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由分数 Brown 运动驱动的随机泛函微分方程的解的存在唯一性及平均原理*

马　丽,　常　洪,　梁　青

(海南师范大学 数学与统计学院, 海口 571158)

摘要: 本文研究了由 Hurst 指数 $H > 1/2$ 的分数 Brown 运动和 Lévy 过程同时驱动的带 Markov 切换和随机比例时间的分布依赖的随机泛函微分方程.首先利用 Carathédory 逼近建立了方程解的存在唯一性,然后在一定的平均条件下,证明了分布依赖随机泛函微分方程的解被其平均化随机泛函微分方程的解在 p-阶矩意义下逼近.

关　键　词: 随机泛函微分方程; 分数 Brown 运动; Lévy 过程; 平均原理

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Existence and Uniqueness With the Averaging Principle for Solutions to Stochastic Functional Differential Equations Driven by Fractional Brownian Motion

MA Li,　CHANG Hong,　LIANG Qing

(*School of Mathematics and Statistics*, *Hainan Normal University*, *Haikou* 571158, *P.R. China*)

Abstract: The distribution-dependent stochastic functional differential equations, driven simultaneously by a fractional Brownian motion with Hurst index $H > 1/2$ and a Lévy process and with the Markov switching and the random proportional times, were studied. Firstly, the existence and uniqueness of the solutions to the equations were established through the Carathédory approximation. Then, under certain averaging conditions, it is proved that the solution to the distribution-dependent stochastic differential equation is approximated (in the sense of the p-th moment convergence) by the solution of its averaged stochastic functional differential equation.

Key words: stochastic functional differential equation; fractional Brownian motion; Lévy process; averaging principle

0　引　　言

近年来,线性随机微分方程解的存在唯一性理论已经比较成熟.但是对于非线性随机微分方程,尤其是

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作者简介: 马丽(1979—),女,博士(E-mail: malihnsd@163.com);
常洪(2000—),男,硕士(E-mail: changhong8240@163.com);
梁青(1980—),男,硕士(通信作者. E-mail: liangqing1112@sina.com).
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具有强非线性、高维复杂结构的随机微分方程,其解的存在唯一性仍然有待解决.本文考虑分数 Brown 运动和 Lévy 过程驱动的带马氏切换和随机比例时间的分布依赖的随机泛函微分方程:

$$\begin{cases} \mathrm{d}\boldsymbol{X}(t)=\boldsymbol{f}(t,\boldsymbol{X}(t),\boldsymbol{X}_t,\mathscr{L}(\boldsymbol{X}(t)),r(t))\mathrm{d}t+ \\ \quad \boldsymbol{g}(t,\boldsymbol{X}(t),\boldsymbol{X}_t,\mathscr{L}(\boldsymbol{X}(t)),r(t))\mathrm{d}\boldsymbol{B}(t)+\boldsymbol{m}(t,\mathscr{L}(\boldsymbol{X}(t)))\mathrm{d}\boldsymbol{B}^H(t)+ \\ \quad \int_{|z|<c}\boldsymbol{h}(t,\boldsymbol{X}(t-),\boldsymbol{X}_t,\mathscr{L}(\boldsymbol{X}(t)),z,r(t))\tilde{N}(\mathrm{d}t,\mathrm{d}z), \qquad t\in[t_0,T], \\ \boldsymbol{X}(t)=\boldsymbol{\xi}(t), \qquad t\in[\underline{\theta}t_0,t_0], \end{cases} \tag{1}$$

其中 $\boldsymbol{X}_t(\theta)=\boldsymbol{X}(\theta t)$, $0<\underline{\theta}\leqslant\theta\leqslant 1$, $\mathscr{L}(\boldsymbol{X}(t))$ 是 $\boldsymbol{X}(t)$ 的分布, $\boldsymbol{f}$: $[t_0,T]\times\mathbb{R}^d\times L^p(\boldsymbol{\Omega};C([\underline{\theta},1];\mathbb{R}^d))\times\mathscr{P}(\mathbb{R}^d)\times\mathbb{S}\to\mathbb{R}^d$, $\boldsymbol{g}$: $[t_0,T]\times\mathbb{R}^d\times\mathscr{P}(\mathbb{R}^d)\times L^p(\boldsymbol{\Omega};C([\underline{\theta},1];\mathbb{R}^d))\times\mathbb{S}\to\mathbb{R}^{d\times l}$, $\boldsymbol{m}$: $[t_0,T]\times\mathscr{P}(\mathbb{R}^d)\to\mathbb{R}^{d\times l}$, $\boldsymbol{h}$: $[t_0,T]\times\mathbb{R}^d\times L^p(\boldsymbol{\Omega};C([\underline{\theta},1];\mathbb{R}^d))\times\mathscr{P}(\mathbb{R}^d)\times\mathbb{R}^d\times\mathbb{S}\to\mathbb{R}^d$ 是 Borel 可测函数.$\boldsymbol{B}(t)$ 是定义在完备概率空间 $(\boldsymbol{\Omega},\mathscr{F},\{\mathscr{F}_t\}_{t\geqslant 0},\mathbb{P})$ 上的 l 维 Brown 运动.$\boldsymbol{B}^H(t)$ 是具有 Hurst 指数 $H\in(1/2,1)$ 的 l 维分数 Brown 运动; $N(\mathrm{d}t,\mathrm{d}z)$ 是定义在 $[0,\infty]\times(\mathbb{R}^d\setminus\{0\})$ 上关于 $(\mathscr{F}_t)_{t\geqslant 0}$- 适应的 Poisson 随机测度, $\tilde{N}(\mathrm{d}t,\mathrm{d}z)=N(\mathrm{d}t,\mathrm{d}z)-\nu(\mathrm{d}z)\mathrm{d}t$ 是其补偿 Poisson 测度,其中 ν 为一个 Lévy 测度即满足 $\int_{\mathbb{R}^d\setminus\{0\}}\frac{\boldsymbol{x}^2}{1+\boldsymbol{x}^2}\nu(\mathrm{d}\boldsymbol{x})<\infty$.本文假设 $V=\nu(|z|<c)<\infty$.对任意的 $t\in[\underline{\theta}t_0,t_0]$, $\boldsymbol{\xi}(t)$ 是 $\mathbb{R}^d$ 值随机变量,且 $\mathbb{E}[\|\boldsymbol{\xi}\|^p]:=\mathbb{E}[\sup_{t\in[\underline{\theta}t_0,t_0]}|\boldsymbol{\xi}(t)|^p]<\infty$, $\boldsymbol{\xi}(t)$ 关于 t 是 Lipschitz 连续的.

分数 Brown 运动驱动的随机微分方程解的存在唯一性方面已经有许多结果.文献[1]得到了由 Hurst 指数 $H>1/2$ 的分数 Brown 运动驱动的多维随机微分方程解的存在唯一性.文献[2]通过 Euler 方法得到了由 Hurst 指数 $H>1/2$ 在零点反射的分数 Brown 运动驱动的随机泛函微分方程解的存在性.文献[3]得到了由分数 Brown 运动和标准 Brown 运动共同驱动的随机微分方程强解的存在唯一性.文献[4]得到了由 Hurst 指数 $H>1/2$ 的分数 Brown 运动驱动的分布依赖的随机微分方程解的存在唯一性.平均原理为简化动力学系统和微分方程的近似解提供了一个强有力的工具.在系数不依赖于分布的情况下,文献[5-7]分别探讨了由 Brown 运动、Lévy 过程、分数 Brown 运动等驱动的随机微分方程的平均原理.在分布依赖的情况下, 文献[8]得到了具有快慢时间尺度的随机微分方程的平均原理; 文献[9]得到了具有快慢时间尺度的随机偏微分方程的平均原理. 文献[10]在一定平均条件下, 建立了由 Brown 运动驱动的随机微分方程的平均原理. 文献[11]得到了由分数 Brown 运动和标准 Brown 运动共同驱动的随机微分方程的平均原理. 带 Markov 切换的随机泛函微分方程越来越受到研究者的关注, 文献[12]研究了含随机比例时间的随机泛函微分方程的解的稳定性.

目前,尚未有相关文献研究由分数 Brown 运动和 Lévy 过程驱动的带 Markov 切换和随机比例时间的分布依赖的随机微分方程.分数 Brown 运动的增量具有长期依赖性(Hurst 指数 $H>1/2$), 适合描述具有记忆效应或持续性的现象(如金融市场波动率、水文时间序列等),无法体现波动的突变.Lévy 过程能够刻画随机现象的突然跳跃或极端波动(如金融危机中的价格崩盘、地震能量释放等)行为,但缺乏对长期依赖关系的刻画.现实生活中许多复杂系统(如金融市场、湍流、生物运动)同时具有长期依赖和离散跳跃行为.单一驱动源(仅分数 Brown 运动或仅 Lévy)无法全面刻画这些复杂系统的动力学特性,联合驱动的随机泛函微分方程能更灵活地刻画这类混合特征.例如,在金融中,Brown 运动因缺乏记忆性不适合作为波动率模型,而 Hurst 指数 $H>1/2$ 的分数 Brown 运动可能过于"平滑"(即粗糙度不足).由 Brown 运动和分数 Brown 运动同时驱动的随机微分方程则可规避这些缺陷.本文利用 Carathédory 逼近建立了方程解的存在唯一性.然后在一定的平均条件下,证明了方程的解能被其平均化随机泛函微分方程的解在 p-阶矩意义下逼近.

本文主要内容安排如下:第 1 节给出了一些预备知识;第 2 节证明了分数 Brown 运动和 Lévy 过程同时驱动的带 Markov 切换和随机比例时间的分布依赖的随机泛函微分方程解的存在唯一性;第 3 节建立了分布依赖的随机泛函微分方程解的平均原理;第 4 节为本文的结论.

1　预备知识

设 $(\boldsymbol{\Omega},\mathscr{F},\{\mathscr{F}_t\}_{t\geqslant 0},\mathbb{P})$ 是一个满足通常条件的完备的带流的概率空间, $|\cdot|$ 和 $\langle\cdot,\cdot\rangle$ 分别表示欧氏

范数和内积，设 $\mathrm{tr}(\boldsymbol{A})$ 和 $\boldsymbol{A}^{\mathrm{T}}$ 分别为矩阵 $\boldsymbol{A}$ 的迹和转置，$\|\boldsymbol{A}\| = \sqrt{\mathrm{tr}(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}})}$.设 $\mathcal{P}(\mathbb{R}^d)$ 是 $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ 上概率测度组成的集合，其中 $\mathcal{B}(\mathbb{R}^d)$ 为 $\mathbb{R}^d$ 上的 Borel $\boldsymbol{\sigma}$-代数.设 $p \in [2,\infty)$，定义 $\mathcal{P}_p(\mathbb{R}^d) := \left\{\mu \in \mathcal{P}(\mathbb{R}^d): \int_{\mathbb{R}^d} |\boldsymbol{x}|^p \mu(\mathrm{d}\boldsymbol{x}) < \infty\right\}$，则 $\mathcal{P}_p(\mathbb{R}^d)$ 在如下 L^p-Wasserstein 距离下为波兰空间：

$$\mathbb{W}_p(\mu_1,\mu_2) := \inf_{\pi\in\mathcal{C}(\mu_1,\mu_2)} \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |\boldsymbol{x}-\boldsymbol{y}|^p \pi(\mathrm{d}\boldsymbol{x},\mathrm{d}\boldsymbol{y})\right)^{1/p}, \qquad \mu_1,\mu_2 \in \mathcal{P}_p(\mathbb{R}^d),$$

其中，$\mathcal{C}(\mu_1,\mu_2)$ 是 $\mathbb{R}^d\times\mathbb{R}^d$ 上具有边缘分布 μ_1 和 μ_2 的概率测度构成的集合.此外，若 $\mu_1 = \mathcal{L}(\boldsymbol{X})$，$\mu_2 = \mathcal{L}(\boldsymbol{Y})$ 分别是随机变量 $\boldsymbol{X}$ 和 $\boldsymbol{Y}$ 的分布，$\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$ 是 $(\boldsymbol{X},\boldsymbol{Y})$ 的联合分布，则

$$\mathbb{W}_p^p(\mu_1,\mu_2) \leqslant \int_{\mathbb{R}^d\times\mathbb{R}^d} |\boldsymbol{x}-\boldsymbol{y}|^p \mathcal{L}(\boldsymbol{X},\boldsymbol{Y})(\mathrm{d}\boldsymbol{x},\mathrm{d}\boldsymbol{y}) = \mathbb{E}|\boldsymbol{X}-\boldsymbol{Y}|^p.$$

对于任意 $T > 0$，设 $C([\underline{\theta}t_0,T];\mathbb{R}^d)$ 是 $[\underline{\theta}t_0,T]$ 上所有 $\mathbb{R}^d$ 值连续函数构成的集合，赋予上确界范数.令 $L^p(\Omega;C([\underline{\theta},1];\mathbb{R}^d))$ 表示满足 $\mathbb{E}[\sup_{t_0\leqslant t\leqslant T}\sup_{\underline{\theta}\leqslant\theta\leqslant1}|\boldsymbol{X}_t(\theta)|^p] < \infty$ 的 $C([\underline{\theta},1];\mathbb{R}^d)$ 值随机过程 $\boldsymbol{X}_t$ 的全体，则 $L^p(\Omega;C([\underline{\theta},1];\mathbb{R}^d))$ 在 $\|\boldsymbol{X}\|_{L^p} = [\mathbb{E}(\sup_{t_0\leqslant t\leqslant T}\sup_{\underline{\theta}\leqslant\theta\leqslant1}|\boldsymbol{X}_t(\theta)|^p)]^{1/p}$ 下为 Banach 空间.

设 $r(t)$ 是一个取值于 $\mathbb{S} = \{1,2,\cdots,N\}$ 的连续时间马氏链，其生成元 $\boldsymbol{\Gamma} = (\gamma_{ij})_{N\times N}$ 为

$$\mathbb{P}\{r(t+\Delta)=j \mid r(t)=i\} = \begin{cases}\gamma_{ij}\Delta + o(\Delta), & i\neq j,\\ 1+\gamma_{ii}\Delta+o(\Delta), & i=j,\end{cases}$$

其中，$\Delta > 0$，$\gamma_{ij} \geqslant 0$ 为 i 到 j 的转移速率，$\gamma_{ij} = -\sum_{i\neq j}\gamma_{ij}$.$r(t)$ 的样本轨道是一个右连续的阶梯函数，在$[0,\infty)$的任何有限区间上只有有限多个跳跃点.假设马氏链 $r(t)$ 与 $\tilde{N}(\mathrm{d}t,\mathrm{d}z)$、Brown 运动 $\boldsymbol{B}(t)$、分数 Brown 运动 $\boldsymbol{B}^H(t)$ 都独立.

下面，我们给出关于分数 Brown 运动的 Wiener 积分的定义.对于 $H > 1/2$，令 $\boldsymbol{\Psi}$ 表示积分算子

$$\boldsymbol{\Psi} f(t) = H(2H-1)\int_0^\infty f(s)\,|s-t|^{2H-2}\mathrm{d}s,$$

并定义内积

$$\langle f_1,f_2\rangle_{\Psi} = \langle f_1,\boldsymbol{\Psi} f_2\rangle = H(2H-1)\int_0^\infty\int_0^\infty f_1(s)f_2(t)\,|s-t|^{2H-2}\mathrm{d}s\mathrm{d}t,$$

其中$\langle\cdot,\cdot\rangle$表示 $L^2([0,\infty))$ 上的通常内积.记 L^2_{Ψ}（相应地，$L^2_{\Psi}([0,T])$）为满足 $\langle f,f\rangle_{\Psi} < \infty$（相应地，$\langle f1_{[0,T]},f1_{[0,T]}\rangle_{\Psi} < \infty$）的可测函数$f$的等价类构成的空间.令 ε 为 $[0,\infty)$（相应地，$[0,T]$）上的阶梯函数空间，则 $\overline{(L^2([0,\infty)),\langle,\rangle_H)} = \overline{(\varepsilon,\langle,\rangle_H)}$（相应地，$\overline{(L^2([0,T]),\langle,\rangle_H)} = \overline{(\varepsilon,\langle,\rangle_H)}$）.注意到 Hurst 指数为 H 的分数 Brown 运动 $\boldsymbol{B}^H(t)$ 的协方差函数为

$$E(\boldsymbol{B}^H(t)\boldsymbol{B}^H(s)) = R_H(t,s) = \frac{1}{2}(t^{2H}+s^{2H}-|s-t|^{2H}) =$$
$$H(2H-1)\int_0^\infty\int_0^\infty 1_{[0,s]}(u)1_{[0,t]}(v)\,|u-v|^{2H-2}\mathrm{d}u\mathrm{d}v = \langle 1_{[0,s]},1_{[0,t]}\rangle_{\Psi},$$

从而，映射 $j:1_{[0,t]}\to\boldsymbol{B}^H(t)$ 为函数空间 $\overline{(L^2([0,\infty)),\langle,\rangle_H)}$（相应地，$\overline{(L^2([0,T]),\langle,\rangle_H)}$）到由随机变量 $\boldsymbol{B}^H(t)$，$t\geqslant0$（相应地，$t\leqslant T$），生成的 Gauss 空间的一个子空间的等距映射.对于 $f\in L^2_{\Psi}$，积分 $\int_0^\infty f(t)\,\mathrm{d}\boldsymbol{B}^H(t)$ 定义为f在该等距映射下的像.特别地，对于任意的f_1，$f_2 \in L^2_{\Psi}([0,T])$ 有

$$\mathbb{E}\left(\int_0^T f_1(u)\,\mathrm{d}\boldsymbol{B}^H(u)\int_0^T f_2(v)\,\mathrm{d}\boldsymbol{B}^H(v)\right) = H(2H-1)\int_0^T\int_0^T f_1(u)f_2(v)\,|u-v|^{2H-2}\mathrm{d}u\mathrm{d}v$$

和

$$\mathbb{E}\left(\int_s^t f(u)\,\mathrm{d}\boldsymbol{B}_u^H\right)^2 = H(2H-1)\int_s^t\int_s^t f(u)f(v)\,|u-v|^{2H-2}\mathrm{d}u\mathrm{d}v.$$

引理 1（Bihari's inequality，文献[13]Theorem 8.2） 设 $T > 0$，$c > 0$，$\psi(\cdot):\mathbb{R}^+\to\mathbb{R}^+$ 是一个连续非递减的函数，对于所有 $t > 0$，有 $\psi(t) > 0$.设 $u(\cdot)$ 是 $[0,T]$ 上的有界 Borel 可测非负函数，$v(\cdot)$ 是 $[0,T]$ 上的

非负可积函数.如果对于 $0 \leqslant t \leqslant T$ 有

$$u(t) \leqslant c + \int_0^t v(s)\psi(u(s))\mathrm{d}s,$$

则对任意 $t \in [0,T]$ 有

$$u(t) \leqslant G^{-1}\left(G(c) + \int_0^t v(s)\mathrm{d}s\right),$$

且 $G(c) + \int_0^t v(s)\mathrm{d}s \in \mathrm{Dom}(G^{-1})$, 其中 $G(r) = \int_1^r \frac{\mathrm{d}s}{\psi(s)}$, $r > 0$, G^{-1} 是 G 的反函数, $\mathrm{Dom}(G^{-1})$ 为 G^{-1} 的定义域.

2　解的存在唯一性

本节我们将在假设 1 下利用 Carathéodory 逼近,建立由分数 Brown 运动和 Lévy 过程驱动的带 Markov 切换和随机比例的分布依赖的随机泛函微分方程(1)的解的存在唯一性.对于任意整数 $k \geqslant 1$, 当 $\underline{\theta} t_0 \leqslant t \leqslant t_0$, 定义 $\boldsymbol{X}^k(t) = \boldsymbol{\xi}(t)$; 当 $t \in [t_0, T]$ 时,定义

$$\begin{aligned}\boldsymbol{X}^k(t) = {}& \boldsymbol{\xi}(t_0) + \int_{t_0}^t \boldsymbol{f}\left(s, \boldsymbol{X}^k\left(s - \frac{1}{k}\right), \boldsymbol{X}^k_{s-1/k}, \mathcal{L}\left(\boldsymbol{X}^k\left(s - \frac{1}{k}\right)\right), r\left(s - \frac{1}{k}\right)\right)\mathrm{d}s + \\ & \int_{t_0}^t \boldsymbol{g}\left(s, \boldsymbol{X}^k\left(s - \frac{1}{k}\right), \boldsymbol{X}^k_{s-1/k}, \mathcal{L}\left(\boldsymbol{X}^k\left(s - \frac{1}{k}\right)\right), r\left(s - \frac{1}{k}\right)\right)\mathrm{d}\boldsymbol{B}(s) + \\ & \int_{t_0}^t \boldsymbol{m}\left(s, \mathcal{L}\left(\boldsymbol{X}^k\left(s - \frac{1}{k}\right)\right)\right)\mathrm{d}\boldsymbol{B}^H(s) + \\ & \int_{t_0}^t \int_{|z|<c} \boldsymbol{h}\left(s, \boldsymbol{X}^k\left(s - \frac{1}{k}\right), \boldsymbol{X}^k_{s-1/k}, \mathcal{L}\left(\boldsymbol{X}^k\left(s - \frac{1}{k}\right)\right), z, r\left(s - \frac{1}{k}\right)\right), \tilde{N}(\mathrm{d}s, \mathrm{d}z).\end{aligned} \tag{2}$$

对于 $t_0 \leqslant t \leqslant t_0 + 1/k$, $\boldsymbol{X}^k(t)$ 可以写为

$$\begin{aligned}\boldsymbol{X}^k(t) = {}& \boldsymbol{\xi}(t_0) + \int_{t_0}^t \boldsymbol{f}\left(s, \boldsymbol{\xi}\left(s - \frac{1}{k}\right), \boldsymbol{\xi}\left(\theta\left(s - \frac{1}{k}\right)\right), \mathcal{L}\left(\boldsymbol{\xi}\left(s - \frac{1}{k}\right)\right), r\left(s - \frac{1}{k}\right)\right)\mathrm{d}s + \\ & \int_{t_0}^t \boldsymbol{g}\left(s, \boldsymbol{\xi}\left(s - \frac{1}{k}\right), \boldsymbol{\xi}\left(\theta\left(s - \frac{1}{k}\right)\right), \mathcal{L}\left(\boldsymbol{\xi}\left(s - \frac{1}{k}\right)\right), r\left(s - \frac{1}{k}\right)\right)\mathrm{d}\boldsymbol{B}(s) + \\ & \int_{t_0}^t \boldsymbol{m}(s, \mathcal{L}(\boldsymbol{\xi}))\mathrm{d}\boldsymbol{B}^H(s) + \\ & \int_{t_0}^t \int_{|z|<c} \boldsymbol{h}\left(s, \boldsymbol{\xi}\left(s - \frac{1}{k}\right), \boldsymbol{\xi}\left(\theta\left(s - \frac{1}{k}\right)\right), \mathcal{L}\left(\boldsymbol{\xi}\left(s - \frac{1}{k}\right)\right), z, r\left(s - \frac{1}{k}\right)\right)\tilde{N}(\mathrm{d}s, \mathrm{d}z).\end{aligned} \tag{3}$$

当 $t_0 + 1/k \leqslant t \leqslant t_0 + 2/k$ 时,若 $t_0 + 1/k \leqslant s \leqslant t$, 则 $t_0 \leqslant s - 1/k \leqslant t_0 + 1/k$, 此时 $\boldsymbol{X}^k(s - 1/k)$ 可由上式定义,从而

$$\begin{aligned}\boldsymbol{X}^k(t) = {}& \boldsymbol{X}^k\left(t_0 - \frac{1}{k}\right) + \int_{t_0+1/k}^t \boldsymbol{f}\left(s, \boldsymbol{X}^k\left(s - \frac{1}{k}\right), \boldsymbol{X}^k_{s-1/k}, \mathcal{L}\left(\boldsymbol{X}^k\left(s - \frac{1}{k}\right)\right), r\left(s - \frac{1}{k}\right)\right)\mathrm{d}s + \\ & \int_{t_0+1/k}^t \boldsymbol{g}\left(s, \boldsymbol{X}^k\left(s - \frac{1}{k}\right), \boldsymbol{X}^k_{s-1/k}, \mathcal{L}\left(\boldsymbol{X}^k\left(s - \frac{1}{k}\right)\right), r\left(s - \frac{1}{k}\right)\right)\mathrm{d}\boldsymbol{B}(s) + \\ & \int_{t_0+1/k}^t \boldsymbol{m}\left(s, \mathcal{L}\left(\boldsymbol{X}^k\left(s - \frac{1}{k}\right)\right)\right)\mathrm{d}\boldsymbol{B}^H(s) + \\ & \int_{t_0+1/k}^t \int_{|z|<c} \boldsymbol{h}\left(s, \boldsymbol{X}^k\left(s - \frac{1}{k}\right), \boldsymbol{X}^k_{s-1/k}, \mathcal{L}\left(\boldsymbol{X}^k\left(s - \frac{1}{k}\right)\right), z, r\left(s - \frac{1}{k}\right)\right)\tilde{N}(\mathrm{d}s, \mathrm{d}z).\end{aligned}$$

众所周知,与 Picard 逐次逼近相比,Carathéodory 逼近的优点是不需要通过 $\boldsymbol{X}^1(t), \boldsymbol{X}^2(t), \cdots, \boldsymbol{X}^{k-1}(t)$ 来计算 $\boldsymbol{X}^k(t)$.事实上,我们可以在长度为 $1/k$ 的区间上直接计算 $\boldsymbol{X}^k(t)$.

假设 1　存在一个有界非减函数 $K(t)$, 使得

(i) 对任意 $t \in [\underline{\theta} t_0, T]$, $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^d$, $\boldsymbol{y}_1, \boldsymbol{y}_2 \in L^p(\Omega; C([\underline{\theta}, 1]; \mathbb{R}^d))$, $\mu_1, \mu_2 \in \mathcal{P}_p(\mathbb{R}^d)$, $i \in \mathbb{S} = \{1, 2, \cdots, N\}$, 有

$$|f(t,x_1,y_1,\mu_1,i)-f(t,x_2,y_2,\mu_2,i)|^p \le K(t)\psi(|x_1-x_2|^p+\|y_1-y_2\|^p+\mathbb{W}_p^p(\mu_1,\mu_2)),$$
$$\|g(t,x_1,y_1,\mu_1,i)-g(t,x_2,y_2,\mu_2,i)\|^p \le K(t)\psi(|x_1-x_2|^p+\|y_1-y_2\|^p+\mathbb{W}_p^p(\mu_1,\mu_2)),$$
$$\|m(t,\mu_1)-m(t,\mu_2)\|^p \le K(t)\psi(\mathbb{W}_p^p(\mu,\nu)),$$
$$\int_{|z|<c}|h(t,x_1,y_1,\mu_1,z,i)-h(t,x_2,y_2,\mu_2,z,i)|^p\nu(\mathrm{d}z)\le$$
$$K(t)\psi(|x_1-x_2|^p+\|y_1-y_2\|^p+\mathbb{W}_p^p(\mu_1,\mu_2)),$$

其中, $\psi(\cdot)$ 是一个连续非递减的非负凹函数, $\psi(0)=0$, 且 $\int_{0^+}\frac{1}{\psi(x)}\mathrm{d}x=+\infty$.

(ii) 任意 $x\in\mathbb{R}^d, y\in L^p(\Omega;C([\underline{\theta},1];\mathbb{R}^d)), \mu\in\mathscr{P}_p(\mathbb{R}^d), i\in\mathbb{S}=\{1,2,\cdots,N\}$, 有

$$|f(t,x,y,\mu,i)|^p\le K(t)(1+|x|^p+\|y\|^p+\mathbb{W}_p^p(\mu,\delta_0)),$$
$$\|g(t,x,y,\mu,i)\|^p\le K(t)(1+|x|^p+\|y\|^p+\mathbb{W}_p^p(\mu,\delta_0)),$$
$$\|m(t,\mu)\|^p\le K(t)(1+\mathbb{W}_p^p(\mu,\delta_0)),$$
$$\int_{|z|<c}|h(t,x,y,\mu,z,i)|^p\nu(\mathrm{d}z)\le K(t)(1+|x|^p+\|y\|^p+\mathbb{W}_p^p(\mu,\delta_0)),$$

其中 δ_0 为 Dirac 测度,即如果 $x\in A$, 则 $\delta_0(A)=1$, 否则 $\delta_0(A)=0$.

定义 1　设 $X=\{X(t)\}_{\underline{\theta}t_0\le t\le T}$ 为一个 $\mathbb{R}^d$ 值的随机过程,若 $\mathbb{E}(\sup_{t\in[\underline{\theta}t_0,T]}|X(t)|)<\infty$, 且

$$\begin{cases}X(t)=\xi(t_0)+\int_{t_0}^t f(t,X(t),X_t,\mathscr{L}(X(t)),r(t))\mathrm{d}t+\\ \quad\int_{t_0}^t g(t,X(t),X_t,\mathscr{L}(X(t)),r(t))\mathrm{d}B(t)+\int_{t_0}^t m(t,\mathscr{L}(X(t)))\mathrm{d}B^H(t)+\\ \quad\int_{t_0}^t\int_{|z|<c}h(t,X(t-),X_t,\mathscr{L}(X(t)),z,r(t))\tilde{N}(\mathrm{d}t,\mathrm{d}z),\qquad t\in[t_0,T],\ \mathbb{P}-\text{a.s.},\\ X(t)=\xi(t),\qquad t\in[\underline{\theta}t_0,t_0],\ \mathbb{P}-\text{a.s.},\end{cases}\tag{4}$$

则称 X 为方程(1)的解.

此外,若 $Y=\{Y(t)\}_{\underline{\theta}t_0\le t\le T}$ 是方程(1)的另一个解,且 $\mathbb{P}$(对所有的 $t\in[t_0,T]$ 有 $X(t)=Y(t)$)$=1$, 则称 $\mathbb{R}^d$ 值随机过程 $X=\{X(t)\}_{\underline{\theta}t_0\le t\le T}$ 为方程(1)的唯一解.

接下来,我们将证明由式(2)定义的随机序列 $\{X^k(t),k\ge1\}$ 的一致有界性.

引理 2　若假设 1 成立, $p\in[2,\infty)$, $t\in[t_0,T]$, 则对于所有 $k\ge1$, 有

$$\mathbb{E}(\sup_{t_0\le s\le t}|X^k(s)|^p)<\infty.$$

证明　由 c_p 不等式

$$|x_1+x_2+\cdots+x_k|^p\le k^{p-1}(|x_1|^p+|x_2|^p+\cdots+|x_k|^p)$$

和式(2)可得

$$\begin{aligned}&\mathbb{E}(\sup_{t_0\le s\le t}|X^k(s)|^p)\le\\&\quad 5^{p-1}\mathbb{E}\|\xi\|^p+5^{p-1}\mathbb{E}\left(\sup_{t_0\le s\le t}\left|\int_{t_0}^s f\left(u,X^k\left(u-\frac{1}{k}\right),X^k_{u-1/k},\mathscr{L}\left(X^k\left(u-\frac{1}{k}\right)\right),r\left(u-\frac{1}{k}\right)\right)\mathrm{d}u\right|^p\right)+\\&\quad 5^{p-1}\mathbb{E}\left(\sup_{t_0\le s\le t}\left|\int_{t_0}^s g\left(u,X^k\left(u-\frac{1}{k}\right),X^k_{u-1/k},\mathscr{L}\left(X^k\left(u-\frac{1}{k}\right)\right),r\left(u-\frac{1}{k}\right)\right)\mathrm{d}B(u)\right|^p\right)+\\&\quad 5^{p-1}\mathbb{E}\left(\sup_{t_0\le s\le t}\left|\int_{t_0}^s m\left(u,\mathscr{L}\left(X^k\left(u-\frac{1}{k}\right)\right)\right)\mathrm{d}B^H(u)\right|^p\right)+\\&\quad 5^{p-1}\mathbb{E}\left(\sup_{t_0\le s\le t}\left|\int_{t_0}^s\int_{|z|<c}h\left(u,X^k\left(u-\frac{1}{k}\right),X^k_{u-1/k},\mathscr{L}\left(X^k\left(u-\frac{1}{k}\right)\right),z,r\left(u-\frac{1}{k}\right)\right)\tilde{N}(\mathrm{d}u,\mathrm{d}z)\right|^p\right):=\\&\quad 5^{p-1}(\mathbb{E}\|\xi\|^p+I_1+I_2+I_3+I_4).\end{aligned}$$

对于 I_1, 由 Hölder 不等式和假设 1 得

$$I_1 \leqslant T^{p-1}\,\mathbb{E}\left(\int_{t_0}^{t}\left|\boldsymbol{f}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{k}\right),\boldsymbol{X}^k_{u-1/k},\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),r\left(u-\frac{1}{k}\right)\right)\right|^p\mathrm{d}u\right) \leqslant$$

$$(2T)^{p-1}\,\mathbb{E}\int_{t_0}^{t}\left[\left|\boldsymbol{f}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{k}\right),\boldsymbol{X}^k_{u-1/k},\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),r\left(u-\frac{1}{k}\right)\right)-\right.\right.$$

$$\left.\left.\boldsymbol{f}\left(u,0,0,\delta_0,r\left(u-\frac{1}{k}\right)\right)\right|^p+\left|\boldsymbol{f}\left(u,0,0,\delta_0,r\left(u-\frac{1}{k}\right)\right)\right|^p\right]\mathrm{d}u \leqslant$$

$$(2T)^{p-1}\,\mathbb{E}\int_{t_0}^{t}K(u)\left[\psi\left(\left|\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right|^p+\|\boldsymbol{X}^k_{u-1/k}\|^p+\right.\right.$$

$$\left.\left.\mathbb{W}_p^p\left(\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),\delta_0\right)\right)+1\right]\mathrm{d}u.$$

对于 I_2，由 BDG 不等式、Hölder 不等式、假设 1,类似于 I_1 得

$$I_2 \leqslant C_p\,\mathbb{E}\left[\int_{t_0}^{t}\left\|\boldsymbol{g}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{k}\right),\boldsymbol{X}^k_{u-1/k},\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),r\left(u-\frac{1}{k}\right)\right)\right\|^2\mathrm{d}u\right]^{p/2} \leqslant$$

$$C_pT^{p/2-1}\,\mathbb{E}\int_{t_0}^{t}\left\|\boldsymbol{g}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{k}\right),\boldsymbol{X}^k_{u-1/k},\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),r\left(u-\frac{1}{k}\right)\right)\right\|^p\mathrm{d}u \leqslant$$

$$2^{p-1}C_pT^{p/2-1}\,\mathbb{E}\int_{t_0}^{t}K(u)\left[\psi\left(\left|\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right|^p+\|\boldsymbol{X}^k_{u-1/k}\|^p+\right.\right.$$

$$\left.\left.\mathbb{W}_p^p\left(\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),\delta_0\right)\right)+1\right]\mathrm{d}u.$$

对于 I_3，由文献[4]的式(3.5)、假设 1 得

$$I_3 \leqslant \mathbb{E}\left[\sup_{t_0\leqslant s\leqslant t}\left|\int_{t_0}^{s}\boldsymbol{m}\left(u,\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right)\right)\mathrm{d}\boldsymbol{B}^H(u)\right|^p\right] \leqslant$$

$$C_{\lambda,p,H}T^{pH-1}\int_{t_0}^{t}\left\|\boldsymbol{m}\left(u,\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right)\right)\right\|^p\mathrm{d}u \leqslant$$

$$2^{p-1}C_{\lambda,p,H}T^{pH-1}\int_{t_0}^{t}\left[\left\|\boldsymbol{m}\left(u,\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right)\right)-\boldsymbol{m}(u,\delta_0)\right\|^p+\|\boldsymbol{m}(u,\delta_0)\|^p\right]\mathrm{d}u \leqslant$$

$$2^{p-1}C_{\lambda,p,H}T^{pH-1}\,\mathbb{E}\int_{t_0}^{t}K(u)\left[\psi\left(\mathbb{W}_p^p\left(\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),\delta_0\right)\right)+1\right]\mathrm{d}u,$$

其中，$1-H<\lambda<1-1/p$，$C_{\lambda,p,H}$ 是依赖于 λ,p,H 的正常数.

对于 I_4，由文献[14]中的 Kunita 第一不等式、Hölder 不等式、假设 1 得

$$I_4 \leqslant \mathbb{E}\left[\sup_{t_0\leqslant s\leqslant t}\left|\int_{t_0}^{s}\int_{|z|<c}\boldsymbol{h}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{k}\right),\boldsymbol{X}^k_{u-1/k},\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),z,r\left(u-\frac{1}{k}\right)\right)\tilde{N}(\mathrm{d}u,\mathrm{d}z)\right|^p\right] \leqslant$$

$$D_p\,\mathbb{E}\left[\int_{t_0}^{t}\int_{|z|<c}\left|\boldsymbol{h}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{k}\right),\boldsymbol{X}^k_{u-1/k},\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),z,r\left(u-\frac{1}{k}\right)\right)\right|^2\nu(\mathrm{d}z)\mathrm{d}u\right]^{p/2}+$$

$$D_p\,\mathbb{E}\left[\int_{t_0}^{t}\int_{|z|<c}\left|\boldsymbol{h}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{k}\right),\boldsymbol{X}^k_{u-1/k},\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),z,r\left(u-\frac{1}{k}\right)\right)\right|^p\nu(\mathrm{d}z)\mathrm{d}u\right] \leqslant$$

$$D_p(VT)^{p/2-1}\,\mathbb{E}\left[\int_{t_0}^{t}\int_{|z|<c}\left|\boldsymbol{h}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{k}\right),\boldsymbol{X}^k_{u-1/k},\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),z,r\left(u-\frac{1}{k}\right)\right)\right|^p\nu(\mathrm{d}z)\mathrm{d}u\right]+$$

$$D_p\,\mathbb{E}\left[\int_{t_0}^{t}\int_{|z|<c}\left|\boldsymbol{h}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{k}\right),\boldsymbol{X}^k_{u-1/k},\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),z,r\left(u-\frac{1}{k}\right)\right)\right|^p\nu(\mathrm{d}z)\mathrm{d}u\right] \leqslant$$

$$2^{p-1}D_p((VT)^{p/2-1}+1)\times$$

$$\mathbb{E}\int_{t_0}^{t}K(u)\left[\psi\left(\left|\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right|^p+\|\boldsymbol{X}^k_{u-1/k}\|^p+\mathbb{W}_p^p\left(\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),\delta_0\right)\right)+1\right]\mathrm{d}u,$$

其中，D_p 是依赖于 p 的正常数.

注意到，$\psi(\cdot)$ 是一个连续非递减的非负凹函数,因此存在一个正数 a，使得 $\psi(x)\leqslant a(1+x)$.此外，

$$\mathbb{E}\int_{t_0}^{t}\|\boldsymbol{X}_{u-1/k}^{k}\|^{p}\mathrm{d}u\leqslant\mathbb{E}\int_{t_0}^{t}\sup_{\underline{\theta}\leqslant\theta\leqslant 1}\left|\boldsymbol{X}^{k}\left(\theta\left(u-\frac{1}{k}\right)\right)\right|^{p}\mathrm{d}u\leqslant$$
$$\mathbb{E}\|\boldsymbol{\xi}\|^{p}T+\mathbb{E}\int_{t_0}^{t}\sup_{t_0\leqslant e\leqslant u}|\boldsymbol{X}^{k}(e)|^{p}\mathrm{d}u.$$

综上

$$\begin{aligned}&\mathbb{E}\left(\sup_{t_0\leqslant s\leqslant t}|\boldsymbol{X}^{k}(s)|^{p}\right)\leqslant\\&\quad 10^{p-1}(\mathbb{E}\|\boldsymbol{\xi}\|^{p}+T^{p-1}+C_{p}T^{p/2-1}+C_{\lambda,p,H}T^{pH-1}+D_{p}((VT)^{p/2-1}+1))\times\\&\quad\mathbb{E}\int_{t_0}^{t}K(u)\left[\psi\left(\left|\boldsymbol{X}^{k}\left(u-\frac{1}{k}\right)\right|^{p}+\|\boldsymbol{X}_{u-1/k}{}^{k}\|^{p}+\mathbb{W}_{p}^{p}\left(\mathcal{L}\left(\boldsymbol{X}^{k}\left(u-\frac{1}{k}\right)\right),\delta_0\right)\right)+1\right]\mathrm{d}u\leqslant\\&\quad 10^{p-1}(\mathbb{E}\|\boldsymbol{\xi}\|^{p}+T^{p-1}+C_{p}T^{p/2-1}+C_{\lambda,p,H}T^{pH-1}+D_{p}((VT)^{p/2-1}+1))\times\\&\quad\mathbb{E}\int_{t_0}^{t}K(u)\left[a\left|\boldsymbol{X}^{k}\left(u-\frac{1}{k}\right)\right|^{p}+a\|\boldsymbol{X}_{u-1/k}{}^{k}\|^{p}+a\,\mathbb{E}\left|\boldsymbol{X}^{k}\left(u-\frac{1}{k}\right)\right|^{p}+a+1\right]\mathrm{d}u\leqslant\\&\quad 10^{p-1}K(T)(\mathbb{E}\|\boldsymbol{\xi}\|^{p}+T^{p-1}+C_{p}T^{p/2-1}+C_{\lambda,p,H}T^{pH-1}+D_{p}((VT)^{p/2-1}+1))\times\\&\quad\left(3aT\,\mathbb{E}\|\boldsymbol{\xi}\|^{p}+aT+T+3a\,\mathbb{E}\int_{t_0}^{t}\sup_{t_0\leqslant e\leqslant u}|\boldsymbol{X}^{k}(e)|^{p}\mathrm{d}u\right)\leqslant\\&\quad C_1+C_2\mathbb{E}\int_{t_0}^{t}\sup_{t_0\leqslant e\leqslant u}|\boldsymbol{X}^{k}(e)|^{p}\mathrm{d}u,\end{aligned}$$

其中, $C_1=10^{p-1}K(T)(\mathbb{E}\|\boldsymbol{\xi}\|^{p}+T^{p-1}+C_{p}T^{p/2-1}+C_{\lambda,p,H}T^{pH-1}+D_{p}((VT)^{p/2-1}+1))(aT+T+3aT\mathbb{E}\|\boldsymbol{\xi}\|^{p})$, $C_2=3\times 10^{p-1}aK(T)(\mathbb{E}\|\boldsymbol{\xi}\|^{p}+T^{p-1}+C_{p}T^{p/2-1}+C_{\lambda,p,H}T^{pH-1}+D_{p}((VT)^{p/2-1}+1))$.由 Gronwall 不等式可得

$$\mathbb{E}\left(\sup_{t_0\leqslant s\leqslant t}|\boldsymbol{X}^{k}(s)|^{p}\right)<\infty.$$

引理 3 若假设 1 成立, $t_0\leqslant s<t\leqslant T$, 则

$$\mathbb{E}|\boldsymbol{X}^{k}(t)-\boldsymbol{X}^{k}(s)|^{p}\leqslant C_3[(t-s)^{p}+(t-s)^{p/2}+(t-s)^{pH}+(t-s)],$$

其中, $C_3=8^{p-1}(1\vee C_p\vee C_{\lambda,p,H}\vee D_p)K(T)(a+1+3Ta\mathbb{E}\|\boldsymbol{\xi}\|^{p}+3a\mathbb{E}(\sup t_0\leqslant e\leqslant T|\boldsymbol{X}^{k}(e)|^{p}))$.

证明

$$\begin{aligned}&\mathbb{E}|\boldsymbol{X}^{k}(t)-\boldsymbol{X}^{k}(s)|^{p}\leqslant\\&\quad 4^{p-1}\,\mathbb{E}\left|\int_{s}^{t}\boldsymbol{f}\left(u,\boldsymbol{X}^{k}\left(u-\frac{1}{k}\right),\boldsymbol{X}_{u-1/k}^{k},\mathcal{L}\left(\boldsymbol{X}^{k}\left(u-\frac{1}{k}\right)\right),r\left(u-\frac{1}{k}\right)\right)\mathrm{d}u\right|^{p}+\\&\quad 4^{p-1}\,\mathbb{E}\left|\int_{s}^{t}\boldsymbol{g}\left(u,\boldsymbol{X}^{k}\left(u-\frac{1}{k}\right),\boldsymbol{X}_{u-1/k}^{k},\mathcal{L}\left(\boldsymbol{X}^{k}\left(u-\frac{1}{k}\right)\right),r\left(u-\frac{1}{k}\right)\right)\mathrm{d}\boldsymbol{B}(u)\right|^{p}+\\&\quad 4^{p-1}\,\mathbb{E}\left|\int_{s}^{t}\boldsymbol{m}\left(u,\mathcal{L}\left(\boldsymbol{X}^{k}\left(u-\frac{1}{k}\right)\right)\right)\mathrm{d}\boldsymbol{B}^{H}(u)\right|^{p}+\\&\quad 4^{p-1}\,\mathbb{E}\left|\int_{s}^{t}\int_{|z|<c}\boldsymbol{h}\left(u,\boldsymbol{X}^{k}\left(u-\frac{1}{k}\right),\boldsymbol{X}_{u-1/k}^{k},\mathcal{L}\left(\boldsymbol{X}^{k}\left(u-\frac{1}{k}\right)\right),z,r\left(u-\frac{1}{k}\right)\right)\tilde{N}(\mathrm{d}u,\mathrm{d}z)\right|^{p}:=\\&\quad 4^{p-1}(J_1+J_2+J_3+J_4).\end{aligned}$$

对于 J_1, 由 Hölder 不等式、假设 1,类似于 I_1 得

$$\begin{aligned}&J_1\leqslant(2(t-s))^{p-1}\times\\&\quad\mathbb{E}\int_{s}^{t}K(u)\left[\psi\left(\left|\boldsymbol{X}^{k}\left(u-\frac{1}{k}\right)\right|^{p}+\|\boldsymbol{X}_{u-1/k}^{k}\|^{p}+\mathbb{W}_{p}^{p}\left(\mathcal{L}\left(\boldsymbol{X}^{k}\left(u-\frac{1}{k}\right)\right),\delta_0\right)\right)+1\right]\mathrm{d}u.\end{aligned}$$

对于 J_2, 由 BDG 不等式、Hölder 不等式、假设 1,类似于 I_1 得

$$\begin{aligned}&J_2\leqslant\mathbb{E}\left(\sup_{s\leqslant u_1\leqslant t}\left|\int_{s}^{u_1}\boldsymbol{g}\left(u,\boldsymbol{X}^{k}\left(u-\frac{1}{k}\right),\boldsymbol{X}_{u-1/k}^{k},\mathcal{L}\left(\boldsymbol{X}^{k}\left(u-\frac{1}{k}\right)\right),r\left(u-\frac{1}{k}\right)\right)\mathrm{d}\boldsymbol{B}(u)\right|^{p}\right)\leqslant\\&\quad C_p\,\mathbb{E}\left[\int_{s}^{t}\left\|\boldsymbol{g}\left(u,\boldsymbol{X}^{k}\left(u-\frac{1}{k}\right),\boldsymbol{X}_{u-1/k}^{k},\mathcal{L}\left(\boldsymbol{X}^{k}\left(u-\frac{1}{k}\right)\right),r\left(u-\frac{1}{k}\right)\right)\right\|^{2}\mathrm{d}u\right]^{p/2}\leqslant\\&\quad C_p(t-s)^{p/2-1}\,\mathbb{E}\int_{s}^{t}\left\|\boldsymbol{g}\left(u,\boldsymbol{X}^{k}\left(u-\frac{1}{k}\right),\boldsymbol{X}_{u-1/k}^{k},\mathcal{L}\left(\boldsymbol{X}^{k}\left(u-\frac{1}{k}\right)\right),r\left(u-\frac{1}{k}\right)\right)\right\|^{p}\mathrm{d}u\leqslant\end{aligned}$$

$$2^{p-1}C_p(t-s)^{p/2-1}\times$$
$$\mathbb{E}\int_s^t K(u)\left[\psi\left(\left|\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right|^p+\|\boldsymbol{X}^k_{u-1/k}\|^p+\mathbb{W}_p^p\left(\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),\delta_0\right)\right)+1\right]\mathrm{d}u.$$

对于 J_3, 由文献[11]的式(3.16)、假设 1 得

$$J_3=\mathbb{E}\left|\int_s^t\boldsymbol{m}\left(u,\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right)\right)\mathrm{d}\boldsymbol{B}^H(u)\right|^p\leqslant$$
$$C_{\lambda,p,H}(t-s)^{pH-1}\int_s^t\left\|\boldsymbol{m}\left(u,\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right)\right)\right\|^p\mathrm{d}u\leqslant$$
$$2^{p-1}C_{\lambda,p,H}(t-s)^{pH-1}\int_{t_s}^t\left[\left\|\boldsymbol{m}\left(u,\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right)\right)-\boldsymbol{m}(u,\delta_0)\right\|^p+\|\boldsymbol{m}(u,\delta_0)\|^p\right]\mathrm{d}u\leqslant$$
$$2^{p-1}C_{\lambda,p,H}(t-s)^{pH-1}\int_s^t K(u)\left[\psi\left(\mathbb{W}_p^p\left(\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),\delta_0\right)\right)+1\right]\mathrm{d}u.$$

对于 J_4, 由 Kunita 第一不等式、Hölder 不等式、假设 1,类似于 I_1 得

$$J_4=\mathbb{E}\left|\int_s^t\int_{|z|<c}\boldsymbol{h}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{k}\right),\boldsymbol{X}^k_{u-1/k},\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),z,r\left(u-\frac{1}{k}\right)\right)\tilde{N}(\mathrm{d}u,\mathrm{d}z)\right|^p\leqslant$$
$$D_p\,\mathbb{E}\left[\int_s^t\int_{|z|<c}\left|\boldsymbol{h}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{k}\right),\boldsymbol{X}^k_{u-1/k},\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),z,r\left(u-\frac{1}{k}\right)\right)\right|^2\nu(\mathrm{d}z)\mathrm{d}u\right]^{p/2}+$$
$$D_p\,\mathbb{E}\left[\int_s^t\int_{|z|<c}\left|\boldsymbol{h}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{k}\right),\boldsymbol{X}^k_{u-1/k},\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),z,r\left(u-\frac{1}{k}\right)\right)\right|^p\nu(\mathrm{d}z)\mathrm{d}u\right]\leqslant$$
$$D_pV^{p/2-1}(t-s)^{p/2-1}\times$$
$$\mathbb{E}\left[\int_s^t\int_{|z|<c}\left|\boldsymbol{h}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{k}\right),\boldsymbol{X}^k_{u-1/k},\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),z,r\left(u-\frac{1}{k}\right)\right)\right|^p\nu(\mathrm{d}z)\mathrm{d}u\right]+$$
$$D_p\,\mathbb{E}\left[\int_s^t\int_{|z|<c}\left|\boldsymbol{h}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{k}\right),\boldsymbol{X}^k_{u-1/k},\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),z,r\left(u-\frac{1}{k}\right)\right)\right|^p\nu(\mathrm{d}z)\mathrm{d}u\right]\leqslant$$
$$D_p(V^{p/2-1}(t-s)^{p/2-1}+1)\times$$
$$\mathbb{E}\left[\int_s^t\int_{|z|<c}\left|\boldsymbol{h}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{k}\right),\boldsymbol{X}^k_{u-1/k},\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),z,r\left(u-\frac{1}{k}\right)\right)\right|^p\nu(\mathrm{d}z)\mathrm{d}u\right]\leqslant$$
$$2^{p-1}D_p(V^{p/2-1}(t-s)^{p/2-1}+1)\times$$
$$\mathbb{E}\left\{\int_s^t\int_{|z|<c}\left[\left|\boldsymbol{h}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{k}\right),\boldsymbol{X}^k_{u-1/k},\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),z,r\left(u-\frac{1}{k}\right)\right)-\right.\right.\right.$$
$$\left.\left.\left.\boldsymbol{h}\left(u,0,0,\delta_0,z,r\left(u-\frac{1}{k}\right)\right)\right|^p+\left|\boldsymbol{h}\left(u,0,0,\delta_0,z,r\left(u-\frac{1}{k}\right)\right)\right|^p\right]\nu(\mathrm{d}z)\mathrm{d}u\right\}\leqslant$$
$$2^{p-1}D_p(V^{p/2-1}(t-s)^{p/2-1}+1)\times$$
$$\mathbb{E}\int_s^t K(u)\left[\psi\left(\left|\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right|^p+\|\boldsymbol{X}^k_{u-1/k}\|^p+\mathbb{W}_p^p\left(\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),\delta_0\right)\right)+1\right]\mathrm{d}u.$$

综上

$$\mathbb{E}|\boldsymbol{X}^k(t)-\boldsymbol{X}^k(s)|^p\leqslant$$
$$8^{p-1}[(t-s)^{p-1}+C_p(t-s)^{p/2-1}+C_{\lambda,p,H}(t-s)^{pH-1}+D_p(V^{p/2-1}(t-s)^{p/2-1}+1)]\times$$
$$\mathbb{E}\int_s^t K(u)\left[\psi\left(\left|\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right|^p+\|\boldsymbol{X}^k_{u-1/k}\|^p+\mathbb{W}_p^p\left(\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),\delta_0\right)\right)+1\right]\mathrm{d}u\leqslant$$
$$8^{p-1}((t-s)^{p-1}+C_p(t-s)^{p/2-1}+C_{\lambda,p,H}(t-s)^{pH-1}+D_p(V^{p/2-1}(t-s)^{p/2-1}+1))\times$$
$$\int_s^t[K(u)(a+1+3aT\mathbb{E}\|\boldsymbol{\xi}\|^p+3a\mathbb{E}(\sup_{t_0\leqslant e\leqslant t}|\boldsymbol{X}^k(e)|^p))]\mathrm{d}u\leqslant$$
$$C_3[(t-s)^p+(t-s)^{p/2}+(t-s)^{pH}+(t-s)],$$

其中

$$C_3 = 8^{p-1}(1 \vee C_p \vee C_{\lambda,p,H} \vee D_p V^{p/2-1} \vee D_p)K(T)(a + 1 + 3aT\mathbb{E}\|\boldsymbol{\xi}\|^p + 3a\mathbb{E}(\sup_{t_0 \leqslant e \leqslant T}|\boldsymbol{X}^k(e)|^p)).$$

定理 1 若假设 1 成立,则方程(1)有唯一的解.

证明 我们首先证明 $(\boldsymbol{X}^k)_{k\geqslant 1}$ 是 Cauchy 列.设 $n > k \geqslant 1$,

$$\begin{aligned}
&\sup_{t_0\leqslant s\leqslant t}|\boldsymbol{X}^n(s) - \boldsymbol{X}^k(s)|^p \leqslant \\
&4^{p-1}\sup_{t_0\leqslant s\leqslant t}\left|\int_{t_0}^s\left[\boldsymbol{f}\left(u,\boldsymbol{X}^n\left(u-\frac{1}{n}\right),\boldsymbol{X}^n_{u-1/n},\mathcal{L}\left(\boldsymbol{X}^n\left(u-\frac{1}{n}\right)\right),r\left(u-\frac{1}{n}\right)\right) - \right.\right.\\
&\left.\left.\boldsymbol{f}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{k}\right),\boldsymbol{X}_{u-1/k}{}^k,\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),r\left(u-\frac{1}{k}\right)\right)\right]\mathrm{d}u\right|^p + \\
&4^{p-1}\sup_{t_0\leqslant s\leqslant t}\left|\int_{t_0}^s\left[\boldsymbol{g}\left(u,\boldsymbol{X}^n\left(u-\frac{1}{n}\right),\boldsymbol{X}^n_{u-1/n},\mathcal{L}\left(\boldsymbol{X}^n\left(u-\frac{1}{n}\right)\right),r\left(u-\frac{1}{n}\right)\right) - \right.\right.\\
&\left.\left.\boldsymbol{g}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{k}\right),\boldsymbol{X}_{u-1/k}{}^k,\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),r\left(u-\frac{1}{k}\right)\right)\right]\mathrm{d}\boldsymbol{B}(u)\right|^p + \\
&4^{p-1}\sup_{t_0\leqslant s\leqslant t}\left|\int_{t_0}^s\left[\boldsymbol{m}\left(u,\mathcal{L}\left(\boldsymbol{X}^n\left(u-\frac{1}{n}\right)\right)\right) - \boldsymbol{m}\left(u,\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right)\right)\right]\mathrm{d}\boldsymbol{B}^H(u)\right|^p + \\
&4^{p-1}\sup_{t_0\leqslant s\leqslant t}\left|\int_{t_0}^s\int_{|z|<c}\left[\boldsymbol{h}\left(u,\boldsymbol{X}^n\left(u-\frac{1}{n}\right),\boldsymbol{X}^n_{u-1/n},\mathcal{L}\left(\boldsymbol{X}^n\left(u-\frac{1}{n}\right)\right),z,r\left(u-\frac{1}{n}\right)\right) - \right.\right.\\
&\left.\left.\boldsymbol{h}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{k}\right),\boldsymbol{X}_{u-1/k}{}^k,\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),z,r\left(u-\frac{1}{k}\right)\right)\right]\tilde{N}(\mathrm{d}u,\mathrm{d}z)\right|^p := \\
&4^{p-1}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4).
\end{aligned}$$

由 Hölder 不等式、假设 1、引理 3 得

$$\begin{aligned}
&\mathbb{E}\alpha_1 \leqslant T^{p-1}\,\mathbb{E}\int_{t_0}^t\left|\boldsymbol{f}\left(u,\boldsymbol{X}^n\left(u-\frac{1}{n}\right),\boldsymbol{X}^n_{u-1/n},\mathcal{L}\left(\boldsymbol{X}^n\left(u-\frac{1}{n}\right)\right),r\left(u-\frac{1}{n}\right)\right) - \right.\\
&\left.\boldsymbol{f}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{k}\right),\boldsymbol{X}_{u-1/k}{}^k,\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),r\left(u-\frac{1}{k}\right)\right)\right|^p\mathrm{d}u \leqslant \\
&(2T)^{p-1}\,\mathbb{E}\int_{t_0}^t\left|\boldsymbol{f}\left(u,\boldsymbol{X}^n\left(u-\frac{1}{n}\right),\boldsymbol{X}^n_{u-1/n},\mathcal{L}\left(\boldsymbol{X}^n\left(u-\frac{1}{n}\right)\right),r\left(u-\frac{1}{n}\right)\right) - \right.\\
&\left.\boldsymbol{f}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{n}\right),\boldsymbol{X}^k_{u-1/n},\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{n}\right)\right),r\left(u-\frac{1}{n}\right)\right)\right|^p\mathrm{d}u + \\
&(2T)^{p-1}\,\mathbb{E}\int_{t_0}^t\left|\boldsymbol{f}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{n}\right),\boldsymbol{X}^k_{u-1/n},\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{n}\right)\right),r\left(u-\frac{1}{n}\right)\right) - \right.\\
&\left.\boldsymbol{f}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{k}\right),\boldsymbol{X}_{u-1/k}{}^k,\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),r\left(u-\frac{1}{k}\right)\right)\right|^p\mathrm{d}u \leqslant \\
&(2T)^{p-1}\int_{t_0}^t K(u)\psi\left(2\,\mathbb{E}\left|\boldsymbol{X}^n\left(u-\frac{1}{n}\right) - \boldsymbol{X}^k\left(u-\frac{1}{n}\right)\right|^p + \mathbb{E}\|\boldsymbol{X}^n_{u-1/n} - \boldsymbol{X}^k_{u-1/n}\|^p\right)\mathrm{d}u + \\
&(2T)^{p-1}\,\mathbb{E}\int_{t_0}^t K(u)\psi\left[3C_3\left(\left(\frac{1}{k}-\frac{1}{n}\right)^p + \left(\frac{1}{k}-\frac{1}{n}\right)^{p/2} + \left(\frac{1}{k}-\frac{1}{n}\right)^{pH} + \left(\frac{1}{k}-\frac{1}{n}\right)\right)\right]\mathrm{d}u.
\end{aligned}$$

由 Hölder 不等式、BDG 不等式、假设 1、引理 3,类似于 $\mathbb{E}\alpha_1$ 得

$$\begin{aligned}
&\mathbb{E}\alpha_2 \leqslant C_p\mathbb{E}\left[\int_{t_0}^t\left\|\boldsymbol{g}\left(u,\boldsymbol{X}^n\left(u-\frac{1}{n}\right),\boldsymbol{X}^n_{u-1/n},\mathcal{L}\left(\boldsymbol{X}^n\left(u-\frac{1}{n}\right)\right),r\left(u-\frac{1}{n}\right)\right) - \right.\right.\\
&\left.\left.\boldsymbol{g}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{k}\right),\boldsymbol{X}_{u-1/k}{}^k,\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),r\left(u-\frac{1}{k}\right)\right)\right\|^2\mathrm{d}u\right]^{p/2} \leqslant \\
&C_pT^{p/2-1}\,\mathbb{E}\left(\int_{t_0}^t\left\|\boldsymbol{g}\left(u,\boldsymbol{X}^n\left(u-\frac{1}{n}\right),\boldsymbol{X}^n_{u-1/n},\mathcal{L}\left(\boldsymbol{X}^n\left(u-\frac{1}{n}\right)\right),r\left(u-\frac{1}{n}\right)\right) - \right.\right.
\end{aligned}$$

$$\boldsymbol{g}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{k}\right),\boldsymbol{X}_{u-1/k}{}^k,\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),r\left(u-\frac{1}{k}\right)\right)\Big\|^p\mathrm{d}u\Big)\leqslant$$

$$C_p2^{p-1}T^{p/2-1}\int_{t_0}^t K(u)\psi\left(2\,\mathbb{E}\left|\boldsymbol{X}^n\left(u-\frac{1}{n}\right)-\boldsymbol{X}^k\left(u-\frac{1}{n}\right)\right|^p+\mathbb{E}\,\|\boldsymbol{X}_{u-1/n}^n-\boldsymbol{X}_{u-1/n}^k\|^p\right)\mathrm{d}u+$$

$$C_p2^{p-1}T^{p/2-1}\int_{t_0}^t K(u)\psi\left[3C_3\left(\left(\frac{1}{k}-\frac{1}{n}\right)^p+\left(\frac{1}{k}-\frac{1}{n}\right)^{p/2}+\left(\frac{1}{k}-\frac{1}{n}\right)^{pH}+\left(\frac{1}{k}-\frac{1}{n}\right)\right)\right]\mathrm{d}u.$$

由文献[11]的式(3.16)、假设 1、引理 3,类似于 $\mathbb{E}\alpha_1$ 可得

$$\mathbb{E}\alpha_3=\mathbb{E}\left[\sup_{t_0\leqslant s\leqslant t}\left|\int_{t_0}^s\boldsymbol{m}\left(u,\mathcal{L}\left(\boldsymbol{X}^n\left(u-\frac{1}{n}\right)\right)\right)-\boldsymbol{m}\left(u,\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right)\right)\mathrm{d}\boldsymbol{B}^H(u)\right|^p\right]\leqslant$$

$$C_{\lambda,p,H}T^{pH-1}\,\mathbb{E}\left(\int_{t_0}^t\left\|\boldsymbol{m}\left(u,\mathcal{L}\left(\boldsymbol{X}^n\left(u-\frac{1}{n}\right)\right)\right)-\boldsymbol{m}\left(u,\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right)\right)\right\|^p\mathrm{d}u\right)\leqslant$$

$$2^{p-1}C_{\lambda,p,H}T^{pH-1}\int_{t_0}^t K(u)\psi\left(\mathbb{E}\left|\boldsymbol{X}^n\left(u-\frac{1}{n}\right)-\boldsymbol{X}^k\left(u-\frac{1}{n}\right)\right|^p\right)\mathrm{d}u+$$

$$2^{p-1}C_{\lambda,p,H}T^{pH-1}\int_{t_0}^t K(u)\psi\left[C_3\left(\left(\frac{1}{k}-\frac{1}{n}\right)^p+\left(\frac{1}{k}-\frac{1}{n}\right)^{p/2}+\left(\frac{1}{k}-\frac{1}{n}\right)^{pH}+\left(\frac{1}{k}-\frac{1}{n}\right)\right)\right]\mathrm{d}u.$$

由假设 1 和引理 3 得

$$\mathbb{E}\alpha_4=\mathbb{E}\left[\sup_{t_0\leqslant s\leqslant t}\left|\int_{t_0}^s\int_{|z|<c}\left[\boldsymbol{h}\left(u,\boldsymbol{X}^n\left(u-\frac{1}{n}\right),\boldsymbol{X}_{u-1/n}^n,\mathcal{L}\left(\boldsymbol{X}^n\left(u-\frac{1}{n}\right)\right),z,r\left(u-\frac{1}{n}\right)\right)-\right.\right.\right.$$

$$\left.\left.\left.\boldsymbol{h}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{k}\right),\boldsymbol{X}_{u-1/k}^k,\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),z,r\left(u-\frac{1}{k}\right)\right)\right]\tilde{N}(\mathrm{d}u,\mathrm{d}z)\right|^p\right]\leqslant$$

$$D_p\,\mathbb{E}\left(\int_{t_0}^t\int_{|z|<c}\left|\boldsymbol{h}\left(u,\boldsymbol{X}^n\left(u-\frac{1}{n}\right),\boldsymbol{X}_{u-1/n}^n,\mathcal{L}\left(\boldsymbol{X}^n\left(u-\frac{1}{n}\right)\right),z,r\left(u-\frac{1}{n}\right)\right)-\right.\right.$$

$$\left.\left.\boldsymbol{h}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{k}\right),\boldsymbol{X}_{u-1/k}^k,\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),z,r\left(u-\frac{1}{k}\right)\right)\right|^2\nu(\mathrm{d}z)\mathrm{d}u\right)^{p/2}+$$

$$D_p\,\mathbb{E}\left(\int_{t_0}^t\int_{|z|<c}\left|\boldsymbol{h}\left(u,\boldsymbol{X}^n\left(u-\frac{1}{n}\right),\boldsymbol{X}_{u-1/n}^n,\mathcal{L}\left(\boldsymbol{X}^n\left(u-\frac{1}{n}\right)\right),z,r\left(u-\frac{1}{n}\right)\right)-\right.\right.$$

$$\left.\left.\boldsymbol{h}\left(u,\boldsymbol{X}^k\left(u-\frac{1}{k}\right),\boldsymbol{X}_{u-1/k}^k,\mathcal{L}\left(\boldsymbol{X}^k\left(u-\frac{1}{k}\right)\right),z,r\left(u-\frac{1}{k}\right)\right)\right|^p\nu(\mathrm{d}z)\mathrm{d}u\right)\leqslant$$

$$2^{p-1}(D_p(VT)^{p/2-1}+D_p)\times$$

$$\int_{t_0}^t K(u)\psi\left(2\,\mathbb{E}\left|\boldsymbol{X}^n\left(u-\frac{1}{n}\right)-\boldsymbol{X}^k\left(u-\frac{1}{n}\right)\right|^p+\mathbb{E}\,\|\boldsymbol{X}_{u-1/n}^n-\boldsymbol{X}_{u-1/n}^k\|^p\right)\mathrm{d}u+$$

$$2^{p-1}(D_p(VT)^{p/2-1}+D_p)\times$$

$$\int_{t_0}^t K(u)\psi\left[3C_3\left(\left(\frac{1}{k}-\frac{1}{n}\right)^p+\left(\frac{1}{k}-\frac{1}{n}\right)^{p/2}+\left(\frac{1}{k}-\frac{1}{n}\right)^{pH}+\left(\frac{1}{k}-\frac{1}{n}\right)\right)\right]\mathrm{d}u.$$

注意

$$\mathbb{E}\,\|\boldsymbol{X}_{u-1/n}^n-\boldsymbol{X}_{u-1/n}^k\|^p\leqslant\mathbb{E}\left(\sup_{\underline{\theta}\leqslant\theta\leqslant 1}\left|\boldsymbol{X}^n\left(\theta\left(u-\frac{1}{n}\right)\right)-\boldsymbol{X}^k\left(\theta\left(u-\frac{1}{n}\right)\right)\right|^p\right)\leqslant$$

$$\mathbb{E}\left(\sup_{t_0\leqslant e\leqslant u}|\boldsymbol{X}^n(e)-\boldsymbol{X}^k(e)|^p\right).$$

综上

$$\mathbb{E}\left(\sup_{t_0\leqslant s\leqslant t}\mathbb{E}|\boldsymbol{X}^n(s)-\boldsymbol{X}^k(s)|^p\right)\leqslant$$

$$C_4\int_{t_0}^t K(u)\psi\left(3\mathbb{E}\left(\sup_{t_0\leqslant e\leqslant u}|\boldsymbol{X}^n(e)-\boldsymbol{X}^k(e)|^p\right)\right)\mathrm{d}u+$$

$$C_4K(T)\int_{t_0}^t\psi\left(3C_3\left(\left(\frac{1}{k}-\frac{1}{n}\right)^p+\left(\frac{1}{k}-\frac{1}{n}\right)^{p/2}+\left(\frac{1}{k}-\frac{1}{n}\right)^{pH}+\left(\frac{1}{k}-\frac{1}{n}\right)\right)\right)\mathrm{d}u,$$

其中，$C_4 = 8^{p-1}(T^{p-1} + C_p T^{p/2-1} + C_{\lambda,p,H} T^{pH-1} + D_p(VT)^{p/2-1} + D_p)$.

令 $Z(t) = \lim_{n,k\to\infty} \mathbb{E}(\sup_{t_0\leqslant s\leqslant t} |\boldsymbol{X}^n(s) - \boldsymbol{X}^k(s)|^p)$，由 $\psi(0)=0$ 可得对任意 $\epsilon \geqslant 0$ 有

$$Z(t) \leqslant \epsilon + C_4 K(T)\int_{t_0}^{t}\psi(3Z(u))\mathrm{d}u.$$

由引理 1 可得

$$Z(t) \leqslant \frac{1}{3}G^{-1}(G(3\epsilon) + 3C_4K(T)t),$$

且 $G^{-1}(G(3\epsilon) + 3C_4K(T)t) \in \mathrm{Dom}(G^{-1})$，其中 $G(r) = \int_1^r \frac{\mathrm{d}s}{\psi(s)}, r > 0$.由 $\psi(0)=0$ 可得

$$\lim_{\epsilon\to 0} G(3\epsilon) = -\infty,\ \mathrm{Dom}(G^{-1}) = (-\infty, G(\infty)).$$

因此，当 $\epsilon \to 0$，$Z(t)=0$，即

$$\lim_{n,k\to\infty} \mathbb{E}(\sup_{t_0\leqslant s\leqslant t} |\boldsymbol{X}^n(s) - \boldsymbol{X}^k(s)|^p) = 0. \tag{5}$$

因此，$\{\boldsymbol{X}^k\}_{k\geqslant 1}$ 是 $L^p(\Omega;\ C([t_0,T]);\mathbb{R}^d)$ 中的 Cauchy 序列，其极限记为 $\boldsymbol{X}$，即当 $n\to\infty$ 时，有

$$\lim_{k\to\infty} \mathbb{E}(\sup_{t_0\leqslant s\leqslant t} |\boldsymbol{X}(s) - \boldsymbol{X}^k(s)|^p) = 0. \tag{6}$$

现在，我们将证明 $\boldsymbol{X}$ 是式(1)的一个解.对于所有 $\underline{\theta}t_0 \leqslant t \leqslant T$，我们有

$$\mathbb{E}\left(\left|\boldsymbol{X}(t) - \boldsymbol{X}^k\left(t-\frac{1}{k}\right)\right|^p\right) = \mathbb{E}\left(\left|\boldsymbol{X}(t) - \boldsymbol{X}^k(t) + \boldsymbol{X}^k(t) - \boldsymbol{X}^k\left(t-\frac{1}{k}\right)\right|^p\right) \leqslant$$
$$2^{p-1}\mathbb{E}(|\boldsymbol{X}(t) - \boldsymbol{X}^k(t)|^p) + 2^{p-1}\mathbb{E}\left(\left|\boldsymbol{X}^k(t) - \boldsymbol{X}^k\left(t-\frac{1}{k}\right)\right|^p\right).$$

由引理 3 和式(6)可得，当 $k\to\infty$ 时，$\mathbb{E}(|\boldsymbol{X}(t) - \boldsymbol{X}^k(t-1/k)|^p) = 0$，因此，式(2)两侧取极限可得

$$\boldsymbol{X}(t) = \boldsymbol{\xi}(t_0) + \int_{t_0}^{t}\boldsymbol{f}(s,\boldsymbol{X}(s),\boldsymbol{X}_s,\mathcal{L}(\boldsymbol{X}(s)),r(s))\mathrm{d}s +$$
$$\int_{t_0}^{t}\boldsymbol{g}(s,\boldsymbol{X}(s),\boldsymbol{X}_s,\mathcal{L}(\boldsymbol{X}(s)),r(s))\mathrm{d}\boldsymbol{B}(s) + \int_{t_0}^{t}\boldsymbol{m}(s,\mathcal{L}(\boldsymbol{X}(s)))\mathrm{d}\boldsymbol{B}^H(s) +$$
$$\int_{t_0}^{t}\int_{|z|<c}\boldsymbol{h}(s,\boldsymbol{X}(s-),\boldsymbol{X}_s,\mathcal{L}(\boldsymbol{X}(s)),z,r(s))\tilde{N}(\mathrm{d}s,\mathrm{d}z).$$

唯一性：设 $\boldsymbol{X}(t)$ 和 $\boldsymbol{Y}(t)$ 是式(1)的两个解，则

$$\mathbb{E}(\sup_{t_0\leqslant s\leqslant t}\mathbb{E}|\boldsymbol{X}(s) - \boldsymbol{Y}(s)|^p) \leqslant$$
$$4^{p-1}\mathbb{E}\left[\sup_{t_0\leqslant s\leqslant t}\left|\int_{t_0}^{s}(\boldsymbol{f}(u,\boldsymbol{X}(u),\boldsymbol{X}_u,\mathcal{L}(\boldsymbol{X}(u)),r(u)) - \boldsymbol{f}(u,\boldsymbol{Y}(u),\boldsymbol{Y}_u,\mathcal{L}(\boldsymbol{Y}(u)),r(u)))\mathrm{d}u\right|^p\right] +$$
$$4^{p-1}\mathbb{E}\left[\sup_{t_0\leqslant s\leqslant t}\left|\int_{t_0}^{s}(\boldsymbol{g}(u,\boldsymbol{X}(u),\boldsymbol{X}_u,\mathcal{L}(\boldsymbol{X}(u)),r(u)) -\right.\right.$$
$$\left.\left.\boldsymbol{g}(u,\boldsymbol{Y}(u),\boldsymbol{Y}_u,\mathcal{L}(\boldsymbol{Y}(u)),r(u)))\mathrm{d}\boldsymbol{B}(u)\right|^p\right] +$$
$$4^{p-1}\mathbb{E}\left[\sup_{t_0\leqslant s\leqslant t}\left|\int_{t_0}^{s}(\boldsymbol{m}(u,\mathcal{L}(\boldsymbol{X}(u)) - \boldsymbol{m}(u,\mathcal{L}(\boldsymbol{Y}(u)))\mathrm{d}\boldsymbol{B}^H(u)\right|^p\right] +$$
$$4^{p-1}\mathbb{E}\left[\sup_{t_0\leqslant s\leqslant t}\left|\int_{t_0}^{s}\int_{|z|<c}(\boldsymbol{h}(u,\boldsymbol{X}(u),\boldsymbol{X}_u,\mathcal{L}(\boldsymbol{X}(u)),z,r(u)) -\right.\right.$$
$$\left.\left.\boldsymbol{h}(u,\boldsymbol{Y}(u),\boldsymbol{Y}_u,\mathcal{L}(\boldsymbol{Y}(u)),z,r(u)))\tilde{N}(\mathrm{d}u,\mathrm{d}z)\right|^p\right] :=$$
$$4^{p-1}(\beta_1 + \beta_2 + \beta_3 + \beta_4).$$

对于 β_1，由 Hölder 不等式和假设 1 得

$$\beta_1 \leqslant T^{p-1}\mathbb{E}\int_{t_0}^{t}|\boldsymbol{f}(u,\boldsymbol{X}(u),\boldsymbol{X}_u,\mathcal{L}(\boldsymbol{X}(u)),r(u)) -$$
$$\boldsymbol{f}(u,\boldsymbol{Y}(u),\boldsymbol{Y}_u,\mathcal{L}(\boldsymbol{Y}(u)),r(u))|^p\mathrm{d}u \leqslant$$

$$T^{p-1}\,\mathbb{E}\int_{t_0}^{t}K(u)\psi(|\boldsymbol{X}(u)-\boldsymbol{Y}(u)|^p+\|\boldsymbol{X}_u-\boldsymbol{Y}_u\|+\mathbb{W}_p^p(\mathcal{L}(\boldsymbol{X}(u)),\mathcal{L}(\boldsymbol{Y}(u))))\mathrm{d}u\leqslant$$

$$T^{p-1}K(T)\int_{t_0}^{t}\psi(3\mathbb{E}(\sup_{t_0\leqslant u\leqslant s}|\boldsymbol{X}(u)-\boldsymbol{Y}(u)|^p))\mathrm{d}s.$$

对于 β_2, 由 BDG 不等式、Hölder 不等式和假设 1 得

$$\beta_2\leqslant C_p\,\mathbb{E}\left[\int_{t_0}^{t}\|\boldsymbol{g}(u,\boldsymbol{X}(u),\boldsymbol{X}_u,\mathcal{L}(\boldsymbol{X}(u)),r(u))-\boldsymbol{g}(u,\boldsymbol{Y}(u),\boldsymbol{Y}_u,\mathcal{L}(\boldsymbol{Y}(u)),r(u))\|^2\mathrm{d}u\right]^{p/2}\leqslant$$

$$C_pT^{p/2-1}\,\mathbb{E}\int_{t_0}^{t}\|\boldsymbol{g}(u,\boldsymbol{X}(u),\boldsymbol{X}_u,\mathcal{L}(\boldsymbol{X}(u)),r(u))-\boldsymbol{g}(u,\boldsymbol{Y}(u),\boldsymbol{Y}_u,\mathcal{L}(\boldsymbol{Y}(u)),r(u))\|^p\mathrm{d}u\leqslant$$

$$C_pT^{p/2-1}\,\mathbb{E}\int_{t_0}^{t}K(u)\psi(|\boldsymbol{X}(u)-\boldsymbol{Y}(u)|^p+\|\boldsymbol{X}_u-\boldsymbol{Y}_u\|+\mathbb{W}_p^p(\mathcal{L}(\boldsymbol{X}(u)),\mathcal{L}(\boldsymbol{Y}(u))))\mathrm{d}u\leqslant$$

$$C_pT^{p/2-1}K(T)\int_{t_0}^{t}\psi(3\mathbb{E}(\sup_{t_0\leqslant u\leqslant s}|\boldsymbol{X}(u)-\boldsymbol{Y}(u)|^p))\mathrm{d}s.$$

对于 β_3, 由文献[4]的式(3.5)和假设 1 可得

$$\beta_3\leqslant C_{\lambda,p,H}T^{pH-1}\,\mathbb{E}\int_{t_0}^{t}\|\boldsymbol{m}(u,\mathcal{L}(\boldsymbol{X}(u)))-\boldsymbol{m}(u,\mathcal{L}(\boldsymbol{Y}(u)))\|^p\mathrm{d}u\leqslant$$

$$C_{\lambda,p,H}T^{pH-1}\,\mathbb{E}\int_{t_0}^{t}K(u)\psi(\mathbb{W}_p^p(\mathcal{L}(\boldsymbol{X}(u)),\mathcal{L}(\boldsymbol{Y}(u))))\mathrm{d}u\leqslant$$

$$C_{\lambda,p,H}T^{pH-1}K(T)\int_{t_0}^{t}\psi(\mathbb{E}(\sup_{t_0\leqslant u\leqslant s}|\boldsymbol{X}(u)-\boldsymbol{Y}(u)|^p))\mathrm{d}s.$$

对于 β_4, 由 Kunita 第一不等式、Hölder 不等式、假设 1 得

$$\beta_4=\mathbb{E}\left[\sup_{t_0\leqslant s\leqslant t}\left|\int_{t_0}^{s}\int_{|z|<c}(\boldsymbol{h}(u,\boldsymbol{X}(u),\boldsymbol{X}_u,\mathcal{L}(\boldsymbol{X}(u)),z,r(u))-\right.\right.$$

$$\left.\left.\boldsymbol{h}(u,\boldsymbol{Y}(u),\boldsymbol{Y}_u,\mathcal{L}(\boldsymbol{Y}(u)),z,r(u)))\tilde{N}(\mathrm{d}u,\mathrm{d}z)\right|^p\right]\leqslant$$

$$D_p((VT)^{p/2-1}+1)\,\mathbb{E}\left(\int_{t_0}^{t}\int_{|z|<c}[\boldsymbol{h}(u,\boldsymbol{X}(u),\boldsymbol{X}_u,\mathcal{L}(\boldsymbol{X}(u)),z,r(u))-\right.$$

$$\left.\boldsymbol{h}(u,\boldsymbol{Y}(u),\boldsymbol{Y}_u,\mathcal{L}(\boldsymbol{Y}(u)),z,r(u))]^p\nu(\mathrm{d}z)\mathrm{d}u\right)\leqslant$$

$$D_p((VT)^{p/2-1}+1)K(T)\int_{t_0}^{t}\psi(3\mathbb{E}(\sup_{t_0\leqslant u\leqslant s}|\boldsymbol{X}(u)-\boldsymbol{Y}(u)|^p))\mathrm{d}s.$$

综上

$$\mathbb{E}(\sup_{t_0\leqslant s\leqslant t}|\boldsymbol{X}(s)-\boldsymbol{Y}(s)|^p)\leqslant$$

$$4^{p-1}(T^{p-1}+C_pT^{p/2-1}+C_{\lambda,p,H}T^{pH-1}+D_p(VT)^{p/2-1}+D_p)K(T)\times$$

$$\int_{t_0}^{t}\psi(3\mathbb{E}(\sup_{t_0\leqslant u\leqslant s}|\boldsymbol{X}(u)-\boldsymbol{Y}(u)|^p))\mathrm{d}s.$$

则由引理 1 可得 $\boldsymbol{X}(t)=\boldsymbol{Y}(t),t\in[\underline{\theta}t_0,T],\mathbb{P}-\mathrm{a.s.}$.

3 平 均 原 理

设 $\epsilon\in(0,\varepsilon_0)$ 是一个正参数, ε_0 为固定值,

$$\begin{cases}\boldsymbol{X}^{\varepsilon}(t)=\boldsymbol{\xi}(t_0)+\displaystyle\int_{t_0}^{t}\boldsymbol{f}\left(\frac{s}{\epsilon},\boldsymbol{X}^{\varepsilon}(s),\boldsymbol{X}_s^{\varepsilon},\mathcal{L}(\boldsymbol{X}^{\varepsilon}(s)),r(s)\right)\mathrm{d}s+\\ \quad\displaystyle\int_{t_0}^{t}\boldsymbol{g}\left(\frac{s}{\epsilon},\boldsymbol{X}^{\varepsilon}(s),\boldsymbol{X}_s^{\varepsilon},\mathcal{L}(\boldsymbol{X}^{\varepsilon}(s)),r(s)\right)\mathrm{d}\boldsymbol{B}(s)+\int_{t_0}^{t}\boldsymbol{m}\left(\frac{s}{\epsilon},\mathcal{L}(\boldsymbol{X}^{\varepsilon}(s))\right)\mathrm{d}\boldsymbol{B}^{H}(s)+\\ \quad\displaystyle\int_{t_0}^{t}\int_{|z|<c}\boldsymbol{h}\left(\frac{s}{\epsilon},\boldsymbol{X}^{\varepsilon}(s),\boldsymbol{X}_s^{\varepsilon},\mathcal{L}(\boldsymbol{X}^{\varepsilon}(s)),z,r(s)\right)\tilde{N}(\mathrm{d}s,\mathrm{d}z),\qquad t\in[t_0,T],\\ \boldsymbol{X}^{\varepsilon}(t)=\boldsymbol{\xi}(t),\qquad t\in[\underline{\theta}t_0,t_0].\end{cases}$$

本节的目的是证明当 $t \in [\underline{\theta}t_0, T]$ 时,解 $\boldsymbol{X}^\epsilon(t)$ 能在一定意义下逼近以下的平均方程的解:

$$\begin{cases}\bar{\boldsymbol{X}}(t)=\boldsymbol{\xi}(t_0)+\int_{t_0}^{t}\bar{\boldsymbol{f}}(\bar{\boldsymbol{X}}(s),\bar{\boldsymbol{X}}_s,\mathcal{L}(\bar{\boldsymbol{X}}(s)),r(s))\mathrm{d}s+\\ \quad\int_{t_0}^{t}\bar{\boldsymbol{g}}(\bar{\boldsymbol{X}}(s),\bar{\boldsymbol{X}}_s,\mathcal{L}(\bar{\boldsymbol{X}}(s)),r(s))\mathrm{d}\boldsymbol{B}(s)+\int_{t_0}^{t}\bar{\boldsymbol{m}}(\mathcal{L}(\bar{\boldsymbol{X}}(s)))\mathrm{d}\boldsymbol{B}^H(s)+\\ \quad\int_{t_0}^{t}\int_{|z|<c}\bar{\boldsymbol{h}}(\bar{\boldsymbol{X}}(s),\bar{\boldsymbol{X}}_s,\mathcal{L}(\bar{\boldsymbol{X}}(s)),z,r(s))\tilde{N}(\mathrm{d}s,\mathrm{d}z),\qquad t\in[t_0,T],\\ \bar{\boldsymbol{X}}(t)=\boldsymbol{\xi}(t),\qquad t\in[\underline{\theta}t_0,t_0],\end{cases}\tag{7}$$

其中,$\bar{\boldsymbol{f}}$: $\mathbb{R}^d\times L^p(\Omega;C([\underline{\theta},1];\mathbb{R}^d))\times\mathcal{P}_p(\mathbb{R}^d)\times\mathbb{S}\to\mathbb{R}^d$,$\bar{\boldsymbol{g}}$: $\mathbb{R}^d\times L^p(\Omega;C([\underline{\theta},1];\mathbb{R}^d))\times\mathcal{P}_p(\mathbb{R}^d)\times\mathbb{S}\to\mathbb{R}^{d\times l}$,$\bar{\boldsymbol{m}}$: $\mathcal{P}_p(\mathbb{R}^d)\to\mathbb{R}^{d\times l}$,$\bar{\boldsymbol{h}}$: $\mathbb{R}^d\times L^p(\Omega; C([\underline{\theta},1];\mathbb{R}^d))\times\mathcal{P}_p(\mathbb{R}^d)\times\mathbb{R}^d\times\mathbb{S}\to\mathbb{R}^d$满足以下平均条件.

假设 2(平均条件) 存在一个有界正函数 φ: $(0,+\infty)\to(0,+\infty)$, $\lim_{T\to\infty}\varphi(T)=0$, 对任意的 $\boldsymbol{x}\in\mathbb{R}^d$,$\boldsymbol{y}\in L^p(\Omega; C([\underline{\theta},1];\mathbb{R}^d))$,$\mu\in\mathcal{P}_p(\mathbb{R}^d)$,$i\in\mathbb{S}$, 有

$$\sup_{t\geqslant 0}\left|\frac{1}{T}\int_t^{t+T}[\boldsymbol{f}(s,\boldsymbol{x},\boldsymbol{y},\mu,i)-\bar{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{y},\mu,i)]\mathrm{d}s\right|^p\leqslant\varphi(T)\kappa(|\boldsymbol{x}|^p+\|\boldsymbol{y}\|^p+\mathbb{W}_p^p(\mu,\delta_0)),$$

$$\sup_{t\geqslant 0}\frac{1}{T}\int_t^{t+T}\|\boldsymbol{g}(s,\boldsymbol{x},\boldsymbol{y},\mu,i)-\bar{\boldsymbol{g}}(\boldsymbol{x},\boldsymbol{y},\mu,i)\|^p\mathrm{d}s\leqslant\varphi(T)\kappa(|\boldsymbol{x}|^p+\|\boldsymbol{y}\|^p+\mathbb{W}_p^p(\mu,\delta_0)),$$

$$\sup_{t\geqslant 0}\frac{1}{T}\int_t^{t+T}\|\boldsymbol{m}(s,\mu)-\bar{\boldsymbol{m}}(\mu)\|^p\mathrm{d}s\leqslant\varphi(T)\kappa(\mathbb{W}_p^p(\mu,\delta_0)),$$

$$\sup_{t\geqslant 0}\frac{1}{T}\int_t^{t+T}\int_{|z|<c}|\boldsymbol{h}(s,\boldsymbol{x},\boldsymbol{y},\mu,z,i)-\bar{\boldsymbol{h}}(\boldsymbol{x},\boldsymbol{y},\mu,z,i)|^p\nu(\mathrm{d}z)\mathrm{d}s\leqslant$$
$$\varphi(T)\kappa(|\boldsymbol{x}|^p+\|\boldsymbol{y}\|^p+\mathbb{W}_p^p(\mu,\delta_0)),$$

其中 $\kappa(\cdot)$ 是一个非递减的非负凹函数,且 $\kappa(0)=0$,$\int_{0^+}\frac{1}{\kappa(x)}\mathrm{d}x=+\infty$.

注 1 (i) 注意到

$$\sup_{t\geqslant 0}\left|\frac{1}{T}\int_t^{t+T}[\boldsymbol{f}(s,\boldsymbol{x},\boldsymbol{y},\mu,i)-\bar{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{y},\mu,i)]\mathrm{d}s\right|^p\leqslant\sup_{t\geqslant 0}\frac{1}{T}\int_t^{t+T}|\boldsymbol{f}(s,\boldsymbol{x},\boldsymbol{y},\mu,i)-\bar{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{y},\mu,i)|^p\mathrm{d}s.$$

因此,假设 2 比如下的平均条件要更弱:

$$\sup_{t\geqslant 0}\frac{1}{T}\int_t^{t+T}|\boldsymbol{f}(s,\boldsymbol{x},\boldsymbol{y},\mu,i)-\bar{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{y},\mu,i)|^p\mathrm{d}s\leqslant\varphi(T)\kappa(|\boldsymbol{x}|^p+\|\boldsymbol{y}\|^p+\mathbb{W}_p^p(\mu,\delta_0)).$$

(ii) 由假设 1 和假设 2 得,对任意 $\boldsymbol{x}_1,\boldsymbol{x}_2\in\mathbb{R}^d$,$\boldsymbol{y}_1,\boldsymbol{y}_2\in L^p(\Omega; C([\underline{\theta},1];\mathbb{R}^d))$,$\mu_1,\mu_2\in\mathcal{P}_p(\mathbb{R}^d)$,$i\in\mathbb{S}$, 有

$$\begin{aligned}&|\bar{\boldsymbol{f}}(\boldsymbol{x}_1,\boldsymbol{y}_1,\mu_1,i)-\bar{\boldsymbol{f}}(\boldsymbol{x}_2,\boldsymbol{y}_2,\mu_2,i)|^p\leqslant\\&\quad 3^{p-1}\left|\frac{1}{T}\int_0^T[\bar{\boldsymbol{f}}(\boldsymbol{x}_1,\boldsymbol{y}_1,\mu_1,i)-\boldsymbol{f}(s,\boldsymbol{x}_1,\boldsymbol{y}_1,\mu_1,i)]\mathrm{d}s\right|^p+\\&\quad 3^{p-1}\left|\frac{1}{T}\int_0^T[\boldsymbol{f}(s,\boldsymbol{x}_1,\boldsymbol{y}_1,\mu_1,i)-\boldsymbol{f}(s,\boldsymbol{x}_2,\boldsymbol{y}_2,\mu_2,i)]\mathrm{d}s\right|^p+\\&\quad 3^{p-1}\left|\frac{1}{T}\int_0^T[\boldsymbol{f}(s,\boldsymbol{x}_2,\boldsymbol{y}_2,\mu_2,i)-\bar{\boldsymbol{f}}(\boldsymbol{x}_2,\boldsymbol{y}_2,\mu_2,i)]\mathrm{d}s\right|^p\leqslant\\&\quad 3^{p-1}\varphi(T)\kappa(|\boldsymbol{x}_1|^p+\|\boldsymbol{y}_1\|^p+\mathbb{W}_p^p(\mu_1,\delta_0))+\\&\quad 3^{p-1}K(T)\psi(|\boldsymbol{x}_1-\boldsymbol{x}_2|^p+\|\boldsymbol{y}_1-\boldsymbol{y}_2\|^p+\mathbb{W}_p^p(\mu_1,\mu_2))+\\&\quad 3^{p-1}\varphi(T)\kappa(|\boldsymbol{x}_2|^p+\|\boldsymbol{y}_2\|^p+\mathbb{W}_p^p(\mu_2,\delta_0)).\end{aligned}$$

对任意 $\boldsymbol{x}\in\mathbb{R}^d$,$\boldsymbol{y}\in L^p(\Omega;C([\underline{\theta},1];\mathbb{R}^d))$,$\mu\in\mathcal{P}_p(\mathbb{R}^d)$, 有

$$\begin{aligned}&|\bar{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{y},\mu,i)|^p=\left|\frac{1}{T}\int_0^T\bar{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{y},\mu,i)\mathrm{d}s\right|^p\leqslant\\&\quad 2^{p-1}\left|\frac{1}{T}\int_0^T[\boldsymbol{f}(s,\boldsymbol{x},\boldsymbol{y},\mu,i)-\bar{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{y},\mu,i)]\mathrm{d}s\right|^p+2^{p-1}\left|\frac{1}{T}\int_0^T\boldsymbol{f}(s,\boldsymbol{x},\boldsymbol{y},\mu,i)\,\mathrm{d}s\right|^p\leqslant\end{aligned}$$

$$2^{p-1}\varphi(T)\kappa(|\boldsymbol{x}|^p+\|\boldsymbol{y}\|^p+\mathbb{W}_p^p(\mu,\delta_0))+2^{p-1}K(T)(1+|\boldsymbol{x}|^p+\|\boldsymbol{y}\|^p+\mathbb{W}_p^p(\mu,\delta_0)).$$

同理可得

$$\begin{aligned}
&\|\bar{\boldsymbol{g}}(\boldsymbol{x}_1,\boldsymbol{y}_1,\mu_1,i)-\bar{\boldsymbol{g}}(\boldsymbol{x}_2,\boldsymbol{y}_2,\mu_2,i)\|^p\leqslant 3^{p-1}\varphi(T)\kappa(|\boldsymbol{x}_1|^p+\|\boldsymbol{y}_1\|^p+\mathbb{W}_p^p(\mu_1,\delta_0))+\\
&\quad 3^{p-1}K(T)\psi(|\boldsymbol{x}_1-\boldsymbol{x}_2|^p+\|\boldsymbol{y}_1-\boldsymbol{y}_2\|^p+\mathbb{W}_p^p(\mu_1,\mu_2))+\\
&\quad 3^{p-1}\varphi(T)\kappa(|\boldsymbol{x}_2|^p+\|\boldsymbol{y}_2\|^p+\mathbb{W}_p^p(\mu_2,\delta_0)),\\
&\|\bar{\boldsymbol{g}}(\boldsymbol{x},\boldsymbol{y},\mu,i)\|^p\leqslant 2^{p-1}\varphi(T)\kappa(|\boldsymbol{x}|^p+\|\boldsymbol{y}\|^p+\mathbb{W}_p^p(\mu,\delta_0))+\\
&\quad 2^{p-1}K(T)(1+|\boldsymbol{x}|^p+\|\boldsymbol{y}\|^p+\mathbb{W}_p^p(\mu,\delta_0)),\\
&\|\bar{\boldsymbol{m}}(\mu_1)-\bar{\boldsymbol{m}}(\mu_2)\|^p\leqslant 3^{p-1}\varphi(T)\kappa(\mathbb{W}_p^p(\mu_1,\delta_0))+\\
&\quad 3^{p-1}K(T)\psi(\mathbb{W}_p^p(\mu_1,\mu_2))+3^{p-1}\varphi(T)\kappa(\mathbb{W}_p^p(\mu_2,\delta_0)),\\
&\|\bar{\boldsymbol{m}}(\mu)\|^p\leqslant 2^{p-1}\varphi(T)\kappa(\mathbb{W}_p^p(\mu,\delta_0))+2^{p-1}K(T)(1+\mathbb{W}_p^p(\mu,\delta_0)),\\
&\int_{|z|<c}|\bar{\boldsymbol{h}}(\boldsymbol{x}_1,\boldsymbol{y}_1,\mu_1,z,i)-\bar{\boldsymbol{h}}(\boldsymbol{x}_2,\boldsymbol{y}_2,\mu_2,z,i)|^p\nu(\mathrm{d}z)\leqslant\\
&\quad 3^{p-1}\varphi(T)\kappa(|\boldsymbol{x}_1|^p+\|\boldsymbol{y}_1\|^p+\mathbb{W}_p^p(\mu_1,\delta_0))+\\
&\quad 3^{p-1}K(T)\psi(|\boldsymbol{x}_1-\boldsymbol{x}_2|^p+\|\boldsymbol{y}_1-\boldsymbol{y}_2\|^p+\mathbb{W}_p^p(\mu_1,\mu_2))+\\
&\quad 3^{p-1}\varphi(T)\kappa(|\boldsymbol{x}_2|^p+\|\boldsymbol{y}_2\|^p+\mathbb{W}_p^p(\mu_2,\delta_0)),\\
&\int_{|z|<c}|\bar{\boldsymbol{h}}(\boldsymbol{x},\boldsymbol{y},\mu,z,i)|^p\nu(\mathrm{d}z)\leqslant\\
&\quad 2^{p-1}\varphi(T)\kappa(|\boldsymbol{x}|^p+\|\boldsymbol{y}\|^p+\mathbb{W}_p^p(\mu,\delta_0))+2^{p-1}K(T)(1+|\boldsymbol{x}|^p+\|\boldsymbol{y}\|^p+\mathbb{W}_p^p(\mu,\delta_0)).
\end{aligned}$$

令 $T\to\infty$ 时,由于 $K(T)$ 是有界的和 $\lim_{T\to\infty}\varphi(T)=0$,因此存在一个常数 L,使得

$$\begin{aligned}
&|\bar{\boldsymbol{f}}(\boldsymbol{x}_1,\boldsymbol{y}_1,\mu_1,i)-\bar{\boldsymbol{f}}(\boldsymbol{x}_2,\boldsymbol{y}_2,\mu_2,i)|^p\leqslant L\psi(|\boldsymbol{x}_1-\boldsymbol{x}_2|^p+\|\boldsymbol{y}_1-\boldsymbol{y}_2\|^p+\mathbb{W}_p^p(\mu_1,\mu_2)),\\
&\|\bar{\boldsymbol{g}}(\boldsymbol{x}_1,\boldsymbol{y}_1,\mu_1,i)-\bar{\boldsymbol{g}}(\boldsymbol{x}_2,\boldsymbol{y}_2,\mu_2,i)\|^p\leqslant L\psi(|\boldsymbol{x}_1-\boldsymbol{x}_2|^p+\|\boldsymbol{y}_1-\boldsymbol{y}_2\|^p+\mathbb{W}_p^p(\mu_1,\mu_2)),\\
&\|\bar{\boldsymbol{m}}(\mu_1)-\bar{\boldsymbol{m}}(\mu_2)\|^p\leqslant L\psi(\mathbb{W}_p^p(\mu_1,\mu_2)),\\
&\int_{|z|<c}|\bar{\boldsymbol{h}}(\boldsymbol{x}_1,\boldsymbol{y}_1,\mu_1,z,i)-\bar{\boldsymbol{h}}(\boldsymbol{x}_2,\boldsymbol{y}_2,\mu_2,z,i)|^p\nu(\mathrm{d}z)\leqslant\\
&\quad L\psi(|\boldsymbol{x}_1-\boldsymbol{x}_2|^p+\|\boldsymbol{y}_1-\boldsymbol{y}_2\|^p+\mathbb{W}_p^p(\mu_1,\mu_2)),\\
&|\bar{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{y},\mu,i)|^p\leqslant L(1+|\boldsymbol{x}|^p+\|\boldsymbol{y}\|^p+\mathbb{W}_p^p(\mu,\delta_0)),\\
&\|\bar{\boldsymbol{g}}(\boldsymbol{x},\boldsymbol{y},\mu,i)\|^p\leqslant L(1+|\boldsymbol{x}|^p+\|\boldsymbol{y}\|^p+\mathbb{W}_p^p(\mu,\delta_0)),\\
&\|\bar{\boldsymbol{m}}(\mu)\|^p\leqslant L(1+\mathbb{W}_p^p(\mu,\delta_0)),\\
&\int_{|z|<c}|\bar{\boldsymbol{h}}(\boldsymbol{x},\boldsymbol{y},\mu,z,i)|^p\nu(\mathrm{d}z)\leqslant L(1+|\boldsymbol{x}|^p+\|\boldsymbol{y}\|^p+\mathbb{W}_p^p(\mu,\delta_0)).
\end{aligned}$$

因此,系数 $\bar{\boldsymbol{f}},\bar{\boldsymbol{g}},\bar{\boldsymbol{m}},\bar{\boldsymbol{h}}$ 满足假设 1.所以,在假设 2 条件下,式(7)有唯一解 $\{\bar{\boldsymbol{X}}(t)\}_{t\in[\theta t_0,T]}$.

注 2　采用引理 2、引理 3 类似的证明方法,可得

$$\mathbb{E}(\sup_{t_0\leqslant t\leqslant T}|\boldsymbol{X}^\epsilon(t)|^p)\leqslant C_5K\left(\frac{T}{\epsilon}\right),$$

$$\mathbb{E}|\boldsymbol{X}^\epsilon(t)-\boldsymbol{X}^\epsilon(s)|^p\leqslant C_6K\left(\frac{T}{\epsilon}\right)[(t-s)^p+(t-s)^{p/2}+(t-s)^{pH}+(t-s)],$$

其中,C_5,C_6 是依赖于 ϵ 的正常数.

引理 4　若假设 1 和假设 2 成立,则

$$\lim_{\epsilon\to0}\mathbb{E}\left[\sup_{t_0\leqslant t\leqslant T}\left|\int_{t_0}^t\left[\boldsymbol{f}\left(\frac{s}{\epsilon},\boldsymbol{X}^\epsilon(s),\boldsymbol{X}_s^\epsilon,\mathcal{L}(\boldsymbol{X}^\epsilon(s)),i\right)-\bar{\boldsymbol{f}}(\boldsymbol{X}^\epsilon(s),\boldsymbol{X}_s^\epsilon,\mathcal{L}(\boldsymbol{X}^\epsilon(s)),i)\right]\mathrm{d}s\right|^p\right]=0.$$

证明　$\{t_1,t_2,\cdots,t_N\}$ 是 $[t_0,T]$ 的划分,$t_i=t_0+i\sqrt{\epsilon}$,$0\leqslant i\leqslant N-1$,$0<T-t_{N-1}\leqslant\sqrt{\epsilon}$,$t_N=T$,则 $T\leqslant t_0+N\sqrt{\epsilon}<T+\sqrt{\epsilon}$,有

$$\begin{aligned}
&\left|\int_{t_0}^t\left[\boldsymbol{f}\left(\frac{s}{\epsilon},\boldsymbol{X}^\epsilon(s),\boldsymbol{X}_s^\epsilon,\mathcal{L}(\boldsymbol{X}^\epsilon(s)),i\right)-\bar{\boldsymbol{f}}(\boldsymbol{X}^\epsilon(s),\boldsymbol{X}_s^\epsilon,\mathcal{L}(\boldsymbol{X}^\epsilon(s)),i)\right]\mathrm{d}s\right|^p\leqslant N^{p-1}\sum_{i=0}^{N-2}|\boldsymbol{X}_i|^p+\\
&\quad N^{p-1}\left|\int_{[(t-t_0)/\sqrt{\epsilon}]\sqrt{\epsilon}}^t\left[\boldsymbol{f}\left(\frac{s}{\epsilon},\boldsymbol{X}^\epsilon(s),\boldsymbol{X}_s^\epsilon,\mathcal{L}(\boldsymbol{X}^\epsilon(s)),i\right)-\bar{\boldsymbol{f}}(X^\epsilon(s),\boldsymbol{X}_s^\epsilon,\mathcal{L}(\boldsymbol{X}^\epsilon(s)),i)\right]\mathrm{d}s\right|^p,
\end{aligned}$$

其中 $X_i := \int_{t_i}^{t_{i+1}} \left[\boldsymbol{f}\left(\frac{s}{\epsilon}, \boldsymbol{X}^\epsilon(s), X_s^\epsilon, \mathcal{L}(\boldsymbol{X}^\epsilon(s)), i\right) - \bar{\boldsymbol{f}}(\boldsymbol{X}^\epsilon(s), \boldsymbol{X}_s^\epsilon, \mathcal{L}(\boldsymbol{X}^\epsilon(s)), i) \right] \mathrm{d}s$.由 Hölder 不等式和假设 2 得

$$
\begin{aligned}
&\left| \int_{[(t-t_0)/\sqrt{\epsilon}]\sqrt{\epsilon}}^{t} \left[\boldsymbol{f}\left(\frac{s}{\epsilon}, \boldsymbol{X}^\epsilon(s), \boldsymbol{X}_s^\epsilon, \mathcal{L}(\boldsymbol{X}^\epsilon(s)), i\right) - \bar{\boldsymbol{f}}(\boldsymbol{X}^\epsilon(s), \boldsymbol{X}_s^\epsilon, \mathcal{L}(\boldsymbol{X}^\epsilon(s)), i) \right] \mathrm{d}s \right|^p \leqslant \\
&\quad 2^{p-1}\left(t - \left[\frac{t-t_0}{\sqrt{\epsilon}}\right]\sqrt{\epsilon} \right)^{p-1} \times \\
&\quad \int_{[(t-t_0)/\sqrt{\epsilon}]\sqrt{\epsilon}}^{t} \left(\left| \boldsymbol{f}\left(\frac{s}{\epsilon}, \boldsymbol{X}^\epsilon(s), \boldsymbol{X}_s^\epsilon, \mathcal{L}(\boldsymbol{X}^\epsilon(s)), i\right) \right|^p + | \bar{\boldsymbol{f}}(\boldsymbol{X}^\epsilon(s), \boldsymbol{X}_s^\epsilon, \mathcal{L}(\boldsymbol{X}^\epsilon(s)), i) |^p \right) \mathrm{d}s \leqslant \\
&\quad 2^{p}\left(t - \left[\frac{t-t_0}{\sqrt{\epsilon}}\right]\sqrt{\epsilon} \right)^{p-1} \times \\
&\quad \int_{[(t-t_0)/\sqrt{\epsilon}]\sqrt{\epsilon}}^{t} \left(K\left(\frac{s}{\epsilon}\right) + L \right) [\psi(|\boldsymbol{X}^\epsilon(s)|^p + \|\boldsymbol{X}_s^\epsilon\|^p + \mathbb{E}|\boldsymbol{X}^\epsilon(s)|^p) + 1] \mathrm{d}s \leqslant \\
&\quad 2^p\left(K\left(\frac{T}{\epsilon}\right) + L \right) (\sqrt{\epsilon})^p \left(a + 1 + a\sup_{t_0 \leqslant s \leqslant t} |\boldsymbol{X}^\epsilon(s)|^p + a\sup_{t_0 \leqslant s \leqslant t} \|\boldsymbol{X}_s^\epsilon\|^p + a\sup_{t_0 \leqslant s \leqslant t} \mathbb{E}|\boldsymbol{X}^\epsilon(s)|^p\right).
\end{aligned}
$$

由于假设 2、注 1 和注 2 可得

$$
\begin{aligned}
&|X_i|^p = \left| \int_{t_i}^{t_{i+1}} \left[\boldsymbol{f}\left(\frac{s}{\epsilon}, \boldsymbol{X}^\epsilon(s), \boldsymbol{X}_s^\epsilon, \mathcal{L}(\boldsymbol{X}^\epsilon(s)), i\right) - \bar{\boldsymbol{f}}(\boldsymbol{X}^\epsilon(s), \boldsymbol{X}_s^\epsilon, \mathcal{L}(\boldsymbol{X}^\epsilon(s)), i) \right] \mathrm{d}s \right|^p \leqslant \\
&\quad 3^{p-1}\left| \int_{t_i}^{t_{i+1}} \left[\boldsymbol{f}\left(\frac{s}{\epsilon}, X^\epsilon(t_i), \boldsymbol{X}_{t_i}^\epsilon, \mathcal{L}(\boldsymbol{X}^\epsilon(t_i)), i\right) - \bar{\boldsymbol{f}}(\boldsymbol{X}^\epsilon(t_i), \boldsymbol{X}_{t_i}^\epsilon, \mathcal{L}(\boldsymbol{X}^\epsilon(t_i)), i) \right] \mathrm{d}s \right|^p + \\
&\quad 3^{p-1}\left| \int_{t_i}^{t_{i+1}} \left[\boldsymbol{f}\left(\frac{s}{\epsilon}, \boldsymbol{X}^\epsilon(s), \boldsymbol{X}_s^\epsilon, \mathcal{L}(\boldsymbol{X}^\epsilon(s)), i\right) - \boldsymbol{f}\left(\frac{s}{\epsilon}, \boldsymbol{X}^\epsilon(t_i), \boldsymbol{X}_{t_i}^\epsilon, \mathcal{L}(\boldsymbol{X}^\epsilon(t_i)), i\right) \right] \mathrm{d}s \right|^p + \\
&\quad 3^{p-1}\left| \int_{t_i}^{t_{i+1}} [\bar{\boldsymbol{f}}(\boldsymbol{X}^\epsilon(t_i), \boldsymbol{X}_{t_i}^\epsilon, \mathcal{L}(\boldsymbol{X}^\epsilon(t_i)), i) - \bar{\boldsymbol{f}}(\boldsymbol{X}^\epsilon(s), \boldsymbol{X}_s^\epsilon, \mathcal{L}(\boldsymbol{X}^\epsilon(s)), i)] \mathrm{d}s \right|^p \leqslant \\
&\quad 3^{p-1}\left| \int_{t_i/\epsilon}^{t_{i+1}/\epsilon} \left[\boldsymbol{f}\left(\frac{s}{\epsilon}, \boldsymbol{X}^\epsilon(t_i), \boldsymbol{X}_{t_i}^\epsilon, \mathcal{L}(\boldsymbol{X}^\epsilon(t_i)), i\right) - \bar{\boldsymbol{f}}(\boldsymbol{X}^\epsilon(t_i), \boldsymbol{X}_{t_i}^\epsilon, \mathcal{L}(\boldsymbol{X}^\epsilon(t_i)), i) \right] \mathrm{d}s \right|^p + \\
&\quad 3^{p-1}(\sqrt{\epsilon})^{p-1}\left(K\left(\frac{s}{\epsilon}\right) + L \right) \times \\
&\quad \int_{t_i}^{t_{i+1}} \psi(|\boldsymbol{X}^\epsilon(s) - \boldsymbol{X}^\epsilon(t_i)|^p + \|\boldsymbol{X}_s^\epsilon - \boldsymbol{X}_{t_i}^\epsilon\|^p + \mathbb{E}|\boldsymbol{X}^\epsilon(s) - \boldsymbol{X}^\epsilon(t_i)|^p) \mathrm{d}s \leqslant \\
&\quad 3^{p-1}(\sqrt{\epsilon})^p \varphi\left(\frac{1}{\sqrt{\epsilon}}\right) \kappa\left(\sup_{t_0 \leqslant s \leqslant t} |\boldsymbol{X}^\epsilon(s)|^p + \sup_{t_0 \leqslant s \leqslant t} \|\boldsymbol{X}_s^\epsilon\|^p + \sup_{t_0 \leqslant s \leqslant t} \mathbb{E}|\boldsymbol{X}^\epsilon(s)|^p\right) + \\
&\quad 3^{p-1}(\sqrt{\epsilon})^{p-1}\left(K\left(\frac{T}{\epsilon}\right) + L \right) \times \\
&\quad \int_{t_i}^{t_{i+1}} \psi(|\boldsymbol{X}^\epsilon(s) - \boldsymbol{X}^\epsilon(t_i)|^p + \|\boldsymbol{X}_s^\epsilon - \boldsymbol{X}_{t_i}^\epsilon\|^p + \mathbb{E}|\boldsymbol{X}^\epsilon(s) - \boldsymbol{X}^\epsilon(t_i)|^p) \mathrm{d}s.
\end{aligned}
$$

因此

$$
\begin{aligned}
&N^{p-1} \sum_{i=0}^{N-2} \mathbb{E}|\boldsymbol{X}_i|^p = \\
&\quad 3^{p-1}(\sqrt{\epsilon})^{p-1} N^{p-1} \sum_{i=0}^{N-2} \varphi\left(\frac{1}{\sqrt{\epsilon}}\right) \left(\kappa\left(a\mathbb{E}\|\boldsymbol{\xi}\|^p + 3a\mathbb{E}\left(\sup_{t_0 \leqslant s \leqslant t} |\boldsymbol{X}^\epsilon(s)|^p\right)\right)\right) + \\
&\quad 3^{p-1}(\sqrt{\epsilon})^{p-1} N^{p-1} \left(K\left(\frac{T}{\epsilon}\right) + L \right) \sum_{i=0}^{N-2} \psi\left(3\mathbb{E}\left(\sup_{t_0 \leqslant s \leqslant t} |\boldsymbol{X}^\epsilon(s) - \boldsymbol{X}^\epsilon(t_i)|^p\right)\right) \leqslant \\
&\quad 3^{p-1}(\sqrt{\epsilon})^{p} N^{p-1} \sum_{i=0}^{N-2} \varphi\left(\frac{1}{\sqrt{\epsilon}}\right) \left(\kappa\left(a\mathbb{E}\|\boldsymbol{\xi}\|^p + 3a\mathbb{E}\left(\sup_{t_0 \leqslant s \leqslant t} |\boldsymbol{X}^\epsilon(s)|^p\right)\right)\right) +
\end{aligned}
$$

$$3^{p-1}(\sqrt{\epsilon})^{p-1}N^{p-1}\left(K\left(\frac{T}{\epsilon}\right)+L\right)\sum_{i=0}^{N-2}\int_{t_i}^{t_{i+1}}\psi(3C_6((\sqrt{\epsilon})^{p-1}+(\sqrt{\epsilon})^{p/2}+(\sqrt{\epsilon})^{pH}+\sqrt{\epsilon}))\mathrm{d}s\leqslant$$
$$3^{p-1}(\sqrt{\epsilon})^{p}N^{p}\varphi\left(\frac{1}{\sqrt{\epsilon}}\right)(\kappa(a\mathbb{E}\|\boldsymbol{\xi}\|^{p}+3a\mathbb{E}(\sup_{t_0\leqslant s\leqslant t}|\boldsymbol{X}^{\epsilon}(s)|^{p})))+$$
$$3^{p-1}(\sqrt{\epsilon})^{p}N^{p}\left(K\left(\frac{T}{\epsilon}\right)+L\right)\psi(3C_6((\sqrt{\epsilon})^{p-1}+(\sqrt{\epsilon})^{p/2}+(\sqrt{\epsilon})^{pH}+\sqrt{\epsilon})).$$

从而

$$\mathbb{E}\left(\sup_{t_0\leqslant t\leqslant T}\left|\int_{t_0}^{t}\left[\boldsymbol{f}\left(\frac{s}{\epsilon},\boldsymbol{X}^{\epsilon}(s),\boldsymbol{X}_s^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(s)),i\right)-\bar{\boldsymbol{f}}(\boldsymbol{X}^{\epsilon}(s),\boldsymbol{X}_s^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(s)),i)\right]\mathrm{d}s\right|^{p}\right)\leqslant$$
$$CN^{p-1}(\sqrt{\epsilon})^{p}(a+1+a\mathbb{E}\|\boldsymbol{\xi}\|^{p}+3a\mathbb{E}(\sup_{t_0\leqslant s\leqslant t}|\boldsymbol{X}^{\epsilon}(s)|^{p}))+CN^{p}(\sqrt{\epsilon})^{p}\varphi\left(\frac{1}{\sqrt{\epsilon}}\right)+$$
$$CN^{p}(\sqrt{\epsilon})^{p}\left(K\left(\frac{T}{\epsilon}\right)+L\right)\psi((\sqrt{\epsilon})^{p-1}+(\sqrt{\epsilon})^{p/2}+(\sqrt{\epsilon})^{pH}+\sqrt{\epsilon})\leqslant$$
$$C(T+\sqrt{\epsilon})^{p-1}\sqrt{\epsilon}+C(T+\sqrt{\epsilon})^{p}\left(\varphi\left(\frac{1}{\sqrt{\epsilon}}\right)+\psi(C(\sqrt{\epsilon})^{p-1}+(\sqrt{\epsilon})^{p/2}+(\sqrt{\epsilon})^{pH}+\sqrt{\epsilon})\right).$$

由 $\lim_{\epsilon\to 0}\varphi(1/\sqrt{\epsilon})=0,\psi(0)=0$ 可得当 $\epsilon\to 0$ 时,

$$\mathbb{E}\left(\sup_{t_0\leqslant s\leqslant t}\left|\int_{t_0}^{t}\boldsymbol{f}\left(\frac{s}{\epsilon},\boldsymbol{X}^{\epsilon}(s),\boldsymbol{X}_s^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(s)),i\right)-\bar{\boldsymbol{f}}(\boldsymbol{X}^{\epsilon}(s),\boldsymbol{X}_s^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(s)),i)\mathrm{d}s\right|^{p}\right)\to 0.$$

引理 5　若假设 1 和假设 2 成立,则

$$\lim_{\epsilon\to 0}\mathbb{E}\left(\int_{t_0}^{T}\left\|\boldsymbol{g}\left(\frac{s}{\epsilon},\boldsymbol{X}^{\epsilon}(s),\boldsymbol{X}_s^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(s)),i\right)-\bar{\boldsymbol{g}}(\boldsymbol{X}^{\epsilon}(s),\boldsymbol{X}_s^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(s)),i)\right\|^{p}\mathrm{d}s\right)=0.$$

证明　与引理 4 类似,设 $t_i=t_0+i\sqrt{\varepsilon}$, 其中 $0\leqslant i\leqslant N-1$, $0<T-t_{N-1}\leqslant\sqrt{\varepsilon}$, 且 $t_N=T$.首先定义 Y_i 为

$$Y_i:=\int_{t_i}^{t_{i+1}}\left\|\boldsymbol{g}\left(\frac{s}{\epsilon},\boldsymbol{X}^{\epsilon}(s),\boldsymbol{X}_s^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(s)),i\right)-\bar{\boldsymbol{g}}(\boldsymbol{X}^{\epsilon}(s),\boldsymbol{X}_s^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(s)),i)\right\|^{p}\mathrm{d}s,$$

则有

$$\mathbb{E}Y_i\leqslant 3^{p-1}\,\mathbb{E}\left(\int_{t_i}^{t_{i+1}}\left\|\boldsymbol{g}\left(\frac{s}{\epsilon},\boldsymbol{X}^{\epsilon}(s),\boldsymbol{X}_s^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(s)),i\right)-\boldsymbol{g}\left(\frac{s}{\epsilon},\boldsymbol{X}^{\epsilon}(t_i),\boldsymbol{X}_{t_i}^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(t_i)),i\right)\right\|^{p}\mathrm{d}s\right)+$$
$$3^{p-1}\,\mathbb{E}\left(\int_{t_i}^{t_{i+1}}\left\|\boldsymbol{g}\left(\frac{s}{\epsilon},\boldsymbol{X}^{\epsilon}(t_i),\boldsymbol{X}_{t_i}^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(t_i)),i\right)-\bar{\boldsymbol{g}}(\boldsymbol{X}^{\epsilon}(t_i),\boldsymbol{X}_{t_i}^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(t_i)),i)\right\|^{p}\mathrm{d}s\right)+$$
$$3^{p-1}\,\mathbb{E}\left(\int_{t_i}^{t_{i+1}}\|\bar{\boldsymbol{g}}(\boldsymbol{X}^{\epsilon}(t_i),\boldsymbol{X}_{t_i}^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(t_i)),i)-\bar{\boldsymbol{g}}(\boldsymbol{X}^{\epsilon}(s),\boldsymbol{X}_s^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(s)),i)\|^{p}\mathrm{d}s\right),$$

由假设 2 得

$$\mathbb{E}Y_i\leqslant 3^{p-1}\left(K\left(\frac{T}{\epsilon}\right)+L\right)\int_{t_i}^{t_{i+1}}\psi(|\boldsymbol{X}^{\epsilon}(s)-\boldsymbol{X}^{\epsilon}(t_i)|^{p}+\|\boldsymbol{X}_s^{\epsilon}-\boldsymbol{X}_{t_i}^{\epsilon}\|^{p}+\mathbb{E}|\boldsymbol{X}^{\epsilon}(s)-\boldsymbol{X}^{\epsilon}(t_i)|^{p})\mathrm{d}s+$$
$$3^{p-1}\sqrt{\epsilon}\varphi\left(\frac{1}{\sqrt{\epsilon}}\right)\kappa(a\mathbb{E}\|\boldsymbol{\xi}\|^{p}+3a\mathbb{E}(\sup_{t_0\leqslant s\leqslant t}|\boldsymbol{X}^{\epsilon}(s)|^{p}))\leqslant$$
$$3^{p-1}C\sqrt{\epsilon}\left(K\left(\frac{T}{\epsilon}\right)+L\right)\psi(C_6((\sqrt{\epsilon})^{p-1}+(\sqrt{\epsilon})^{p/2}+(\sqrt{\epsilon})^{pH}+\sqrt{\epsilon}))+$$
$$3^{p-1}\sqrt{\epsilon}\varphi\left(\frac{1}{\sqrt{\epsilon}}\right)\kappa(a\mathbb{E}\|\boldsymbol{\xi}\|^{p}+3a\mathbb{E}(\sup_{t_0\leqslant s\leqslant t}|\boldsymbol{X}^{\epsilon}(s)|^{p})).$$

因此

$$\mathbb{E}\left(\int_{t_0}^{T}\left\|\boldsymbol{g}\left(\frac{s}{\epsilon},\boldsymbol{X}^{\epsilon}(s),\boldsymbol{X}_s^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(s)),i\right)-\bar{\boldsymbol{g}}(\boldsymbol{X}^{\epsilon}(s),\boldsymbol{X}_s^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(s)),i)\right\|^{p}\mathrm{d}s\right)=\sum_{i=1}^{N-1}\mathbb{E}Y_i=$$
$$CN3^{p-1}\sqrt{\epsilon}\left(K\left(\frac{T}{\epsilon}\right)+L\right)\psi(C_6((\sqrt{\epsilon})^{p-1}+(\sqrt{\epsilon})^{p/2}+(\sqrt{\epsilon})^{pH}+\sqrt{\epsilon}))+$$

$$3^{p-1}N\sqrt{\epsilon}\varphi\left(\frac{1}{\sqrt{\epsilon}}\right)\kappa\left(a\mathbb{E}\|\boldsymbol{\xi}\|^{p}+3a\mathbb{E}\left(\sup_{t_0\leqslant s\leqslant t}|\boldsymbol{X}^{\epsilon}(s)|^{p}\right)\right).$$

当 $\epsilon\to 0$ 时,

$$\mathbb{E}\left(\int_{t_0}^{t}\left\|\boldsymbol{g}\left(\frac{s}{\epsilon},\boldsymbol{X}^{\epsilon}(s),\boldsymbol{X}_s^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(s)),i\right)-\bar{\boldsymbol{g}}(\boldsymbol{X}^{\epsilon}(s),\boldsymbol{X}_s^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(s)),i)\right\|^{p}\mathrm{d}s\right)\to 0.$$

引理 6 若假设 1 和假设 2 成立,则

$$\lim_{\epsilon\to 0}\mathbb{E}\left(\sup_{t_0\leqslant s\leqslant T}\int_{t_0}^{t}\left\|\boldsymbol{m}\left(\frac{s}{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(s))\right)-\bar{\boldsymbol{m}}(\mathcal{L}(\boldsymbol{X}^{\epsilon}(s)))\right\|^{p}\mathrm{d}s\right)=0,$$

$$\lim_{\epsilon\to 0}\mathbb{E}\left(\int_{t_0}^{t}\int_{|z|<c}\left|\boldsymbol{h}\left(\frac{s}{\epsilon},\boldsymbol{X}^{\epsilon}(s-),\boldsymbol{X}_s^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(s)),z,i\right)-\bar{\boldsymbol{h}}(\boldsymbol{X}^{\epsilon}(s-),\boldsymbol{X}_s^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(s)),z,i)\right|^{p}\nu(\mathrm{d}z)\mathrm{d}s\right)=0.$$

引理 6 的证明与引理 4 类似,这里我们省略了证明.

定理 2(平均原理) 若假设 1 和假设 2 成立, $t_0\leqslant t\leqslant T$, 则

$$\lim_{\epsilon\to 0}\mathbb{E}\left(\sup_{t_0\leqslant s\leqslant t}|\boldsymbol{X}^{\epsilon}(s)-\bar{\boldsymbol{X}}(s)|^{p}\right)=0.$$

证明

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t_0\leqslant s\leqslant t}|\boldsymbol{X}^{\epsilon}(s)-\bar{\boldsymbol{X}}(s)|^{p}\right)\leqslant\\
&\quad 4^{p-1}\mathbb{E}\left(\sup_{t_0\leqslant s\leqslant t}\left|\int_{t_0}^{s}\left[\boldsymbol{f}\left(\frac{r}{\epsilon},\boldsymbol{X}^{\epsilon}(r),\boldsymbol{X}_r^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(r)),i\right)-\bar{\boldsymbol{f}}(\bar{\boldsymbol{X}}(r),\bar{\boldsymbol{X}}_r,\mathcal{L}(\bar{\boldsymbol{X}}(r)),i)\right]\mathrm{d}r\right|^{p}\right)+\\
&\quad 4^{p-1}\mathbb{E}\left(\sup_{t_0\leqslant s\leqslant t}\left|\int_{t_0}^{s}\left[\boldsymbol{g}\left(\frac{r}{\epsilon},\boldsymbol{X}^{\epsilon}(r),\boldsymbol{X}_r^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(r)),i\right)-\bar{\boldsymbol{g}}(\bar{\boldsymbol{X}}(r),\bar{\boldsymbol{X}}_r,\mathcal{L}(\bar{\boldsymbol{X}}(r)),i)\right]\mathrm{d}\boldsymbol{B}(r)\right|^{p}\right)+\\
&\quad 4^{p-1}\mathbb{E}\left(\sup_{t_0\leqslant s\leqslant t}\left|\int_{t_0}^{s}\left[\boldsymbol{m}\left(\frac{r}{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(r))\right)-\bar{\boldsymbol{m}}(\mathcal{L}(\bar{\boldsymbol{X}}(r)))\right]\mathrm{d}\boldsymbol{B}^{H}(r)\right|^{p}\right)+\\
&\quad 4^{p-1}\mathbb{E}\left(\sup_{t_0\leqslant s\leqslant t}\left|\int_{t_0}^{s}\int_{|z|<c}\left[\boldsymbol{h}\left(\frac{r}{\epsilon},\boldsymbol{X}^{\epsilon}(r-),\boldsymbol{X}_r^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(r)),z,i\right)-\right.\right.\right.\\
&\quad \left.\left.\left.\bar{\boldsymbol{h}}(\bar{\boldsymbol{X}}(r-),\bar{\boldsymbol{X}}_r,\mathcal{L}(\bar{\boldsymbol{X}}(r)),z,i)\right]\tilde{N}(\mathrm{d}r,\mathrm{d}z)\right|^{p}\right):=\\
&\quad 4^{p-1}(V_1+V_2+V_3+V_4).
\end{aligned}$$

对于 V_1, 由 Hölder 不等式和注 1 得

$$\begin{aligned}
V_1\leqslant{}&2^{p-1}\mathbb{E}\left(\sup_{t_0\leqslant s\leqslant t}\left|\int_{t_0}^{s}\left[\boldsymbol{f}\left(\frac{r}{\epsilon},\boldsymbol{X}^{\epsilon}(r),\boldsymbol{X}_r^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(r)),i\right)-\bar{\boldsymbol{f}}(\boldsymbol{X}^{\epsilon}(r),\boldsymbol{X}_r^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(r)),i)\right]\mathrm{d}r\right|^{p}\right)+\\
&(2T)^{p-1}\mathbb{E}\int_{t_0}^{t}|\bar{\boldsymbol{f}}(\boldsymbol{X}^{\epsilon}(r),\boldsymbol{X}_r^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(r)),i)-\bar{\boldsymbol{f}}(\bar{\boldsymbol{X}}(r),\bar{\boldsymbol{X}}_r,\mathcal{L}(\bar{\boldsymbol{X}}(r)),i)|^{p}\mathrm{d}r\leqslant\\
&2^{p-1}\mathbb{E}\left(\sup_{t_0\leqslant s\leqslant t}\left|\int_{t_0}^{s}\left[\boldsymbol{f}\left(\frac{r}{\epsilon},\boldsymbol{X}^{\epsilon}(r),\boldsymbol{X}_r^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(r)),i\right)-\bar{\boldsymbol{f}}(\boldsymbol{X}^{\epsilon}(r),\boldsymbol{X}_r^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(r)),i)\right]\mathrm{d}r\right|^{p}\right)+\\
&(2T)^{p-1}L\int_{t_0}^{t}\psi\left(3\mathbb{E}\left(\sup_{t_0\leqslant u\leqslant r}|\boldsymbol{X}^{\epsilon}(u)-\bar{\boldsymbol{X}}(u)|^{p}\right)\right)\mathrm{d}r.
\end{aligned}$$

对于 V_2, 由 BDG 不等式、Hölder 不等式和注 1 得

$$\begin{aligned}
V_2\leqslant{}&C_p\left(\mathbb{E}\int_{t_0}^{t}\left\|\boldsymbol{g}\left(\frac{r}{\epsilon},\boldsymbol{X}^{\epsilon}(r),\boldsymbol{X}_r^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(r)),i\right)-\bar{\boldsymbol{g}}(\bar{\boldsymbol{X}}(r),\bar{\boldsymbol{X}}_r,\mathcal{L}(\bar{\boldsymbol{X}}(r)),i)\right\|^{2}\mathrm{d}r\right)^{p/2}\leqslant\\
&C_p2^{p-1}T^{(p-1)/2}\mathbb{E}\int_{t_0}^{t}\left\|\boldsymbol{g}\left(\frac{r}{\epsilon},\boldsymbol{X}^{\epsilon}(r),\boldsymbol{X}_r^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(r)),i\right)-\bar{\boldsymbol{g}}(\boldsymbol{X}^{\epsilon}(r),\boldsymbol{X}_r^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(r)),i)\right\|^{p}\mathrm{d}r+\\
&C_p2^{p-1}T^{(p-1)/2}\mathbb{E}\int_{t_0}^{t}\|\bar{\boldsymbol{g}}(\boldsymbol{X}^{\epsilon}(r),\boldsymbol{X}_r^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(r)),i)-\bar{\boldsymbol{g}}(\bar{\boldsymbol{X}}(r),\bar{\boldsymbol{X}}_r,\mathcal{L}(\bar{\boldsymbol{X}}(r)),i)\|^{p}\mathrm{d}r\leqslant\\
&C_p2^{p-1}T^{(p-1)/2}\mathbb{E}\int_{t_0}^{t}\left\|\boldsymbol{g}\left(\frac{r}{\epsilon},\boldsymbol{X}^{\epsilon}(r),\boldsymbol{X}_r^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(r)),i\right)-\bar{\boldsymbol{g}}(\boldsymbol{X}^{\epsilon}(r),\boldsymbol{X}_r^{\epsilon},\mathcal{L}(\boldsymbol{X}^{\epsilon}(r)),i)\right\|^{p}\mathrm{d}r+
\end{aligned}$$

$$C_p 2^{p-1} T^{(p-1)/2} L \int_{t_0}^{t} \psi(3\mathbb{E}(\sup_{t_0 \leqslant u \leqslant r} | \boldsymbol{X}^{\epsilon}(u) - \bar{\boldsymbol{X}}(u) |^p)) \mathrm{d}r.$$

对于 V_3, 由文献[4]的式(3.5)和注 1 得

$$V_3 \leqslant 2^{p-1} C_{\lambda,p,H} T^{pH-1} \mathbb{E} \int_{t_0}^{t} \left\| \boldsymbol{m}\left(\frac{r}{\epsilon}, \mathcal{L}(\boldsymbol{X}^{\epsilon}(r))\right) - \bar{\boldsymbol{m}}(\mathcal{L}(\boldsymbol{X}^{\epsilon}(r))) \right\|^p \mathrm{d}r +$$

$$2^{p-1} C_{\lambda,p,H} T^{pH-1} \mathbb{E} \int_{t_0}^{t} \| \bar{\boldsymbol{m}}(\mathcal{L}(\boldsymbol{X}^{\epsilon}(r))) - \bar{\boldsymbol{m}}(\mathcal{L}(\bar{\boldsymbol{X}}(r))) \|^p \mathrm{d}r \leqslant$$

$$2^{p-1} C_{\lambda,p,H} T^{pH-1} \mathbb{E} \int_{t_0}^{t} \left\| \boldsymbol{m}\left(\frac{r}{\epsilon}, \mathcal{L}(\boldsymbol{X}^{\epsilon}(r))\right) - \bar{\boldsymbol{m}}(\mathcal{L}(\boldsymbol{X}^{\epsilon}(r))) \right\|^p \mathrm{d}r +$$

$$2^{p-1} C_{\lambda,p,H} T^{pH-1} L \int_{t_0}^{t} \psi(\mathbb{E}(\sup_{t_0 \leqslant u \leqslant r} | \boldsymbol{X}^{\epsilon}(u) - \bar{\boldsymbol{X}}(u) |^p)) \mathrm{d}r.$$

对于 V_4, 由 Kunita 第一不等式、Hölder 不等式和注 1 得

$$V_4 \leqslant C_7 \mathbb{E} \int_{t_0}^{t} \int_{|z|<c} \left| \boldsymbol{h}\left(\frac{r}{\epsilon}, \boldsymbol{X}^{\epsilon}(r-), \boldsymbol{X}_r^{\epsilon}, \mathcal{L}(\boldsymbol{X}^{\epsilon}(r)), z, i\right) - \bar{\boldsymbol{h}}(\boldsymbol{X}^{\epsilon}(r-), \boldsymbol{X}_r^{\epsilon}, \mathcal{L}(\boldsymbol{X}^{\epsilon}(r)), z, i) \right|^p \nu(\mathrm{d}z) \mathrm{d}r \leqslant$$

$$2^{p-1} C_7 \int_{t_0}^{t} \int_{|z|<c} \left| \boldsymbol{h}\left(\frac{r}{\epsilon}, \boldsymbol{X}^{\epsilon}(r-), \boldsymbol{X}_r^{\epsilon}, \mathcal{L}(\boldsymbol{X}^{\epsilon}(r)), z, i\right) - \bar{\boldsymbol{h}}(\boldsymbol{X}^{\epsilon}(r-), \boldsymbol{X}_r^{\epsilon}, \mathcal{L}(\boldsymbol{X}^{\epsilon}(r)), z, i) \right|^p \nu(\mathrm{d}z) \mathrm{d}r +$$

$$2^{p-1} C_7 \int_{t_0}^{t} \int_{|z|<c} | \bar{\boldsymbol{h}}(\boldsymbol{X}^{\epsilon}(r-), \boldsymbol{X}_r^{\epsilon}, \mathcal{L}(\boldsymbol{X}^{\epsilon}(r)), z, i) - \bar{\boldsymbol{h}}(\bar{\boldsymbol{X}}(r-), \bar{\boldsymbol{X}}_r, \mathcal{L}(\bar{\boldsymbol{X}}(r)), z, i) |^p \nu(\mathrm{d}z) \mathrm{d}r \leqslant$$

$$2^{p-1} C_7 \int_{t_0}^{t} \int_{|z|<c} \left| \boldsymbol{h}\left(\frac{r}{\epsilon}, \boldsymbol{X}^{\epsilon}(r-), \boldsymbol{X}_r^{\epsilon}, \mathcal{L}(\boldsymbol{X}^{\epsilon}(r)), z, i\right) - \bar{\boldsymbol{h}}(\boldsymbol{X}^{\epsilon}(r-), \boldsymbol{X}_r^{\epsilon}, \mathcal{L}(\boldsymbol{X}^{\epsilon}(r-)), z, i) \right|^p \nu(\mathrm{d}z) \mathrm{d}r +$$

$$2^{p-1} C_7 L \int_{t_0}^{t} \psi(3\mathbb{E}(\sup_{t_0 \leqslant u \leqslant r} | \boldsymbol{X}^{\epsilon}(u) - \bar{\boldsymbol{X}}(u) |^p)) \mathrm{d}r,$$

其中, $C_7 = D_p(VT)^{p/2-1} + D_p$.

综上所述

$$\lim_{\epsilon \to 0} \mathbb{E}(\sup_{t_0 \leqslant s \leqslant t} | \boldsymbol{X}^{\epsilon}(t) - \bar{\boldsymbol{X}}(t) |^p) \leqslant$$

$$8^{p-1} \mathbb{E}\left(\sup_{t_0 \leqslant s \leqslant t} \left| \int_{t_0}^{s} \left[\boldsymbol{f}\left(\frac{r}{\epsilon}, \boldsymbol{X}^{\epsilon}(r), \boldsymbol{X}_r^{\epsilon}, \mathcal{L}(\boldsymbol{X}^{\epsilon}(r)), i\right) - \bar{\boldsymbol{f}}(\boldsymbol{X}^{\epsilon}(r), \boldsymbol{X}_r^{\epsilon}, \mathcal{L}(\boldsymbol{X}^{\epsilon}(r)), i) \right] \mathrm{d}r \right|^p \right) +$$

$$8^{p-1} C_p T^{(p-1)/2} \mathbb{E} \int_{t_0}^{t} \left\| \boldsymbol{g}\left(\frac{r}{\epsilon}, \boldsymbol{X}^{\epsilon}(r), \boldsymbol{X}_r^{\epsilon}, \mathcal{L}(\boldsymbol{X}^{\epsilon}(r)), i\right) - \bar{\boldsymbol{g}}(\boldsymbol{X}^{\epsilon}(r), \boldsymbol{X}_r^{\epsilon}, \mathcal{L}(\boldsymbol{X}^{\epsilon}(r)), i) \right\|^p \mathrm{d}r +$$

$$8^{p-1} C_{\lambda,p,H} T^{pH-1} \mathbb{E} \int_{t_0}^{t} \left\| \boldsymbol{m}\left(\frac{r}{\epsilon}, \mathcal{L}(\boldsymbol{X}^{\epsilon}(r))\right) - \bar{\boldsymbol{m}}(\mathcal{L}(\boldsymbol{X}^{\epsilon}(r))) \right\|^p \mathrm{d}r +$$

$$8^{p-1} C_7 \mathbb{E}\left(\int_{t_0}^{t} \int_{|z|<c} \left| \boldsymbol{h}\left(\frac{r}{\epsilon}, \boldsymbol{X}^{\epsilon}(r-), \boldsymbol{X}_r^{\epsilon}, \mathcal{L}(\boldsymbol{X}^{\epsilon}(r)), z, i\right) - \right.\right.$$

$$\left.\left. \bar{\boldsymbol{h}}(\boldsymbol{X}^{\epsilon}(r-), \boldsymbol{X}_r^{\epsilon}, \mathcal{L}(\boldsymbol{X}^{\epsilon}(r)), z, i) \right|^p \nu(\mathrm{d}z) \mathrm{d}r \right) +$$

$$C_8 \int_{t_0}^{t} \psi(3\mathbb{E}(\sup_{t_0 \leqslant u \leqslant r} | \boldsymbol{X}^{\epsilon}(u) - \bar{\boldsymbol{X}}(u) |^p)) \mathrm{d}r,$$

其中

$$C_8 = 8^{p-1} L(T^{p-1} + C_p T^{(p-1)/2} + C_{\lambda,p,H} T^{pH-1} + C_7).$$

由引理 4—6 得,对于任意 $\epsilon_1 > 0$, 有

$$\lim_{\epsilon \to 0} \mathbb{E}(\sup_{t_0 \leqslant s \leqslant t} | \boldsymbol{X}^{\epsilon}(s) - \bar{\boldsymbol{X}}(s) |) \leqslant \epsilon_1 + C_8 \int_{t_0}^{t} \psi(3 \lim_{\epsilon \to 0} \mathbb{E}(\sup_{t_0 \leqslant u \leqslant r} | \boldsymbol{X}^{\epsilon}(u) - \bar{\boldsymbol{X}}(u) |^p)) \mathrm{d}r.$$

因此,由引理 1 可得

$$\lim_{\epsilon \to 0} \mathbb{E}(\sup_{t_0 \leqslant s \leqslant t} | \boldsymbol{X}^{\epsilon}(s) - \bar{\boldsymbol{X}}(s) |) \leqslant \frac{1}{3} G^{-1}(G(3\epsilon_1) + 2C_8 T),$$

$$G^{-1}(G(3\epsilon_1)+2C_8T)\in \mathrm{Dom}(G^{-1}),\ \lim_{\epsilon_1\to 0}G(3\epsilon_1)=-\infty,\ \mathrm{Dom}(G^{-1})=(-\infty,G(\infty)).$$

令 $\epsilon_1 \to 0$ 可得

$$\lim_{\epsilon\to 0}\mathbb{E}\left(\sup_{t_0\leqslant s\leqslant t}|\boldsymbol{X}^{\epsilon}(s)-\bar{\boldsymbol{X}}(s)|\right)=0.$$

注 3 由 Chebyshev-Markov 不等式和定理 2,对于任意给定的数 $\delta>0$, 我们有

$$\lim_{\epsilon\to 0}\mathbb{P}\left(\sup_{t_0\leqslant t\leqslant T}|\boldsymbol{X}^{\epsilon}(t)-\bar{\boldsymbol{X}}(t)|>\delta\right)\leqslant \frac{1}{\delta^p}\lim_{\epsilon\to 0}\mathbb{E}\left(\sup_{t_0\leqslant t\leqslant T}|\boldsymbol{X}^{\epsilon}(t)-\bar{\boldsymbol{X}}(t)|^p\right)=0,$$

这表明解 $\boldsymbol{X}^{\epsilon}(t)$ 依概率收敛到平均解 $\bar{\boldsymbol{X}}(t)$.

4 结 论

本文研究了由 Hurst 指数 $H>1/2$ 的分数 Brown 运动和 Lévy 过程同时驱动的带 Markov 切换和随机比例时间的分布依赖的随机泛函微分方程.在系数满足非 Lipschitz 条件与多项式增长假设下,采用 Carathéodory 逼近构造逼近解序列,借助 Gronwall 不等式及 Bihari 不等式证明该序列的收敛性,进而证明原方程解的存在唯一性.在此基础上,引入平均条件,建立了该方程的平均原理,证明了分布依赖随机泛函微分方程的解被其平均化随机泛函微分方程的解在 p-阶矩意义下逼近.

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