Q}_n \mathbf{W} \|^{1/2} \| \mathbf{Q}_n \mathbf{W} \|_{1,2}^{1/2} + 4\nu_r \| \mathbf{P}_n \mathbf{U} \| \| \mathbf{Q}_n \mathbf{W} \|_{1,2} + \\ & 2 \| \mathbf{F} \|_{\nu^*} \| \mathbf{Q}_n \mathbf{U} \|_\nu + 2 \| \tilde{\mathbf{F}} \|_{H^{-1}} \| \mathbf{Q}_n \mathbf{W} \|_{1,2} ] d\theta = 0. \end{aligned} \quad (40)$$

进而结合式(39)和(40), 可知  $\varrho(t)$  满足条件(6).

第三步: 证明问题(3)具有有限的确定模个数. 运用引理3和式(24), 可推得

$$\lim_{t \rightarrow +\infty} \zeta(t) = \lim_{t \rightarrow +\infty} ( \| \mathbf{Q}_n \mathbf{U}(t) \|^2 + \| \mathbf{Q}_n \mathbf{W}(t) \|^2 ) = 0,$$

即式(11)成立, 定理1得证.

### 参考文献 (References):

- [1] DUAN H, PENG T, TU Z, et al. Finite-time stabilization and destabilization analysis of quaternion-valued neural networks with discrete delays[J]. *Complexity*, 2020, **2020**(1): 8526030.
- [2] HE X, QIU J, LI X, et al. A brief survey on stability and stabilization of impulsive systems with delayed impulses[J]. *Discrete and Continuous Dynamical Systems S*, 2022, **15**(7): 1797.
- [3] JIANG X, CHEN X, CHI M, et al. On Hopf bifurcation and control for a delay systems[J]. *Applied Mathematics and Computation*, 2020, **370**: 124906.
- [4] SURIGUGA, KAO Y, SHAO C, et al. Stability of high-order delayed Markovian jumping reaction-diffusion HNNs with uncertain transition rates[J]. *Applied Mathematics and Computation*, 2021, **389**: 125559.
- [5] LI T, ROGOVCHENKO Y V. On asymptotic behavior of solutions to higher-order sublinear Emden-Fowler delay differential equations[J]. *Applied Mathematics Letters*, 2017, **67**: 53-59.
- [6] MI L. Asymptotic boundary estimates for solutions to the  $p$ -Laplacian with infinite boundary values[J]. *Boundary Value Problems*, 2019, **2019**(1): 66.
- [7] ZHANG C, AGARWAL R P, BOHNER M, et al. Oscillation of fourth-order delay dynamic equations[J]. *Science China Mathematics*, 2015, **58**(1): 143-160.
- [8] MÁLEK J, NEČAS J, ROKYTA M, et al. *Weak and Measure-Valued Solutions to Evolutionary PDEs*[M]. Boston, MA: Springer US, 1996.
- [9] BAE H O. Existence, regularity, and decay rate of solutions of non-Newtonian flow[J]. *Journal of Mathematical Analysis and Applications*, 1999, **231**(2): 467-491.
- [10] BLOOM F, HAO W. Regularization of a non-Newtonian system in an unbounded channel: existence and uniqueness of solutions[J]. *Nonlinear Analysis: Theory, Methods & Applications*, 2001, **44**(3): 281-309.
- [11] BLOOM F, HAO W. Regularization of a non-Newtonian system in an unbounded channel: existence of a maximal compact attractor[J]. *Nonlinear Analysis: Theory, Methods & Applications*, 2001, **43**(6): 743-766.
- [12] ZHAO C, LI Y, ZHOU S. Regularity of trajectory attractor and upper semicontinuity of global attractor for a 2D non-Newtonian fluid[J]. *Journal of Differential Equations*, 2009, **247**(8): 2331-2363.
- [13] MIGUN N P, PROHORENKO P P. *Hydrodynamics and Heat Exchange for Gradient Flows of Fluid With Microstructure*[M]. Minsk: Nauka i Tekhnika, 1984.
- [14] ABIDIN M Z, MARWAN M, ULLAH N, et al. Well-posedness in variable-exponent function spaces for the three-dimensional micropolar fluid equations[J]. *Journal of Mathematics*, 2023, **2023**: 4083997.
- [15] DONG B Q, CHEN Z M. Global attractors of two-dimensional micropolar fluid flows in some unbounded domains[J]. *Applied Mathematics and Computation*, 2006, **182**(1): 610-620.
- [16] LUKASZEWICZ G. *Micropolar Fluids: Theory and Applications, Modeling and Simulation in Science, Engi-*

- neering and Technology*[M]. Boston: Birkh auser, 1999.
- [17] 赖春林, 孙文龙. 微极流方程组确定结点个数估计[J]. 武汉大学学报(理学版), 2023, **69**(6): 837-844. (LAI Chunlin, SUN Wenlong. Estimation of the number of determining nodes to the non-autonomous micropolar uid equations[J]. *Journal of Wuhan Univesity (Natnral Science Edition)*, 2023, **69**(6): 837-844. (in Chinese))
- [18] SUN W. Micropolar fluid flows with delay on 2D unbounded domains[J]. *Journal of Applied Analysis & Computation*, 2018, **8**(1): 356-378.
- [19] SUN W, CHENG J, HAN X. Random attractors for 2D stochastic micropolar fluid flows on unbounded domains[J]. *Discrete and Continuous Dynamical Systems B*, 2021, **26**(1): 693-716.
- [20] ZHAO C, SUN W. Global well-posedness and pullback attractors for a two-dimensional non-autonomous micropolar fluid flows with infinite delays[J]. *Communications in Mathematical Sciences*, 2017, **15**(1): 97-121.
- [21] AGARWAL R P, BOHNER M, LI T. Oscillatory behavior of second-order half-linear damped dynamic equations[J]. *Applied Mathematics and Computation*, 2015, **254**: 408-418.
- [22] JIA Z, YANG Z. Large time behavior to a chemotaxis-consumption model with singular sensitivity and logistic source[J]. *Mathematical Methods in the Applied Sciences*, 2021, **44**(5): 3630-3645.
- [23] LIU Q, BOHNER M, GRACE S R, et al. Asymptotic behavior of even-order damped differential equations with  $p$ -Laplacian like operators and deviating arguments[J]. *Journal of Inequalities and Applications*, 2016, **2016**(1): 321.
- [24] LIU Z, DING Y, LIU C, et al. Existence and uniqueness of solutions for singular fractional differential equation boundary value problem with  $p$ -Laplacian[J]. *Advances in Difference Equations*, 2020, **2020**(1): 83.
- [25] LEI C, LI H, ZHAO Y. Dynamical behavior of a reaction-diffusion SEIR epidemic model with mass action infection mechanism in a heterogeneous environment[J]. *Discrete and Continuous Dynamical Systems B*, 2024, **29**(7): 3163-3198.
- [26] MI L. Asymptotic behavior for the unique positive solution to a singular elliptic problem[J]. *Communications on Pure and Applied Analysis*, 2015, **14**(3): 1053-1072.
- [27] 马丽, 孙芳芳. 非 Lipschitz 条件下高维 McKean-Vlasov 随机微分方程解的存在唯一性[J]. 应用数学和力学, 2023, **44**(10): 1272-1290. (MA Li, SUN Fangfang. Existence and uniqueness of the solutions to high-dimensional mckean-vlasov SDEs under Non-Lipschitz conditions[J]. *Applied Mathematics and Mechanics*, 2023, **44**(10): 1272-1290. (in Chinese))
- [28] 帅欣, 倪明康. 一类具有转点的右端不连续奇摄动边值问题[J]. 应用数学和力学, 2024, **45**(4): 470-489. (SHUAI Xin, NI Mingkang. A class of right-hand discontinuous singularly perturbed boundary value problems with turning points[J]. *Applied Mathematics and Mechanics*, 2024, **45**(4): 470-489. (in Chinese))
- [29] UTKU N, LI Tongxing, SENEL M. An asymptotic criterion for third-order dynamic equations with positive and negative coefficients[J]. *Hacettepe Journal of Mathematics and Statistics*, 2015, **44**(5): 1157-1162.
- [30] WANG Y, LIU L. Uniqueness and existence of positive solutions for the fractional integro-differential equation [J]. *Boundary Value Problems*, 2017, **2017**(1): 12.
- [31] YANG M, FU Z, SUN J. Existence and large time behavior to coupled chemotaxis-fluid equations in Besov-Morrey spaces[J]. *Journal of Differential Equations*, 2019, **266**(9): 5867-5894.
- [32] YAN D, PANG G, QIU J, et al. Finite-time stability analysis of switched systems with actuator saturation based on event-triggered mechanism[J]. *Discrete and Continuous Dynamical Systems-S*, 2023, **16**(7): 1929-1943.
- [33] YU M, WU S, LI X. Exponential stabilization of nonlinear systems under saturated control involving impulse correction[J]. *Nonlinear Analysis; Hybrid Systems*, 2023, **48**: 101335.
- [34] DE ARAÚJO G M, DE ARAÚJO M A F, LUCENA E F L. Pullback attractors for a class of non-Newtonian micropolar fluids[J]. *Electronic Journal of Differential Equations*, 2018, **2018**: 1-14.
- [35] AI C, TAN Z. Global and exponential attractors for a class of non-Newtonian micropolar fluids[J]. *Mathematical Methods in the Applied Sciences*, 2021, **44**(13): 10032-10052.
- [36] AI C, TAN Z. Pullback exponential attractors for a class of non-Newtonian micropolar fluids[J]. *Journal of*

- Mathematical Analysis and Applications*, 2021, **503**(2): 125320.
- [37] ZHAO C, ZHANG Y, CARABALLO T, et al. Statistical solutions and degenerate regularity for the micropolar fluid with generalized Newton constitutive law[J]. *Mathematical Methods in the Applied Sciences*, 2023, **46**(9): 10311-10331.
- [38] CHEN X, YANG H, HAN X, et al. Existence and degenerate regularity of trajectory statistical solution for the 3D incompressible non-Newtonian micropolar fluids[J]. *Discrete & Continuous Dynamical Systems-Series S*, 2024, **17**(3). DOI:10.3934/dcdss.2023122.
- [39] DE ARAÚJO G M, DE ARAÚJO M A F, LUCENA E F L. On a system of equations of a non-Newtonian micropolar fluid[J]. *Journal of Applied Mathematics*, 2015, **2015**: 481754.
- [40] ELLAHI R, RAHMAN S U, GULZAR M M, et al. A mathematical study of non-Newtonian micropolar fluid in arterial blood flow through composite stenosis[J]. *Applied Mathematics & Information Sciences*, 2014, **8**(4): 1567-1573.
- [41] FOIAS C, TEMAM R. Determination of the solutions of the Navier-Stokes equations by a set of nodal values [J]. *Mathematics of Computation*, 1984, **43**(167): 117-133.
- [42] FOIAS C, MANLEY O, ROSA R, et al. *Navier-Stokes Equations and Turbulence*[M]. Cambridge: Cambridge University Press, 2001.
- [43] CONSTANTIN P, FOIAS C, MANLEY O P, et al. Determining modes and fractal dimension of turbulent flows [J]. *Journal of Fluid Mechanics*, 1985, **150**: 427-440.
- [44] FLANDOLF F, LANGA J A. Determining modes for dissipative random dynamical systems[J]. *Stochastics and Stochastic Reports*, 1999, **66**(1/2): 1-25.
- [45] GALDI G P. Determining modes, nodes and volume elements for stationary solutions of the Navier-Stokes problem past a three-dimensional body[J]. *Archive for Rational Mechanics and Analysis*, 2006, **180**(1): 97-126.
- [46] JONES D A, TITI E S. Upper bounds on the number of determining modes, nodes, and volume elements for the Navier-Stokes equations[J]. *Indiana University Mathematics Journal*, 1993, **42**(3): 875-887.
- [47] KALANTAROV V K, TITI E S. Global attractors and determining modes for the 3D Navier-Stokes-Voigt equations[J]. *Chinese Annals of Mathematics, Series B*, 2009, **30**(6): 697-714.
- [48] SZOPA P. Determining modes for 2-D micropolar fluid flows[J]. *Mathematical and Computer Modelling*, 2005, **42**(9/10): 1079-1088.
- [49] ZHAO C, ZHANG Y L M. Determining nodes of the global attractor for an incompressible non-Newtonian fluid [J]. *Journal of Applied Analysis & Computation*, 2018, **8**(3): 954-964.
- [50] HALE J K, KATO J. Phase space for retarded equations with infinite delay[J]. *Funkcial Ekvac*, 1978, **21**: 11-41.
- [51] LIU G. Pullback asymptotic behavior of solutions for a 2D non-autonomous non-Newtonian fluid[J]. *Journal of Mathematical Fluid Mechanics*, 2017, **19**(4): 623-643.
- [52] SUN W, LAI C, LIANG Y. Determining modes and determining nodes for the 3D non-autonomous regularized magnetohydrodynamics equations[J]. *Discrete and Continuous Dynamical Systems B*, 2023, **28**(6): 3373-3392.
- [53] ZHAO C, ZHOU S, LI Y. Existence and regularity of pullback attractors for an incompressible non-Newtonian fluid with delays[J]. *Quarterly of Applied Mathematics*, 2009, **67**(3): 503-540.
- [54] ZHAO C, LIU G, AN R. Global well-posedness and pullback attractors for an incompressible non-Newtonian fluid with infinite delays[J]. *Differential Equations and Dynamical Systems*, 2017, **25**(1): 39-64.