伴有无穷时滞效应的不可压缩非 Newton 微极流方程组的确定模估计*

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摘要： 本文在二维有界区域上估计一类伴有无穷时滞效应的不可压缩非 Newton 微极流方程组的确定模个数.结果表明, 伴有无穷时滞效应的非 Newton 微极流方程组任意弱解的渐近行为可以完全由其前有限个 Fourier 模的渐近行为所决定.

关　键　词： 不可压缩非 Newton 微极流；　无穷时滞；　确定模；　渐近行为

中图分类号： O175.29　　**文献标志码：** A　　DOI：10.21656/1000-0887.460032

Estimate of Determining Modes for Incompressible Non-Newtonian Micropolar Fluid Equations With Infinite Delay

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Abstract: The number of determining modes was estimated for the incompressible non-Newtonian micropolar fluid with infinite delay in 2D bounded domains. The results show that, the asymptotic behavior of any weak solution to the non-Newtonian micropolar fluid equations with infinite delay depends completely on the asymptotic behavior of their 1st finite number of Fourier modes.

Key words: incompressible non-Newtonian micropolar fluid; infinite delay; determining mode; asymptotic behavior

0　引　　言

时滞微分方程是应用数学的一个重要研究领域, 由于物理原因或非即时传输, 使得时滞现象随处可见, 在许多情况下其影响是不容忽视的.方程中包含某种时滞项的模型得到了广泛的研究, 如文献[1-7].

* **收稿日期：** 2025-02-25; **修订日期：** 2025-04-17
基金项目： 国家自然科学基金(12071484)
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引用格式： 袁艳杰, 王永威. 伴有无穷时滞效应的不可压缩非 Newton 微极流方程组的确定模估计[J]. 应用数学和力学, 2026, **47**(4): 516-528.

本文将在二维有界区域 $\Omega \subset \mathbb{R}^2$ 上研究一类含无穷时滞项的流体动力学模型.在流体动力学理论中,若流体的应力张量与应变速率张量呈线性关系,则称该流体为 Newton 流,否则称为非 Newton 流[8].例如,熔融的塑料、合成的纤维、油漆等均属于非 Newton 流.由于非 Newton 流在现实生活中的广泛应用,受到了许多学者的研究与关注并取得了丰富的研究成果[9-12].当考虑流体流动时的旋转速度(角速度),可以用微极流方程来刻画.微极流方程是 Navier-Stokes 方程的重要推广,可以很好地表征一些经典的 Navier-Stokes 方程无法描述的伴有非均匀应力张量的流体的运动(如动物血液、液晶和稀释水溶性聚合物溶液的运动等)[13].更多相关研究成果可见文献[14-20].

本文考虑一类伴有无穷时滞效应的不可压缩非 Newton 微极流模型,其方程组可表示如下

$$\begin{cases}\dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nabla\cdot\boldsymbol{T}(\boldsymbol{e}(\boldsymbol{u})) + \nabla \boldsymbol{p} = 2\nu_r \nabla\times\boldsymbol{\omega} + \boldsymbol{f}(t,x) + \boldsymbol{g}(t,\boldsymbol{u}_t), & x\in\Omega,\ t>\tau,\\ \nabla\cdot\boldsymbol{u} = 0, & x\in\Omega,\ t>\tau,\\ \dfrac{\partial \boldsymbol{\omega}}{\partial t} - \nu\Delta\boldsymbol{\omega} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} + 4\nu_r\boldsymbol{\omega} = 2\nu_r\nabla\times\boldsymbol{u} + \tilde{\boldsymbol{f}}(t,x) + \tilde{\boldsymbol{g}}(t,\boldsymbol{\omega}_t), & x\in\Omega,\ t>\tau,\end{cases} \tag{1}$$

其中 $\boldsymbol{u}:=\boldsymbol{u}(t,x)$ 表示速度,$\boldsymbol{\omega}:=\boldsymbol{\omega}(t,x)$ 表示角速度,$\boldsymbol{p}:=\boldsymbol{p}(t,x)$ 为流体的压力,$\boldsymbol{f}:=\boldsymbol{f}(t,x)$ 和 $\tilde{\boldsymbol{f}}(t,x)$ 分别表示外力和力矩,$\boldsymbol{g}:=\boldsymbol{g}(t,\boldsymbol{u}_t)$ 和 $\tilde{\boldsymbol{g}}(t,\boldsymbol{\omega}_t)$ 表示含有时滞特征的附加外力,且

$$\boldsymbol{u}_t := \boldsymbol{u}_t(\cdot) = \boldsymbol{u}(t+\cdot),\ \boldsymbol{\omega}_t := \boldsymbol{\omega}_t(\cdot) = \boldsymbol{\omega}(t+\cdot).$$

$\boldsymbol{T}(\boldsymbol{e}(\boldsymbol{u})) := (2\mu_0(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{-\alpha/2}\boldsymbol{e}(\boldsymbol{u}) - 2\mu_1\Delta\boldsymbol{e}(\boldsymbol{u})),\boldsymbol{e}(\boldsymbol{u}) = (e_{jk}(\boldsymbol{u}))_{2\times 2}$ 是应变速率张量,

$$|\boldsymbol{e}(\boldsymbol{u})|^2 = \sum_{j,k=1}^{2} e_{jk}^2(\boldsymbol{u}),\ e_{jk}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right), \quad j,k=1,2.$$

ϵ,μ_0,μ_1 和 $\alpha\in(0,1)$ 为正常数,ν 和 ν_r 为正的黏性系数.另外,

$$\nabla\cdot\boldsymbol{u} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2},\ \nabla\times\boldsymbol{\omega} = \left(\frac{\partial\boldsymbol{\omega}}{\partial x_2}, -\frac{\partial\boldsymbol{\omega}}{\partial x_1}\right).$$

对于非线性发展方程,其研究的一个基本、核心问题是方程解的存在性及其动力学行为[21-33].鉴于不可压缩 Newton 微极流方程组有着重要的理论价值和现实指导意义,其受到了相关学者的广泛关注与研究.例如,在二维区域上,de Araújo 等[34]证明了一类非 Newton 微极流方程组拉回吸引子的存在性及其关于黏性系数 ν_r 的上半连续性,Ai 和 Tan 研究了一类非 Newton 微极流方程组全局吸引子和指数吸引子的存在性[35-36].最近,Zhao 等[37]得到了非 Newton 微极流方程组统计解的存在性及其退化正则性.随后,Chen 等[38]在三维有界区域上证明了该非 Newton 微极流方程组轨道统计解的存在性及其退化正则性.更多相关结果见文献[39-40].

研究方程组的动力学行为(吸引子)固然重要,而进一步揭示吸引子的内部结构特性也有着同等重要的地位.确定模、自由度与确定结点描述的是系统的渐近行为可以被某些有限量所确定,其概念是由数学家 Foias 等[41-42]提出,并阐明了不可压缩 Navier-Stokes 方程组具有有限的确定模与确定结点个数.随后一些相关的研究工作相继出现[43-54].粗略地说,演化系统的全局吸引子是相空间中的紧集,演化系统的解可以在相空间中展开成 Fourier 级数(无穷级数),该 Fourier 级数的前有限(比如说 m)项的和也是相空间中的函数.如果由吸引子中任意两个解的前 m 项的和具有相同的渐近行为可以推出这两个解就具有相同的渐近行为,则称前 m 个 Fourier 模为这个吸引子的确定模,自然数 m 称为该吸引子的自由度.由此可见,演化系统的确定模可以深刻描述系统的动力学行为,自由度则刻画了吸引子的一个本质特征.确定结点则是指:如果吸引子内的两个解在有限个物理空间中的点具有相同的渐近行为,则这两个解在整个空间区域具有相同的渐近行为.因此,关于确定结点的个数从一定程度上刻画了吸引子的物理空间自由度.

本文的主要目的是证明伴有无穷时滞效应的不可压缩非 Newton 微极流方程组具有有限的确定模个数.具体地,我们将证明若系统的任意两个解的若干 Fourier 模具有相同的渐近性质,则其余无穷多个模也具有相同的渐近性质.为此,考虑如下初边值条件:

$$\begin{cases} \boldsymbol{u}_\tau(s,x)=\boldsymbol{\phi}(s,x),\ \boldsymbol{\omega}_\tau(s,x)=\boldsymbol{\psi}(s,x), & s\in(-\infty,0],\ \tau\in\mathbb{R},\ x\in\Omega, \\ \boldsymbol{u}(t,x)\mid_{\partial\Omega}=\boldsymbol{0},\ \boldsymbol{\omega}(t,x)\mid_{\partial\Omega}=\boldsymbol{0},\ 2\mu_1\dfrac{\partial e_{jk}}{\partial x_m}r_k r_m\mid_{\partial\Omega}=0, & j,k,m=1,2,\ \forall t\geqslant\tau, \end{cases} \tag{2}$$

其中 $\boldsymbol{r}=(r_1,r_2)$ 为垂直于边界的单位外法向量.

本文的主要思想来源于文献[42,48].文献[42]和[48]分别估计了 Navier-Stokes 方程组和微极流方程组的确定模个数.与 Navier-Stokes 方程组和微极流方程组相比，无穷时滞不可压缩非 Newton 微极流方程组包含非线性项 $\nabla\cdot(\mu_0(\epsilon+|\boldsymbol{e}(\boldsymbol{u})|^2)^{-\alpha/2}\boldsymbol{e}(\boldsymbol{u}))$ 和高阶项 $\nabla\cdot(-2\mu_1\Delta\boldsymbol{e}(\boldsymbol{u}))$.鉴于此,如何构建解满足微分不等式、如何证明解具有仿压挤性质等并不是容易的问题,这也是证明系统具有有限的确定模个数的关键.另外,无穷时滞项的出现，使得估计问题(1)、(2)的确定模更具困难性.确切地说,考虑到时滞效应可以无界增长,相空间的选择是微妙的(见文献[50]).为此本文引入了空间 $\mathcal{C}_\gamma$，并给出了关于时滞项的假设条件.

本文剩余部分安排如下:第 1 节，做一些准备工作;第 2 节,估计问题(1)、(2)的确定模个数.

1 预 备 知 识

在这一节中，我们主要做一些准备工作，包括引入一些记号、定义及相关的一些结果和预备知识.

在本文中，$\mathbb{N}$,$\mathbb{R}$ 和 $\mathbb{R}_+$ 分别表示自然数集、实数集和正实数集.$L^p(\Omega)$ 和 $W^{m,p}(\Omega)$ 分别表示 Lebesgue 空间和 Sobolev 空间,相应的范数分别记为 $\|\cdot\|_{L^p}$ 和 $\|\cdot\|_{m,p}$.特别地,$H^m(\Omega)\coloneqq W^{m,2}(\Omega)$.

$\mathcal{V}\coloneqq\{\boldsymbol{\varphi}\in\mathcal{C}_0^\infty(\Omega)\times\mathcal{C}_0^\infty(\Omega)\mid\boldsymbol{\varphi}=(\varphi_1,\varphi_2),\ \nabla\cdot\boldsymbol{\varphi}=0\}$;

$H\coloneqq\mathcal{V}$ 在 $L^2(\Omega)\times L^2(\Omega)$ 范数下的完备化空间，范数为 $\|\cdot\|_H$,对偶空间记为 H^*;

$V\coloneqq\mathcal{V}$ 在 $H^2(\Omega)\times H^2(\Omega)$ 范数下的完备化空间，范数为 $\|\cdot\|_V$,对偶空间记为 V^*;

$\hat{H}\coloneqq H\times L^2(\Omega)$,范数为 $\|\cdot\|_{\hat{H}}$,对偶空间记为 $\hat{H}^*$;

$\hat{V}\coloneqq V\times H_0^1(\Omega)$，范数为 $\|\cdot\|_{\hat{V}}$,对偶空间记为 $\hat{V}^*$.

令$(\cdot,\cdot)$和$\langle\cdot,\cdot\rangle$分别表示空间的内积和对偶积.在不引起混淆的情况下，范数 $\|\cdot\|_{L^2}$, $\|\cdot\|_H$ 和 $\|\cdot\|_{\hat{H}}$ 均简记为 $\|\cdot\|$.另外:

$L^p(I;\hat{H})\coloneqq$取值于 $\hat{H}$ 空间中，在区间 I 上 p 次可积的函数全体，其范数为

$$\|\boldsymbol{\varphi}\|_{L^p(I;\hat{H})}=\left(\int_I\|\boldsymbol{\varphi}(t)\|^p\mathrm{d}t\right)^{1/p},\qquad\forall\boldsymbol{\varphi}\in L^p(I;\hat{H}),\ 1\leqslant p<\infty;$$

$\mathcal{C}(I;\hat{H})\coloneqq$取值于 $\hat{H}$ 空间中，在区间 I 上连续的函数全体，其范数为

$$\|\boldsymbol{\varphi}\|_{\mathcal{C}(I;\hat{H})}=\max_{t\in I}\|\boldsymbol{\varphi}(t,x)\|_X;$$

$L^2_{\mathrm{loc}}(I;\hat{H})\coloneqq$取值于 $\hat{H}$空间中，在区间 I 上局部可积的函数全体;“$\hookrightarrow\hookrightarrow$”表示两个空间之间紧嵌入关系.

为将问题(1)、(2)写成抽象的形式，我们引入几个重要的算子.首先，定义算子 A_1 和 A_2 如下:

$$\langle\mathrm{A}_1\boldsymbol{u},\boldsymbol{v}\rangle\coloneqq\sum_{i,j,k=1}^2\int_\Omega\frac{\partial e_{ij}(\boldsymbol{u})}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k}\mathrm{d}x,\qquad\forall\boldsymbol{u},\boldsymbol{v}\in V,$$

$$\langle\mathrm{A}_2\boldsymbol{\omega},\boldsymbol{\xi}\rangle\coloneqq(\nabla\boldsymbol{\omega},\nabla\boldsymbol{\xi}),\qquad\forall\boldsymbol{\omega},\boldsymbol{\xi}\in H_0^1(\Omega).$$

注 1 由以上定义可知，A_1,A_2 为椭圆算子.根据椭圆算子的谱理论，存在由 A_1 的特征值构成的序列 $\{\lambda_n\}_{n=1}^\infty$ 和 H 空间的标准正交基构成的序列 $\{\mathfrak{e}_n\}_{n=1}^\infty\subset D(\mathrm{A}_1)\coloneqq(H^4(\Omega))^2\cap V$, 满足 $\mathrm{span}\{\mathfrak{e}_1,\mathfrak{e}_2,\cdots,\mathfrak{e}_n,\cdots\}$ 在 V 空间中稠密.存在由 A_2 的特征值构成的序列 $\{\lambda_n^*\}_{n=1}^\infty$ 和 $L^2(\Omega)$ 空间的标准正交基构成的序列 $\{\mathfrak{e}_n^*\}_{n=1}^\infty\subset D(\mathrm{A}_2)\coloneqq H^2(\Omega)\cap H_0^1(\Omega)$，满足 $\mathrm{span}\{\mathfrak{e}_1^*,\mathfrak{e}_2^*,\cdots,\mathfrak{e}_n^*,\cdots\}$ 在 $H_0^1(\Omega)$ 空间中稠密.且对任意的 $n\in\mathbb{N}$，成立:

$\mathrm{A}_1\mathfrak{e}_n=\lambda_n\mathfrak{e}_n,\qquad 0<\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_n\leqslant\cdots$, 且当 $n\to\infty$ 时,$\lambda_n\to\infty$;

$\mathrm{A}_2\mathfrak{e}_n^*=\lambda_n^*\mathfrak{e}_n^*,\qquad 0<\lambda_1^*\leqslant\lambda_2^*\leqslant\cdots\leqslant\lambda_n^*\leqslant\cdots$,且当 $n\to\infty$ 时,$\lambda_n^*\to\infty$.

此外，定义算子

$$b_1(\boldsymbol{u},\boldsymbol{v},\boldsymbol{\varphi})\coloneqq\sum_{j,k=1}^2\int_\Omega u_j\frac{\partial v_k}{\partial x_j}\boldsymbol{\varphi}_k\mathrm{d}x,\qquad\forall\boldsymbol{u},\boldsymbol{v},\boldsymbol{\varphi}\in(H_0^1(\Omega))^2,$$

$$b_2(\boldsymbol{u},\boldsymbol{\omega},\boldsymbol{\xi}) := \sum_{i=1}^{2}\int_\Omega u_i \frac{\partial \boldsymbol{\omega}}{\partial x_i}\boldsymbol{\xi}\,\mathrm{d}x, \qquad \forall \boldsymbol{u} \in (H_0^1(\Omega))^2, \boldsymbol{\omega},\boldsymbol{\xi} \in H_0^1(\Omega).$$

不难验证, 算子 $b_1(\cdot,\cdot,\cdot)$ 和 $b_2(\cdot,\cdot,\cdot)$ 分别在 $V\times V\times V$ 和 $V\times H_0^1(\Omega)\times H_0^1(\Omega)$ 上连续.记

$$\langle B_1(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{\varphi}\rangle := b_1(\boldsymbol{u},\boldsymbol{v},\boldsymbol{\varphi}), \qquad \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{\varphi} \in V,$$

$$\langle B_2(\boldsymbol{u},\boldsymbol{\omega}),\boldsymbol{\xi}\rangle := b_2(\boldsymbol{u},\boldsymbol{\omega},\boldsymbol{\xi}), \qquad \forall \boldsymbol{u} \in V, \boldsymbol{\omega},\boldsymbol{\xi} \in H_0^1(\Omega).$$

令 $\mu(\boldsymbol{u}) := 2\mu_0(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{-\alpha/2}$, 定义算子 N 如下:

$$\langle N(\boldsymbol{u}),\boldsymbol{v}\rangle := \sum_{j,k=1}^{2}\int_\Omega \mu(\boldsymbol{u}) e_{jk}(\boldsymbol{u}) e_{jk}(\boldsymbol{v})\,\mathrm{d}x, \qquad \forall \boldsymbol{u},\boldsymbol{v} \in V.$$

可以验证算子 $N(\cdot)$ 是从 V 到 V^* 的连续函数.

最后, 定义算子:

$$\mathcal{K}_1(\boldsymbol{\omega}) := -2\nu_r \nabla\times\boldsymbol{\omega},\ \mathcal{K}_2(\boldsymbol{u},\boldsymbol{\omega}) := -2\nu_r \nabla\times\boldsymbol{u} + 4\nu_r\boldsymbol{\omega}, \qquad \forall \boldsymbol{\omega} \in H_0^1(\Omega),\ \boldsymbol{u} \in V.$$

接下来, 给出相关算子的估计如下(见文献[51]).

引理 1　① 存在正常数 c_1 和 c_2, 使得

$$\begin{cases} c_1\|\boldsymbol{u}\|_V^2 \le \langle \mathrm{A}_1\boldsymbol{u},\boldsymbol{u}\rangle \le c_2\|\boldsymbol{u}\|_V^2, & \forall \boldsymbol{u} \in V, \\ c_1\|\boldsymbol{\omega}\|_{1,2}^2 \le \langle \mathrm{A}_2\boldsymbol{\omega},\boldsymbol{\omega}\rangle \le c_2\|\boldsymbol{\omega}\|_{1,2}^2, & \forall \boldsymbol{\omega} \in H_0^1(\Omega). \end{cases}$$

② 存在仅依赖于区域 Ω 的正常数 c_3 和 c_4, 使得

$$b_1(\boldsymbol{u},\boldsymbol{v},\boldsymbol{\varphi}) = -b_1(\boldsymbol{u},\boldsymbol{\varphi},\boldsymbol{v}),\ b_1(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = 0, \qquad \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{\varphi} \in V,$$

$$b_2(\boldsymbol{u},\boldsymbol{\omega},\boldsymbol{\xi}) = -b_2(\boldsymbol{u},\boldsymbol{\xi},\boldsymbol{\omega}),\ b_2(\boldsymbol{u},\boldsymbol{\omega},\boldsymbol{\omega}) = 0, \qquad \forall \boldsymbol{u} \in V, \boldsymbol{\omega},\boldsymbol{\xi} \in H_0^1(\Omega),$$

$$|b_1(\boldsymbol{u},\boldsymbol{v},\boldsymbol{\varphi})| \le c_3\|\boldsymbol{u}\|^{1/2}\|\nabla\boldsymbol{u}\|^{1/2}\|\boldsymbol{v}\|^{1/2}\|\nabla\boldsymbol{v}\|^{1/2}\|\nabla\boldsymbol{\varphi}\|, \qquad \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{\varphi} \in V,$$

$$|b_2(\boldsymbol{u},\boldsymbol{\omega},\boldsymbol{\xi})| \le c_3\|\boldsymbol{u}\|^{1/2}\|\nabla\boldsymbol{u}\|^{1/2}\|\nabla\boldsymbol{\omega}\|\,\|\boldsymbol{\xi}\|^{1/2}\|\nabla\boldsymbol{\xi}\|^{1/2}, \qquad \forall \boldsymbol{u} \in V, \boldsymbol{\omega},\boldsymbol{\xi} \in H_0^1(\Omega).$$

根据上面算子的定义, 问题(1)、(2)在分布 $\mathcal{D}'(\tau,+\infty;\hat{V})$ 意义下可以写成如下抽象形式:

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + 2\mu_1 \mathrm{A}_1\boldsymbol{u} + B_1(\boldsymbol{u},\boldsymbol{u}) + \mathcal{K}_1(\boldsymbol{\omega}) + N(\boldsymbol{u}) = \boldsymbol{f}(t,x) + \boldsymbol{g}(t,\boldsymbol{u}_t), & x \in \Omega,\ t > \tau, \\ \dfrac{\partial \boldsymbol{\omega}}{\partial t} + \nu \mathrm{A}_2\boldsymbol{\omega} + B_2(\boldsymbol{u},\boldsymbol{\omega}) + \mathcal{K}_2(\boldsymbol{u},\boldsymbol{\omega}) = \tilde{\boldsymbol{f}}(t,x) + \tilde{\boldsymbol{g}}(t,\boldsymbol{\omega}_t), & x \in \Omega,\ t > \tau, \\ \boldsymbol{u}_\tau(s,x) = \boldsymbol{\phi}(s,x),\ \boldsymbol{\omega}_\tau(s,x) = \boldsymbol{\psi}(s,x), & s \in (-\infty,0],\ \tau \in \mathbb{R},\ x \in \Omega, \\ \boldsymbol{u}(t,x)|_{\partial\Omega} = \boldsymbol{0},\ \Omega(t,x)|_{\partial\Omega} = 0,\ 2\mu_1 \dfrac{\partial e_{jk}}{\partial x_m} r_k r_m|_{\partial\Omega} = 0, & j,k,m = 1,2,\ \forall t \ge \tau. \end{cases} \tag{3}$$

为了处理无穷时滞项, 给定 Banach 空间 X, 对于适当的 γ, 引入空间 $\mathcal{C}_\gamma(X)$ 如下:

$$\mathcal{C}_\gamma(X) := \{\boldsymbol{\Phi} \in \mathcal{C}((-\infty,0];X) \mid \exists \lim_{s\to-\infty} \mathrm{e}^{\gamma s}\boldsymbol{\Phi}(s) \in X\},$$

此空间为 Banach 空间, 其范数为

$$\|\boldsymbol{\Phi}\|_\gamma := \sup_{s\in(-\infty,0]} \mathrm{e}^{\gamma s}\|\boldsymbol{\Phi}(s)\|_X.$$

问题(3)的弱解定义如下[20].

定义 1　给定任意的初始条件 $(\boldsymbol{\phi},\boldsymbol{\psi}) \in \mathcal{C}_\gamma(\hat{H})$, 若存在函数对

$$(\boldsymbol{u},\boldsymbol{\omega}) \in \mathcal{C}([\tau,+\infty);\hat{H}) \cap L^2(\tau,+\infty;\hat{V})\ 且 (\boldsymbol{u}_t,\boldsymbol{\omega}_t) \in \mathcal{C}_\gamma(\hat{H}), \qquad \forall t \ge \tau,$$

使得问题(3)在分布 $\mathcal{D}'(\tau,+\infty;\hat{V}^*)$ 意义下成立,则称 $(\boldsymbol{u},\boldsymbol{\omega})$ 是问题(3)的弱解.

给定如下假设条件.

假设(A)　假设 $(\boldsymbol{g}(t,\boldsymbol{u}_t),\tilde{\boldsymbol{g}}(t,\boldsymbol{\omega}_t)): [\tau,+\infty]\times\mathcal{C}_\gamma(\hat{H}) \to (L^2(\Omega))^3$ 满足:

① 对于任意 $(\boldsymbol{\varphi},\boldsymbol{\xi}) \in \mathcal{C}_\gamma(\hat{H})$, 映射 $[\tau,+\infty] \ni t \mapsto (\boldsymbol{g}(t,\boldsymbol{\varphi}),\tilde{\boldsymbol{g}}(t,\boldsymbol{\xi})) \in (L^2(\Omega))^3$ 是可测的;

② $(\boldsymbol{g}(t,0),\tilde{\boldsymbol{g}}(t,0)) = (0,0,0)$;

③ 存在常数 $L > 0$, 使得对于任意 (φ_1,ξ_1), $(\varphi_2,\xi_2) \in \mathcal{C}_\gamma(\hat{H})$, 有

$$\begin{cases}\|\boldsymbol{g}(t,\varphi_1)-\boldsymbol{g}(t,\varphi_2)\| \leqslant L\|\varphi_1-\varphi_2\|_\gamma,\\ \|\tilde{\boldsymbol{g}}(t,\xi_1)-\tilde{\boldsymbol{g}}(t,\xi_2)\| \leqslant L\|\xi_1-\xi_2\|_\gamma,\end{cases}\quad \forall t\geqslant\tau.$$

在上面的假设条件下，如下关于问题(3)弱解的适定性结论可由 Galerkin 方法证明，详细证明过程可参阅文献[20].

引理 2 假设(A)成立且 $2L<\delta<2\gamma$，给定 $(\boldsymbol{\phi},\boldsymbol{\psi})\in\mathcal{C}_\gamma(\hat{H})$，$(\boldsymbol{f},\tilde{\boldsymbol{f}})\in L^2(\tau,+\infty;\hat{V}^*)$，则问题(3)存在唯一的弱解 $(\boldsymbol{u},\boldsymbol{\omega})$，满足

$$(\boldsymbol{u},\boldsymbol{\omega})\in\mathcal{C}([\tau,+\infty);\hat{H})\cap L^2(\tau,+\infty;\hat{V}),\qquad (\boldsymbol{u}_t,\boldsymbol{\omega}_t)\in\mathcal{C}_\gamma(\hat{H}),$$

$$\|(\boldsymbol{u},\boldsymbol{\omega})(t)\|^2\leqslant\|(\boldsymbol{u}_t,\boldsymbol{\omega}_t)\|_\gamma^2\leqslant$$

$$\mathrm{e}^{-(\delta-2L)(t-\tau)}\|(\boldsymbol{\phi},\boldsymbol{\psi})\|_\gamma^2+\frac{1}{\min\{c_1\mu_1,\ c_1\nu\}}\int_\tau^t\mathrm{e}^{-(\delta-2L)(t-\theta)}\|(\boldsymbol{f},\tilde{\boldsymbol{f}})(\theta)\|_{\hat{V}^*}^2\mathrm{d}\theta,\tag{4}$$

其中 $\delta:=\min\{2c_1\mu_1\lambda_1,\ c_1\nu\lambda_1^*+8\nu_r\}$.

最后，我们引入如下引理[47,52].

引理 3 设 $\vartheta(t)$ 和 $\varrho(t)$ 为 $[\tau,+\infty)$ 上的实局部可积函数,且存在 $T_0>0$，使得

$$\liminf_{t\to+\infty}\frac{1}{T_0}\int_t^{t+T_0}\vartheta(\theta)\mathrm{d}\theta>0,\ \limsup_{t\to+\infty}\frac{1}{T_0}\int_t^{t+T_0}\vartheta^-(\theta)\mathrm{d}\theta<+\infty,\tag{5}$$

$$\lim_{t\to+\infty}\frac{1}{T_0}\int_t^{t+T_0}\varrho^+(\theta)\mathrm{d}\theta=0,\tag{6}$$

其中 $\vartheta^-(t)=\max\{-\vartheta(t),0\}$，$\varrho^+(t)=\max\{\varrho(t),0\}$.若 $\zeta(t)$ 是 $[\tau,+\infty)$ 上的非负且绝对连续函数，并且在 $[\tau,+\infty)$ 上 $\mathrm{d}\zeta(t)/\mathrm{d}t+\vartheta(t)\zeta(t)\leqslant\varrho(t)$ 几乎处处成立,则当 $t\to\infty$ 时，$\zeta(t)\to0$.

2 确定模个数估计

本节的主要目的是估计问题(3)的确定模个数.

首先，由注 1 可知，对于任意 $\boldsymbol{u}\in H,\boldsymbol{\omega}\in L^2(\Omega)$，

$$\boldsymbol{u}(t,x)=\sum_{i=1}^\infty(\boldsymbol{u},\mathfrak{e}_i)\mathfrak{e}_i:=\sum_{i=1}^\infty\hat{\boldsymbol{u}}_i(t)\mathfrak{e}_i(x),\ \boldsymbol{\omega}(t,x)=\sum_{i=1}^\infty(\boldsymbol{\omega},\mathfrak{e}_i^*)\mathfrak{e}_i^*:=\sum_{i=1}^\infty\hat{\boldsymbol{\omega}}_i(t)\mathfrak{e}_i^*(x).$$

令 $\boldsymbol{\varepsilon}_n:=\mathrm{span}\{(\mathfrak{e}_1,\mathfrak{e}_1^*),(\mathfrak{e}_2,\mathfrak{e}_2^*),\cdots,(\mathfrak{e}_n,\mathfrak{e}_n^*)\}$，定义 Galerkin 投影算子 P_n 如下

$$\mathrm{P}_n\boldsymbol{u}(t,x)=\sum_{i=1}^n\hat{\boldsymbol{u}}_i(t)\mathfrak{e}_i(x),\ \mathrm{P}_n\boldsymbol{\omega}(t,x)=\sum_{i=1}^n\hat{\boldsymbol{\omega}}_i(t)\mathfrak{e}_i^*(x).$$

接下来，我们给出确定模的定义如下(见文献[48,52]).

定义 2 设 $(\boldsymbol{u}_1(t,x),\boldsymbol{\omega}_1(t,x))$ 和 $(\boldsymbol{u}_2(t,x),\boldsymbol{\omega}_2(t,x))$ 为问题(3)分别对应于外力 $(\boldsymbol{f}_1,\tilde{\boldsymbol{f}}_1)$ 和 $(\boldsymbol{f}_2,\tilde{\boldsymbol{f}}_2)$ 的两个弱解，如果由

$$\begin{cases}\lim\limits_{t\to+\infty}(\|\mathrm{P}_n\boldsymbol{u}_1(t,x)-\mathrm{P}_n\boldsymbol{u}_2(t,x)\|+\|\mathrm{P}_n\boldsymbol{\omega}_1(t,x)-\mathrm{P}_n\boldsymbol{\omega}_2(t,x)\|)=0,\\ \lim\limits_{t\to+\infty}(\|\boldsymbol{f}_1(t,x)-\boldsymbol{f}_2(t,x)\|_{V^*}+\|\tilde{\boldsymbol{f}}_1(t,x)-\tilde{\boldsymbol{f}}_2(t,x)\|_{H^{-1}})=0\end{cases}\tag{7}$$

可推得 $\lim_{t\to+\infty}(\|\boldsymbol{u}_1(t,x)-\boldsymbol{u}_2(t,x)\|+\|\boldsymbol{\omega}_1(t,x)-\boldsymbol{\omega}_2(t,x)\|)=0$，则称与 P_n 相关的前 n 个 Fourier 模为问题(3)的确定模.

本部分的主要结论如下.

定理 1 设 $(\boldsymbol{u}_1,\boldsymbol{\omega}_1)$ 和 $(\boldsymbol{u}_2,\boldsymbol{\omega}_2)$ 为问题(3)分别对应于外力 $(\boldsymbol{f}_1,\tilde{\boldsymbol{f}}_1)$ 和 $(\boldsymbol{f}_2,\tilde{\boldsymbol{f}}_2)$ 的两个弱解，若

$$c_1^2\mu_1\lambda_1\min\{\mu_1,\ \nu\}\geqslant4\max\{8c_3^4/(c_1^3\mu_1^3),\ \tilde{c}\}\ \ \|(\boldsymbol{f}_1,\tilde{\boldsymbol{f}}_1)\|_{L^2(\tau,+\infty;\hat{V}^*)}^4,\tag{8}$$

取 λ_{n+1} (见注 1)满足

$$\begin{cases}\lambda_{n+1}\geqslant\dfrac{4\nu_r^2}{c_1^2\nu\mu_1}+\dfrac{c_4^2}{2c_1\mu_1}+\dfrac{c_5}{2c_1\mu_1}\sum\limits_{i=1}^2\|(\boldsymbol{f}_i,\tilde{\boldsymbol{f}}_i)\|_{L^2(\tau,+\infty;\hat{V}^*)}^2,\\ \lambda_{n+1}^*\geqslant\dfrac{4\nu_r^2}{c_1^2\mu_1\nu}-\dfrac{8\nu_r}{c_1\nu}+\dfrac{c_3^2}{2c_1^3\nu\sqrt{\mu_1\nu}\min\{\mu_1,\ \nu\}}\|(\boldsymbol{f}_2,\tilde{\boldsymbol{f}}_2)\|_{L^2(\tau,+\infty;\hat{V}^*)}^2,\end{cases}\tag{9}$$

则在定义 2 的意义下，与 Galerkin 投影算子 P_n 相关的前 n 个 Fourier 模为问题(3)的确定模，其中 c_4,c_5 和 $\tilde{c}$ 分别来自式(20)、(25)和(34).

证明　设 $(\boldsymbol{u}_1(t,x),\boldsymbol{\omega}_1(t,x))$ 和 $(\boldsymbol{u}_2(t,x),\boldsymbol{\omega}_2(t,x))$ 为问题(3)分别对应于外力 $(\boldsymbol{f}_1,\tilde{\boldsymbol{f}}_1)$ 和 $(\boldsymbol{f}_2,\tilde{\boldsymbol{f}}_2)$ 的两个弱解，其初值分别为 $(\phi_1,\psi_1)\in\mathcal{C}_\gamma(\hat{H})$ 和 $(\phi_2,\psi_2)\in\mathcal{C}_\gamma(\hat{H})$. 记 $\boldsymbol{U}=\boldsymbol{u}_1-\boldsymbol{u}_2,\boldsymbol{W}=\boldsymbol{\omega}_1-\boldsymbol{\omega}_2,\boldsymbol{F}=\boldsymbol{f}_1-\boldsymbol{f}_2$, $\tilde{\boldsymbol{F}}=\tilde{\boldsymbol{f}}_1-\tilde{\boldsymbol{f}}_2$，则在分布 $\mathcal{D}'(\tau,+\infty;\hat{H})$ 意义下，成立：

$$\begin{cases}\dfrac{\partial\boldsymbol{U}}{\partial t}+2\mu_1\mathrm{A}_1\boldsymbol{U}+B_1(\boldsymbol{u}_1,\boldsymbol{u}_1)-B_1(\boldsymbol{u}_2,\boldsymbol{u}_2)+N(\boldsymbol{u}_1)-N(\boldsymbol{u}_2)+\mathcal{K}_1(\boldsymbol{W})=\\ \qquad\boldsymbol{F}+\boldsymbol{g}(t,\boldsymbol{u}_{1,t})-\boldsymbol{g}(t,\boldsymbol{u}_{2,t}),\\ \dfrac{\partial\boldsymbol{W}}{\partial t}+\nu\mathrm{A}_2\boldsymbol{W}+B_2(\boldsymbol{u}_1,\boldsymbol{\omega}_1)-B_2(\boldsymbol{u}_2,\boldsymbol{\omega}_2)+\mathcal{K}_2(\boldsymbol{U},\boldsymbol{W})=\tilde{\boldsymbol{F}}+\tilde{\boldsymbol{g}}(t,\boldsymbol{\omega}_{1,t})-\tilde{\boldsymbol{g}}(t,\boldsymbol{\omega}_{2,t}).\end{cases}\tag{10}$$

令 $\mathrm{Q}_n=\mathrm{I}-\mathrm{P}_n$，其中 I 为单位算子. 根据定义 2，只需证明存在 $n\in\mathbb{N}$，使得在条件(7)下，下式成立：

$$\lim_{t\to+\infty}(\|\mathrm{Q}_n\boldsymbol{U}(t,x)\|+\|\mathrm{Q}_n\boldsymbol{W}(t,x)\|)=0.\tag{11}$$

接下来，我们将证明过程分为三步.

第一步：能量估计. 首先，用 $\mathrm{Q}_n\boldsymbol{U}$ 和 $\mathrm{Q}_n\boldsymbol{W}$ 分别与式(10)的第一式和第二式作对偶积，可得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathrm{Q}_n\boldsymbol{U}\|^2+2\mu_1\langle\mathrm{A}_1\boldsymbol{U},\mathrm{Q}_n\boldsymbol{U}\rangle+b_1(\boldsymbol{U},\boldsymbol{u}_1,\mathrm{Q}_n\boldsymbol{U})+b_1(\boldsymbol{u}_2,\boldsymbol{U},\mathrm{Q}_n\boldsymbol{U})+\langle\mathcal{K}_1(\boldsymbol{W}),\mathrm{Q}_n\boldsymbol{U}\rangle+$$
$$\langle N(\boldsymbol{u}_1)-N(\boldsymbol{u}_2),\mathrm{Q}_n\boldsymbol{U}\rangle=\langle\boldsymbol{F},\mathrm{Q}_n\boldsymbol{U}\rangle+\langle\boldsymbol{g}(t,\boldsymbol{u}_{1,t})-\boldsymbol{g}(t,\boldsymbol{u}_{2,t}),\mathrm{Q}_n\boldsymbol{U}\rangle,\tag{12}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathrm{Q}_n\boldsymbol{W}\|^2+\nu\langle\mathrm{A}_2\boldsymbol{W},\mathrm{Q}_n\boldsymbol{W}\rangle+b_2(\boldsymbol{u}_1,\boldsymbol{W},\mathrm{Q}_n\boldsymbol{W})+b_2(\boldsymbol{U},\boldsymbol{\omega}_2,\mathrm{Q}_n\boldsymbol{W})+\langle\mathcal{K}_2(\boldsymbol{U},\boldsymbol{W}),\mathrm{Q}_n\boldsymbol{W}\rangle=$$
$$\langle\tilde{\boldsymbol{F}},\mathrm{Q}_n\boldsymbol{W}\rangle+\langle\tilde{\boldsymbol{g}}(t,\boldsymbol{\omega}_{1,t})-\tilde{\boldsymbol{g}}(t,\boldsymbol{\omega}_{2,t}),\mathrm{Q}_n\boldsymbol{W}\rangle.\tag{13}$$

由引理 1，可知

$$2c_1\mu_1\|\mathrm{Q}_n\boldsymbol{U}\|_V^2\leqslant2\mu_1\langle\mathrm{A}_1\boldsymbol{U},\mathrm{Q}_n\boldsymbol{U}\rangle=2\mu_1\langle\mathrm{A}_1\mathrm{Q}_n\boldsymbol{U},\mathrm{Q}_n\boldsymbol{U}\rangle,\tag{14}$$

$$c_1\nu\|\mathrm{Q}_n\boldsymbol{W}\|_{1,2}^2\leqslant\nu\langle\mathrm{A}_2\boldsymbol{W},\mathrm{Q}_n\boldsymbol{W}\rangle=\nu\langle\mathrm{A}_2\mathrm{Q}_n\boldsymbol{W},\mathrm{Q}_n\boldsymbol{W}\rangle.\tag{15}$$

由引理 1，可知

$$\begin{aligned}&|b_1(\boldsymbol{U},\boldsymbol{u}_1,\mathrm{Q}_n\boldsymbol{U})+b_1(\boldsymbol{u}_2,\boldsymbol{U},\mathrm{Q}_n\boldsymbol{U})|\leqslant\\&\quad|b_1(\mathrm{P}_n\boldsymbol{U},\boldsymbol{u}_1,\mathrm{Q}_n\boldsymbol{U})+b_1(\mathrm{Q}_n\boldsymbol{U},\boldsymbol{u}_1,\mathrm{Q}_n\boldsymbol{U})|+|b_1(\boldsymbol{u}_2,\mathrm{P}_n\boldsymbol{U},\mathrm{Q}_n\boldsymbol{U})+b_1(\boldsymbol{u}_2,\mathrm{Q}_n\boldsymbol{U},\mathrm{Q}_n\boldsymbol{U})|\leqslant\\&\quad c_3\|\mathrm{P}_n\boldsymbol{U}\|^{1/2}\|\mathrm{P}_n\boldsymbol{U}\|_V^{1/2}\|\boldsymbol{u}_1\|^{1/2}\|\boldsymbol{u}_1\|_V^{1/2}\|\mathrm{Q}_n\boldsymbol{U}\|_V+c_3\|\mathrm{Q}_n\boldsymbol{U}\|\|\mathrm{Q}_n\boldsymbol{U}\|_V\|\boldsymbol{u}_1\|_V+\\&\quad c_3\|\boldsymbol{u}_2\|^{1/2}\|\boldsymbol{u}_2\|_V^{1/2}\|\mathrm{P}_n\boldsymbol{U}\|^{1/2}\|\mathrm{P}_n\boldsymbol{U}\|_V^{1/2}\|\mathrm{Q}_n\boldsymbol{U}\|_V\leqslant\\&\quad c_3\|\mathrm{P}_n\boldsymbol{U}\|^{1/2}\|\mathrm{P}_n\boldsymbol{U}\|_V^{1/2}\|\mathrm{Q}_n\boldsymbol{U}\|_V(\|\boldsymbol{u}_1\|^{1/2}\|\boldsymbol{u}_1\|_V^{1/2}+\|\boldsymbol{u}_2\|^{1/2}\|\boldsymbol{u}_2\|_V^{1/2})+\\&\quad\frac{c_3^2}{c_1\mu_1}\|\mathrm{Q}_n\boldsymbol{U}\|^2\|\boldsymbol{u}_1\|_V^2+\frac{c_1\mu_1}{4}\|\mathrm{Q}_n\boldsymbol{U}\|_V^2,\end{aligned}\tag{16}$$

$$\begin{aligned}&|b_2(\boldsymbol{u}_1,\boldsymbol{W},\mathrm{Q}_n\boldsymbol{W})+b_2(\boldsymbol{U},\boldsymbol{\omega}_2,\mathrm{Q}_n\boldsymbol{W})|\leqslant\\&\quad|b_2(\boldsymbol{u}_1,\mathrm{P}_n\boldsymbol{W},\mathrm{Q}_n\boldsymbol{W})+b_2(\boldsymbol{u}_1,\mathrm{Q}_n\boldsymbol{W},\mathrm{Q}_n\boldsymbol{W})|+|b_2(\mathrm{P}_n\boldsymbol{U},\boldsymbol{\omega}_2,\mathrm{Q}_n\boldsymbol{W})+b_2(\mathrm{Q}_n\boldsymbol{U},\boldsymbol{\omega}_2,\mathrm{Q}_n\boldsymbol{W})|\leqslant\\&\quad c_3\|\boldsymbol{u}_1\|^{1/2}\|\boldsymbol{u}_1\|_V^{1/2}\|\mathrm{P}_n\boldsymbol{W}\|^{1/2}\|\mathrm{P}_n\boldsymbol{W}\|_{1,2}^{1/2}\|\mathrm{Q}_n\boldsymbol{W}\|_{1,2}+\\&\quad c_3\|\mathrm{P}_n\boldsymbol{U}\|^{1/2}\|\mathrm{P}_n\boldsymbol{U}\|_V^{1/2}\|\boldsymbol{\omega}_2\|_{1,2}\|\mathrm{Q}_n\boldsymbol{W}\|^{1/2}\|\mathrm{Q}_n\boldsymbol{W}\|_{1,2}^{1/2}+\\&\quad c_3\|\mathrm{Q}_nU\|^{1/2}\|\mathrm{Q}_nU\|_V^{1/2}\|\boldsymbol{W}\|_{1,2}\|\mathrm{Q}_n\boldsymbol{W}\|^{1/2}\|\mathrm{Q}_n\boldsymbol{W}\|_{1,2}^{1/2}\leqslant\\&\quad c_3\|\boldsymbol{u}_1\|^{1/2}\|\boldsymbol{u}_1\|_V^{1/2}\|\mathrm{P}_n\boldsymbol{W}\|^{1/2}\|\mathrm{P}_n\boldsymbol{W}\|_{1,2}^{1/2}\|\mathrm{Q}_n\boldsymbol{W}\|_{1,2}+\\&\quad c_3\|\mathrm{P}_n\boldsymbol{U}\|^{1/2}\|\mathrm{P}_n\boldsymbol{U}\|_V^{1/2}\|\boldsymbol{\omega}_2\|_{1,2}\|\mathrm{Q}_n\boldsymbol{W}\|^{1/2}\|\mathrm{Q}_n\boldsymbol{W}\|_{1,2}^{1/2}+\\&\quad\frac{c_3^2}{4c_1\sqrt{\mu_1\nu}}\|\boldsymbol{\omega}_2\|_{1,2}^2(\|\mathrm{Q}_n\boldsymbol{U}\|^2+\|\mathrm{Q}_n\boldsymbol{W}\|^2)+\frac{c_1\mu_1}{4}\|\mathrm{Q}_n\boldsymbol{U}\|_V^2+\frac{c_1\nu}{4}\|\mathrm{Q}_n\boldsymbol{W}\|_{1,2}^2.\end{aligned}\tag{17}$$

由算子 $\mathcal{K}_1$ 和 $\mathcal{K}_2$ 的定义，不难推得

$$|\langle\mathcal{K}_1(\boldsymbol{W}),\mathrm{Q}_n\boldsymbol{U}\rangle|\leqslant2\nu_r\|\mathrm{P}_n\boldsymbol{W}\|\|\mathrm{Q}_n\boldsymbol{U}\|_V+2\nu_r\|\mathrm{Q}_n\boldsymbol{W}\|\|\mathrm{Q}_n\boldsymbol{U}\|_V\leqslant$$

$$2\nu_r \|P_n\boldsymbol{W}\| \|Q_n\boldsymbol{U}\|_V + \frac{2\nu_r^2}{c_1\mu_1}\|Q_n\boldsymbol{W}\|^2 + \frac{c_1\mu_1}{2}\|Q_n\boldsymbol{U}\|_V^2, \tag{18}$$

$$\langle \mathcal{K}_2(\boldsymbol{U},\boldsymbol{W}),\ Q_n\boldsymbol{W}\rangle \geqslant -2\nu_r\|P_n\boldsymbol{U}\| \|Q_n\boldsymbol{W}\|_{1,2} - 2\nu_r\|Q_n\boldsymbol{U}\| \|Q_n\boldsymbol{W}\|_{1,2} + 4\nu_r\|Q_n\boldsymbol{W}\|^2 \geqslant$$
$$-2\nu_r\|P_n\boldsymbol{U}\| \|Q_n\boldsymbol{W}\|_{1,2} - \frac{4\nu_r^2}{c_1\nu}\|Q_n\boldsymbol{U}\|^2 - \frac{c_1\nu}{4}\|Q_n\boldsymbol{W}\|_{1,2}^2 + 4\nu_r\|Q_n\boldsymbol{W}\|^2. \tag{19}$$

通过直接计算，可知存在与 μ_0, ϵ 和 α 有关的常数 $c_4 := c_4(\mu_0, \epsilon, \alpha)$，使得

$$-(N(\boldsymbol{u}_1) - N(\boldsymbol{u}_2),\ Q_n\boldsymbol{U}) = -\sum_{j,k=1}^{2}\int_\Omega (\mu(\boldsymbol{u}_1)e_{jk}(\boldsymbol{u}_1) - \mu(\boldsymbol{u}_2)e_{jk}(\boldsymbol{u}_2))e_{jk}(Q_n\boldsymbol{U})\mathrm{d}x =$$
$$-\sum_{j,k=1}^{2}\int_\Omega [(\mu(\boldsymbol{u}_1) - \mu(\boldsymbol{u}_2))e_{jk}(\boldsymbol{u}_1) + \mu(\boldsymbol{u}_2)(e_{jk}(\boldsymbol{u}_1) - e_{jk}(\boldsymbol{u}_2))]e_{jk}(Q_n\boldsymbol{U})\mathrm{d}x =$$
$$-\sum_{j,k=1}^{2}\int_\Omega (\mu(\boldsymbol{u}_1) - \mu(\boldsymbol{u}_2))e_{jk}(\boldsymbol{u}_1)e_{jk}(Q_n\boldsymbol{U})\mathrm{d}x - \sum_{j,k=1}^{2}\int_\Omega \mu(\boldsymbol{u}_2)e_{jk}(P_n\boldsymbol{U} + Q_n\boldsymbol{U})e_{jk}(Q_n\boldsymbol{U})\mathrm{d}x \leqslant$$
$$c_4(\|\boldsymbol{u}_1\|_V + \|\boldsymbol{u}_1\|_V^2 + \|\boldsymbol{u}_2\|_V^2)\|Q_n\boldsymbol{U}\| + c_4(\|\boldsymbol{u}_2\|_V + 1)\|P_n\boldsymbol{U}\| \|Q_n\boldsymbol{U}\|_V. \tag{20}$$

另外，易知

$$\langle \boldsymbol{F},\ Q_n\boldsymbol{U}\rangle \leqslant \|\boldsymbol{F}\|_{V^*}\|Q_n\boldsymbol{U}\|_V,\ \langle \tilde{\boldsymbol{f}},\ Q_n\boldsymbol{W}\rangle \leqslant \|\tilde{\boldsymbol{F}}\|_{H^{-1}}\|Q_n\boldsymbol{W}\|_{1,2}. \tag{21}$$

由假设条件(A)，可得

$$\begin{cases}\langle \boldsymbol{g}(t,\boldsymbol{u}_{1,t}) - \boldsymbol{g}(t,\boldsymbol{u}_{2,t}),\ Q_n\boldsymbol{U}\rangle \leqslant \|\boldsymbol{g}(t,\boldsymbol{u}_{1,t}) - \boldsymbol{g}(t,\boldsymbol{u}_{2,t})\| \|Q_n\boldsymbol{U}\| \leqslant L\|\boldsymbol{u}_t\|_\gamma\|Q_n\boldsymbol{U}\|, \\ \langle \tilde{\boldsymbol{g}}(t,\boldsymbol{\omega}_{1,t}) - \tilde{\boldsymbol{g}}(t,\boldsymbol{\omega}_{2,t}),\ Q_n\boldsymbol{W}\rangle \leqslant L\|\boldsymbol{W}_t\|_\gamma\|Q_n\boldsymbol{W}\|.\end{cases} \tag{22}$$

将式(14)—(22)代入式(12)和(13)，并结合如下 Poincaré 型不等式

$$\lambda_{n+1}\|Q_n\boldsymbol{U}\|^2 \leqslant \|Q_n\boldsymbol{U}\|_V^2,\ \lambda_{n+1}^*\|Q_n\boldsymbol{W}\|^2 \leqslant \|Q_n\boldsymbol{W}\|_{1,2}^2,$$

可得

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|Q_n\boldsymbol{U}\|^2 + \|Q_n\boldsymbol{W}\|^2) + \left(c_1\nu\lambda_{n+1}^* + 8\nu_r - \frac{c_3^2}{2c_1\sqrt{\mu_1\nu}}\|\boldsymbol{\omega}_2\|_{1,2}^2 - \frac{4\nu_r^2}{c_1\mu_1}\right)\|Q_n\boldsymbol{W}\|^2 +$$
$$2\left(c_1\mu_1\lambda_{n+1} - \frac{c_3^2}{c_1\mu_1}\|\boldsymbol{u}_1\|_V^2 - \frac{c_3^2}{4c_1\sqrt{\mu_1\nu}}\|\boldsymbol{\omega}_2\|_{1,2}^2 - \right.$$
$$\left.\frac{4\nu_r^2}{c_1\nu} - c_4(\|\boldsymbol{u}_1\|_V + \|\boldsymbol{u}_1\|_V^2 + \|\boldsymbol{u}_2\|_V^2)\right)\|Q_n\boldsymbol{U}\|^2 \leqslant$$
$$2c_3\|P_n\boldsymbol{U}\|^{1/2}\|P_n\boldsymbol{U}\|_V^{1/2}\|Q_n\boldsymbol{U}\|_V(\|\boldsymbol{u}_1\|^{1/2}\|\boldsymbol{u}_1\|_V^{1/2} + \|\boldsymbol{u}_2\|^{1/2}\|\boldsymbol{u}_2\|_V^{1/2}) +$$
$$2c_3\|\boldsymbol{u}_1\|^{1/2}\|\boldsymbol{u}_1\|_V^{1/2}\|P_n\boldsymbol{W}\|^{1/2}\|P_n\boldsymbol{W}\|_{1,2}^{1/2}\|Q_n\boldsymbol{W}\|_{1,2} +$$
$$2c_3\|P_n\boldsymbol{U}\|^{1/2}\|P_n\boldsymbol{U}\|_V^{1/2}\|\boldsymbol{\omega}_2\|_{1,2}\|Q_n\boldsymbol{W}\|^{1/2}\|Q_n\boldsymbol{W}\|_{1,2}^{1/2} +$$
$$4\nu_r\|P_n\boldsymbol{W}\| \|Q_n\boldsymbol{U}\|_V + 4\nu_r\|P_n\boldsymbol{U}\| \|Q_n\boldsymbol{W}\|_{1,2} +$$
$$2c_4(1 + \|\boldsymbol{u}_2\|_V)\|P_n\boldsymbol{U}\| \|Q_n\boldsymbol{U}\|_V + 2\|\boldsymbol{F}\|_{V^*}\|Q_n\boldsymbol{U}\|_V +$$
$$2\|\tilde{\boldsymbol{F}}\|_{H^{-1}}\|Q_n\boldsymbol{W}\|_{1,2} + 2L\|\boldsymbol{U}_t\|_\gamma\|Q_n\boldsymbol{U}\| + 2L\|\boldsymbol{\omega}_t\|_\gamma\|Q_n\boldsymbol{W}\|. \tag{23}$$

令

$$\zeta(t) := \|Q_n\boldsymbol{U}(t)\|^2 + \|Q_n\boldsymbol{W}(t)\|^2,$$
$$\vartheta(t) := 2\min\left\{c_1\mu_1\lambda_{n+1} - \frac{c_3^2}{c_1\mu_1}\|\boldsymbol{u}_1\|_V^2 - \frac{c_3^2}{4c_1\sqrt{\mu_1\nu}}\|\boldsymbol{\omega}_2\|_{1,2}^2 - \frac{4\nu_r^2}{c_1\nu} - \right.$$
$$\left. c_4(\|\boldsymbol{u}_1\|_V + \|\boldsymbol{u}_1\|_V^2 + \|\boldsymbol{u}_2\|_V^2),\ c_1\nu\lambda_{n+1}^* + 8\nu_r - \frac{c_3^2}{2c_1\sqrt{\mu_1\nu}}\|\boldsymbol{\omega}_2\|_{1,2}^2 - \frac{4\nu_r^2}{c_1\mu_1}\right\},$$
$$\varrho(t) := 2c_3\|P_n\boldsymbol{U}\|^{1/2}\|P_n\boldsymbol{U}\|_V^{1/2}\|Q_n\boldsymbol{U}\|_V(\|\boldsymbol{u}_1\|^{1/2}\|\boldsymbol{u}_1\|_V^{1/2} + \|\boldsymbol{u}_2\|^{1/2}\|\boldsymbol{u}_2\|_V^{1/2}) +$$
$$2c_3\|\boldsymbol{u}_1\|^{1/2}\|\boldsymbol{u}_1\|_V^{1/2}\|P_n\boldsymbol{W}\|^{1/2}\|P_n\boldsymbol{W}\|_{1,2}^{1/2}\|Q_n\boldsymbol{W}\|_{1,2} +$$
$$2c_4(1 + \|\boldsymbol{u}_2\|_V)\|P_n\boldsymbol{U}\| \|Q_n\boldsymbol{U}\|_V +$$

$$4\nu_r \| \mathrm{P}_n \boldsymbol{U} \| \ \| \mathrm{Q}_n \boldsymbol{W} \|_{1,2} + 4\nu_r \| \mathrm{P}_n \boldsymbol{W} \| \ \| \mathrm{Q}_n \boldsymbol{U} \|_V +$$
$$2c_3 \| \mathrm{P}_n \boldsymbol{U} \|^{1/2} \| \mathrm{P}_n \boldsymbol{U} \|_V^{1/2} \| \boldsymbol{\omega}_2 \|_{1,2} \| \mathrm{Q}_n \boldsymbol{W} \|^{1/2} \| \mathrm{Q}_n \boldsymbol{W} \|_{1,2}^{1/2} +$$
$$2 \| \boldsymbol{F} \|_{V^*} \| \mathrm{Q}_n \boldsymbol{U} \|_V + 2 \| \tilde{\boldsymbol{F}} \|_{H^{-1}} \| \mathrm{Q}_n \boldsymbol{W} \|_{1,2} + 2L \| \boldsymbol{U}_t \|_\gamma \| \mathrm{Q}_n \boldsymbol{U} \| + 2L \| \boldsymbol{W}_t \|_\gamma \| \mathrm{Q}_n \boldsymbol{W} \|,$$

则式(23)可表示为

$$\frac{\mathrm{d}\zeta(t)}{\mathrm{d}t} + \vartheta(t)\zeta(t) \leqslant \varrho(t), \qquad \forall t \geqslant \tau. \tag{24}$$

第二步：验证 $\vartheta(t)$ 和 $\varrho(t)$ 分别满足引理 3 中条件(5)和(6).

(I) 验证 $\vartheta(t)$ 满足条件(5).为此，用 $\boldsymbol{u}_i$ 和 $\boldsymbol{\omega}_i$ 分别与式(3)的第一式和第二式作对偶积并相加，可得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\| \boldsymbol{u}_i \|^2 + \| \boldsymbol{\omega}_i \|^2) + 2\mu_1 \langle \mathrm{A}_1 \boldsymbol{u}_i, \boldsymbol{u}_i \rangle + \nu \langle \mathrm{A}_2 \boldsymbol{\omega}_i, \boldsymbol{\omega}_i \rangle +$$
$$\langle B_1(\boldsymbol{u}_i, \boldsymbol{u}_i), \boldsymbol{u}_i \rangle + \langle B_2(\boldsymbol{u}_i, \boldsymbol{\omega}_i), \boldsymbol{\omega}_i \rangle +$$
$$\langle \mathcal{K}_1(\boldsymbol{\omega}_i), \boldsymbol{u}_i \rangle + \langle \mathcal{K}_2(\boldsymbol{u}_i, \boldsymbol{\omega}_i), \boldsymbol{\omega}_i \rangle + \langle N(\boldsymbol{u}_i), \boldsymbol{u}_i \rangle =$$
$$\langle \boldsymbol{f}_i(t,x), \boldsymbol{u}_i \rangle + \langle \tilde{\boldsymbol{f}}_i(t,x), \boldsymbol{\omega}_i \rangle + \langle \boldsymbol{g}(t, \boldsymbol{u}_{i,t}), \boldsymbol{u}_i \rangle + \langle \tilde{\boldsymbol{g}}(t, \boldsymbol{\omega}_{i,t}), \boldsymbol{\omega}_i \rangle, \qquad i = 1,2.$$

结合算子 $\mathcal{K}_1, \mathcal{K}_2$ 和 N 的定义，以及引理 1 和引理 2，可推得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\| \boldsymbol{u}_i \|^2 + \| \boldsymbol{\omega}_i \|^2) + \frac{3}{2}c_1\mu_1 \| \boldsymbol{u}_i \|_V^2 + \frac{1}{2} c_1\nu \| \boldsymbol{\omega}_i \|_{1,2}^2 + 4\nu_r \| \boldsymbol{\omega}_i \|^2 \leqslant$$
$$\frac{1}{2c_1\mu_1} \| \boldsymbol{f}_i \|_{V^*}^2 + \frac{1}{2c_1\nu} \| \tilde{\boldsymbol{f}}_i \|_{H^{-1}}^2 + L \| \boldsymbol{u}_{i,t} \|_\gamma^2 + L \| \boldsymbol{\omega}_{i,t} \|_\gamma^2, \qquad i = 1,2.$$

积分上式并结合式(4)，可得

$$\| \boldsymbol{u}_i(t + T_0) \|^2 + \| \boldsymbol{\omega}_i(t + T_0) \|^2 + 3c_1\mu_1 \int_t^{t+T_0} \| \boldsymbol{u}_i(\theta) \|_V^2 \mathrm{d}\theta + c_1\nu \int_t^{t+T_0} \| \boldsymbol{\omega}_i(\theta) \|_{1,2}^2 \mathrm{d}\theta +$$
$$8\nu_r \int_t^{t+T_0} \| \boldsymbol{\omega}_i(\theta) \|^2 \mathrm{d}\theta \leqslant$$
$$\| \boldsymbol{u}_i(t) \|^2 + \| \boldsymbol{\omega}_i(t) \|^2 + \frac{1}{c_1\mu_1} \int_t^{t+T_0} \| \boldsymbol{f}_i(\theta) \|_{V^*}^2 \mathrm{d}\theta + \frac{1}{c_1\nu} \int_t^{t+T_0} \| \tilde{\boldsymbol{f}}_i(\theta) \|_{H^{-1}}^2 \mathrm{d}\theta +$$
$$2L \int_t^{t+T_0} \| (\boldsymbol{u}_{i,\theta}, \boldsymbol{\omega}_{i,\theta}) \|_\gamma^2 \mathrm{d}\theta \leqslant$$
$$2\mathrm{e}^{-(\delta-2L)(t-\tau)} \| (\boldsymbol{\phi}_i, \boldsymbol{\psi}_i) \|_\gamma^2 + \frac{2}{\min\{c_1\mu_1, c_1\nu\}} \| (\boldsymbol{f}_i, \tilde{\boldsymbol{f}}_i) \|_{L^2(\tau,+\infty;\hat{V}^*)}^2 +$$
$$2L \int_t^{t+T_0} \| (\boldsymbol{u}_{i,\theta}, \boldsymbol{\omega}_{i,\theta}) \|_\gamma^2 \mathrm{d}\theta, \qquad i = 1,2.$$

由上式，可推得

$$\limsup_{t\to+\infty} \frac{1}{T_0} \int_t^{t+T_0} \vartheta(\theta) \mathrm{d}\theta =$$
$$\limsup_{t\to+\infty} \frac{1}{T_0} \int_t^{t+T_0} \min\left\{ 2c_1\mu_1\lambda_{n+1} - \frac{2c_3^2}{c_1\mu_1} \| \boldsymbol{u}_1 \|_V^2 - \frac{c_3^2}{2c_1\sqrt{\mu_1\nu}} \| \boldsymbol{\omega}_2 \|_{1,2}^2 - \frac{8\nu_r^2}{c_1\nu} - \right.$$
$$\left. 2c_4(\| \boldsymbol{u}_1 \|_V + \| \boldsymbol{u}_1 \|_V^2 + \| \boldsymbol{u}_2 \|_V^2),\ c_1\nu\lambda_{n+1}^* + 8\nu_r - \frac{c_3^2}{2c_1\sqrt{\mu_1\nu}} \| \boldsymbol{\omega}_2 \|_{1,2}^2 - \frac{4\nu_r^2}{c_1\mu_1} \right\} \mathrm{d}\theta \leqslant$$
$$\min\left\{ 2c_1\mu_1\lambda_{n+1} + \frac{8\nu_r^2}{c_1\nu} + \right.$$
$$\limsup_{t\to+\infty} \frac{1}{T_0} \int_t^{t+T_0} \left[\frac{2c_3^2}{c_1\mu_1} \| \boldsymbol{u}_1 \|_V^2 + \frac{c_3^2}{2c_1\sqrt{\mu_1\nu}} \| \boldsymbol{\omega}_2 \|_{1,2}^2 + 2c_4(\| \boldsymbol{u}_1 \|_V + \| \boldsymbol{u}_1 \|_V^2 + \| \boldsymbol{u}_2 \|_V^2) \right] \mathrm{d}\theta,$$
$$\left. c_1\nu\lambda_{n+1}^* + 8\nu_r + \frac{4\nu_r^2}{c_1\mu_1} + \frac{c_3^2}{2c_1\sqrt{\mu_1\nu}} \limsup_{t\to+\infty} \frac{1}{T_0} \int_t^{t+T_0} \| \boldsymbol{\omega}_2 \|_{1,2}^2 \mathrm{d}\theta \right\} \leqslant$$

$$\min\left\{2c_1\mu_1\lambda_{n+1}+\frac{8\nu_r^2}{c_1\nu}+c_4+c_5\sum_{i=1}^{2}\|(\boldsymbol{f}_i,\tilde{\boldsymbol{f}}_i)\|_{L^2(\tau,+\infty;\hat{V}^*)}^2,\right.$$
$$\left.c_1\nu\lambda_{n+1}^*+8\nu_r+\frac{4\nu_r^2}{c_1\mu_1}+\frac{c_3^2}{2c_1^2\sqrt{\mu_1\nu}\min\{\mu_1,\nu\}}\|(\boldsymbol{f}_2,\tilde{\boldsymbol{f}}_2)\|_{L^2(\tau,+\infty;\hat{V}^*)}^2\right\}<+\infty, \tag{25}$$

其中 $c_5:=\dfrac{2T_0+2L}{T_0\min\{c_1\mu_1,c_1\nu\}}\left(\dfrac{2c_3^2}{3c_1^2\mu_1^2}+\dfrac{c_4}{c_1\mu_1}+\dfrac{c_3^2}{2c_1^2\nu\sqrt{\mu_1\nu}}\right)$.类似地，可推得

$$\liminf_{t\to+\infty}\frac{1}{T_0}\int_t^{t+T_0}\vartheta(\theta)\mathrm{d}\theta\geqslant$$
$$\min\left\{2c_1\mu_1\lambda_{n+1}-\frac{8\nu_r^2}{c_1\nu}-c_4-c_5\sum_{i=1}^{2}\|(\boldsymbol{f}_i,\tilde{\boldsymbol{f}}_i)\|_{L^2(\tau,+\infty;\hat{V}^*)}^2,\right.$$
$$\left.c_1\nu\lambda_{n+1}^*+8\nu_r-\frac{4\nu_r^2}{c_1\mu_1}-\frac{c_3^2}{2c_1^2\sqrt{\mu_1\nu}\min\{\mu_1,\nu\}}\|(\boldsymbol{f}_2,\tilde{\boldsymbol{f}}_2)\|_{L^2(\tau,+\infty;\hat{V}^*)}^2\right\}.$$

由于当 $n\to+\infty$ 时，$\lambda_n\to+\infty$ 且 $\lambda_n^*\to+\infty$，故可取 λ_n 和 λ_n^* 满足式(9)，进而可得

$$\liminf_{t\to+\infty}\frac{1}{T_0}\int_t^{t+T_0}\vartheta(\theta)\mathrm{d}\theta>0,\qquad T_0>0. \tag{26}$$

由式(25)和式(26)，可知 $\vartheta(t)$ 满足条件(5).

(Ⅱ) 验证 $\varrho(t)$ 满足条件(6).由 $\varrho(t)$ 的表达式可知，我们需要得到 $\|\boldsymbol{U}_t\|_\gamma$ 和 $\|\boldsymbol{W}_t\|_\gamma$ 的相关估计.为此，用 $\boldsymbol{U}$ 和 $\boldsymbol{W}$ 分别与式(10)的第一式和第二式作对偶积，可得

$$\begin{cases}\dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{U}\|^2+\langle 2\mu_1\mathrm{A}_1\boldsymbol{U},\boldsymbol{U}\rangle+\langle B_1(\boldsymbol{u}_1,\boldsymbol{u}_1)-B_1(\boldsymbol{u}_2,\boldsymbol{u}_2),\boldsymbol{U}\rangle+\langle\mathscr{K}_1(\boldsymbol{W}),\boldsymbol{U}\rangle+\\ \qquad\langle N(\boldsymbol{u}_1)-N(\boldsymbol{u}_2),\boldsymbol{U}\rangle=\langle\boldsymbol{F},\boldsymbol{U}\rangle+\langle\boldsymbol{g}(t,\boldsymbol{u}_{1,t})-\boldsymbol{g}(t,\boldsymbol{u}_{2,t}),\boldsymbol{U}\rangle,\\ \dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{W}\|^2+\langle\nu\mathrm{A}_2\boldsymbol{W},\boldsymbol{W}\rangle+\langle B_2(\boldsymbol{u}_1,\boldsymbol{\omega}_1)-B_2(\boldsymbol{u}_2,\boldsymbol{\omega}_2),\boldsymbol{W}\rangle+\langle\mathscr{K}_2(\boldsymbol{U},\boldsymbol{W}),\boldsymbol{W}\rangle=\\ \qquad\langle\tilde{\boldsymbol{F}},\boldsymbol{W}\rangle+\langle\tilde{\boldsymbol{g}}(t,\boldsymbol{\omega}_{1,t})-\tilde{\boldsymbol{g}}(t,\boldsymbol{\omega}_{2,t}),\boldsymbol{W}\rangle.\end{cases} \tag{27}$$

由引理 1，易得

$$2c_1\mu_1\|\boldsymbol{U}\|_V^2\leqslant\langle 2\mu_1\mathrm{A}_1\boldsymbol{U},\boldsymbol{U}\rangle,\ c_1\nu\|\boldsymbol{W}\|_{1,2}^2\leqslant\langle\nu\mathrm{A}_2\boldsymbol{W},\boldsymbol{W}\rangle. \tag{28}$$

根据文献[53]中的式(3.28)，可知

$$\langle N(\boldsymbol{u}_1)-N(\boldsymbol{u}_2),\boldsymbol{U}\rangle\geqslant 0. \tag{29}$$

经直接计算，可得

$$\langle\mathscr{K}_1(\boldsymbol{W}),\boldsymbol{U}\rangle+\langle\mathscr{K}_2(\boldsymbol{U},\boldsymbol{W}),\boldsymbol{W}\rangle=4\nu_r\|\boldsymbol{W}\|^2, \tag{30}$$

$$\langle\boldsymbol{F},\boldsymbol{U}\rangle+\langle\tilde{\boldsymbol{F}},\boldsymbol{W}\rangle\leqslant\|\boldsymbol{F}\|_{V^*}\|\boldsymbol{U}\|_V+\|\tilde{\boldsymbol{F}}\|_{H^{-1}}\|\boldsymbol{W}\|_{1,2}\leqslant$$
$$\frac{1}{c_1\mu_1}\|\boldsymbol{F}\|_{V^*}^2+\frac{1}{c_1\nu}\|\tilde{\boldsymbol{F}}\|_{H^{-1}}^2+\frac{c_1\mu_1}{4}\|\boldsymbol{U}\|_V^2+\frac{c_1\nu}{4}\|\boldsymbol{W}\|_{1,2}^2, \tag{31}$$

$$\langle\boldsymbol{g}(t,\boldsymbol{u}_{1,t})-\boldsymbol{g}(t,\boldsymbol{u}_{2,t}),\boldsymbol{U}\rangle+\langle\tilde{\boldsymbol{g}}(t,\boldsymbol{\omega}_{1,t})-\tilde{\boldsymbol{g}}(t,\boldsymbol{\omega}_{2,t}),\boldsymbol{W}\rangle\leqslant L(\|\boldsymbol{U}_t\|_\gamma^2+\|\boldsymbol{W}_t\|_\gamma^2). \tag{32}$$

另外，由引理 1 及 Gagliardo-Nirenberg 不等式和 Young’s 不等式，可得

$$|\langle B_1(\boldsymbol{u}_1,\boldsymbol{u}_1)-B_1(\boldsymbol{u}_2,\boldsymbol{u}_2),\boldsymbol{U}\rangle|=|\langle B_1(\boldsymbol{U},\boldsymbol{u}_1)-B_1(\boldsymbol{u}_2,\boldsymbol{U}),\boldsymbol{U}\rangle|\leqslant$$
$$c_3\|\boldsymbol{U}\|^{1/2}\|\nabla\boldsymbol{U}\|\|\boldsymbol{u}_1\|\|\mathrm{D}^2\boldsymbol{U}\|^{1/2}\leqslant\frac{64c_3^4}{c_1^3\mu_1^3}\|\boldsymbol{u}_1\|^4\|\boldsymbol{U}\|^2+\frac{c_1\mu_1}{4}\|\boldsymbol{U}\|_V^2, \tag{33}$$

$$|\langle B_2(\boldsymbol{u}_1,\boldsymbol{\omega}_1)-B_2(\boldsymbol{u}_2,\boldsymbol{\omega}_2),\boldsymbol{W}\rangle|=$$
$$|\langle B_2(\boldsymbol{U},\boldsymbol{\omega}_1)-B_2(\boldsymbol{u}_2,\boldsymbol{W}),\boldsymbol{W}\rangle|\leqslant\|\boldsymbol{U}\|_{L^\infty}\|\boldsymbol{\omega}_1\|\|\nabla\boldsymbol{W}\|\leqslant$$
$$c\|\boldsymbol{U}\|^{1/2}\|\mathrm{D}^2\boldsymbol{U}\|^{1/2}\|\boldsymbol{\omega}_1\|\|\nabla W\|\leqslant$$
$$\tilde{c}\|\boldsymbol{\omega}_1\|^4\|\boldsymbol{U}\|^2+\frac{c_1\mu_1}{4}\|\boldsymbol{U}\|_V^2+\frac{c_1\nu}{4}\|\boldsymbol{W}\|_{1,2}^2, \tag{34}$$

其中 $\tilde{c}$ 为与 c_1,μ_1,ν 有关的常数.将式(28)—(34)代入式(27)，得

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\boldsymbol{U}\|^2+\|\boldsymbol{W}\|^2)+8\nu_r\|\boldsymbol{W}\|^2+c_1\nu\|\boldsymbol{W}\|_{1,2}^2+ \frac{5c_1\mu_1}{2}\|\boldsymbol{U}\|_V^2-\frac{16c_3^4}{c_1^3\mu_1^3}\|\boldsymbol{u}_1\|^4\|\boldsymbol{U}\|^2-2\tilde{c}\|\boldsymbol{\omega}_1\|^4\|\boldsymbol{U}\|^2\leqslant 2L(\|\boldsymbol{U}_t\|_\gamma^2+\|\boldsymbol{W}_t\|_\gamma^2)+\frac{2}{c_1\mu_1}\|\boldsymbol{F}\|_{V^*}^2+\frac{2}{c_1\nu}\|\tilde{\boldsymbol{F}}\|_{H^{-1}}{}^2. \tag{35}$$

由式(4)可知，对于 $\forall\varepsilon>0$，存在 $T_1>\tau$，使得

$$\|\boldsymbol{u}_1(t)\|^4+\|\boldsymbol{\omega}_1(t)\|^4<\varepsilon+\frac{1}{\min\{c_1\mu_1,\ c_1\nu\}}\|(\boldsymbol{f}_1,\tilde{\boldsymbol{f}}_1)\|_{L^2(\tau,+\infty;\hat{V}^*)}^4,\qquad \forall t\geqslant T_1,$$

因此，由式(8)和式(35)，可得

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\boldsymbol{U}\|^2+\|\boldsymbol{W}\|^2)+\delta(\|\boldsymbol{U}\|^2+\|\boldsymbol{W}\|^2)\leqslant \frac{2}{c_1\min\{\mu_1,\ \nu\}}\|(\boldsymbol{F},\tilde{\boldsymbol{F}})\|_{\hat{V}^*}^2+2L(\|\boldsymbol{U}_t\|_\gamma^2+\|\boldsymbol{W}_t\|_\gamma^2),\qquad \forall t\geqslant T_1,$$

其中 $\delta:=\min\{2c_1\mu_1\lambda_1,\ c_1\nu\lambda_1^*+8\nu_r\}$.运用与文献[54]中的式(2.34)相同的证明过程，由上式可推得

$$\|\boldsymbol{U}_t\|_\gamma^2+\|\boldsymbol{W}_t\|_\gamma^2\leqslant \mathrm{e}^{-(\delta-2L)(t-\tau)}(\|\boldsymbol{\phi}_1-\boldsymbol{\phi}_2\|_\gamma^2+\|\boldsymbol{\psi}_1-\boldsymbol{\psi}_2\|_\gamma^2)+ \frac{2}{\min\{c_1\mu_1,\ c_1\nu\}}\int_\tau^t \mathrm{e}^{-(\delta-2L)(t-\theta)}\|(\boldsymbol{F},\tilde{\boldsymbol{F}})(\theta)\|_{\hat{V}^*}^2\mathrm{d}\theta,\qquad \forall t\geqslant T_1>\tau. \tag{36}$$

接下来，我们验证 $\varrho(t)$ 满足条件式(6).首先，因为 $2L<\delta$，故由式(4)、(36)和 Hölder 不等式，可得

$$\begin{aligned}
&\lim_{t\to+\infty}\frac{1}{T_0}\int_t^{t+T_0}2L(\|\boldsymbol{U}_\theta\|_\gamma\|\mathrm{Q}_n\boldsymbol{U}(\theta)\|+\|\boldsymbol{W}_\theta\|_\gamma\|\mathrm{Q}_n\boldsymbol{W}(\theta)\|)\mathrm{d}\theta\leqslant\\
&\lim_{t\to+\infty}\frac{2L}{\min\{c_1\mu_1,c_1\nu\}^{1/2}\sqrt{T_0}}\|(\boldsymbol{F},\tilde{\boldsymbol{F}})\|_{L^2(\tau,+\infty;\hat{V}^*)}\left[\int_t^{t+T_0}(\|\boldsymbol{U}_\theta\|_\gamma^2+\|\boldsymbol{W}_\theta\|_\gamma^2)\mathrm{d}\theta\right]^{1/2}\leqslant\\
&\lim_{t\to+\infty}\frac{2L\|(\boldsymbol{F},\tilde{\boldsymbol{F}})\|_{L^2(\tau,+\infty;\hat{V}^*)}}{\min\{c_1\mu_1,c_1\nu\}^{1/2}\sqrt{T_0}}\left[\int_t^{t+T_0}\mathrm{e}^{-(\delta-2L)(\theta-\tau)}(\|\boldsymbol{\phi}_1-\boldsymbol{\phi}_2\|_\gamma^2+\|\boldsymbol{\psi}_1-\boldsymbol{\psi}_2\|_\gamma^2)+\right.\\
&\left.\frac{2}{\min\{c_1\mu_1,c_1\nu\}}\int_t^{t+T_0}\int_\tau^\theta\mathrm{e}^{-(\delta-2L)(\theta-r)}\|(\boldsymbol{F},\tilde{\boldsymbol{F}})\|_{\hat{V}^*}^2\mathrm{d}r\mathrm{d}\theta\right]^{1/2}\leqslant\\
&\lim_{t\to+\infty}\frac{2\sqrt{2}L\|(\boldsymbol{F},\tilde{\boldsymbol{F}})\|_{L^2(\tau,+\infty;\hat{V}^*)}}{\min\{c_1\mu_1,c_1\nu\}\sqrt{T_0}}\left[\int_t^{t+T_0}\int_\tau^{T_2}\mathrm{e}^{-(\delta-2L)(\theta-r)}\|(\boldsymbol{F},\tilde{\boldsymbol{F}})\|_{\hat{V}^*}^2\mathrm{d}r\mathrm{d}\theta+\right.\\
&\left.\int_t^{t+T_0}\int_{T_2}^\theta\mathrm{e}^{-(\delta-2L)(\theta-r)}\|(\boldsymbol{F},\tilde{\boldsymbol{F}})\|_{\hat{V}^*}^2\mathrm{d}r\mathrm{d}\theta\right]^{1/2}\leqslant\\
&\lim_{t\to+\infty}\frac{2\sqrt{2}L\|(\boldsymbol{F},\tilde{\boldsymbol{F}})\|_{L^2(\tau,+\infty;\hat{V}^*)}}{\min\{c_1\mu_1,c_1\nu\}\sqrt{T_0}}\left[\frac{1}{\delta-2L}\|(\boldsymbol{F},\tilde{\boldsymbol{F}})\|_{L^2(\tau,+\infty;\hat{V}^*)}^2\times\right.\\
&\left.(\mathrm{e}^{-(\delta-2L)(t-T_2)}-\mathrm{e}^{-(\delta-2L)(t+T_0-T_2)})+\frac{T_0}{\delta-2L}\|(\boldsymbol{F},\tilde{\boldsymbol{F}})\|_{L^\infty(T_2,+\infty;\hat{V}^*)}^2\right]^{1/2}\leqslant\\
&\lim_{t\to+\infty}\frac{2\sqrt{2}L\|(\boldsymbol{F},\tilde{\boldsymbol{F}})\|_{L^2(\tau,+\infty;\hat{V}^*)}}{\min\{c_1\mu_1,c_1\nu\}(\delta-2L)^{1/2}}\|(\boldsymbol{F},\tilde{\boldsymbol{F}})\|_{L^\infty(T_2,+\infty;\hat{V}^*)},\qquad\forall T_2\geqslant\tau.
\end{aligned}\tag{37}$$

由式(7)可知，当 $t\to+\infty$ 时，$\|(\boldsymbol{F},\tilde{\boldsymbol{F}})(t)\|_{\hat{V}^*}\to 0$.所以，对于 $\forall\varepsilon>0$，可在式(37)中选取 T_2(充分大)，使得 $T_2>T_1>\tau$ 且

$$\|(\boldsymbol{F},\tilde{\boldsymbol{F}})\|_{L^\infty(T_2,+\infty;\hat{V}^*)}<\varepsilon. \tag{38}$$

由 ε 的任意性，并结合式(37)和式(38)，可知

$$\lim_{t\to+\infty}\frac{1}{T_0}\int_t^{t+T_0}2L(\|\boldsymbol{U}_\theta\|_\gamma\|\mathrm{Q}_n\boldsymbol{U}(\theta)\|+\|\boldsymbol{W}_\theta\|_\gamma\|\mathrm{Q}_n\boldsymbol{W}(\theta)\|)\mathrm{d}\theta=0. \tag{39}$$

另外，在式(7)的条件下，易得

$$\begin{aligned}\lim_{t\to+\infty}\frac{1}{T_0}\int_t^{t+T_0}[&2c_3\|\mathrm{P}_n\boldsymbol{U}\|^{1/2}\|\mathrm{P}_n\boldsymbol{U}\|_V^{1/2}\|\mathrm{Q}_n\boldsymbol{U}\|_V(\|\boldsymbol{u}_1\|^{1/2}\|\boldsymbol{u}_1\|_V^{1/2}+\|\boldsymbol{u}_2\|^{1/2}\|\boldsymbol{u}_2\|_V^{1/2})+\\&2c_3\|\boldsymbol{u}_1\|^{1/2}\|\boldsymbol{u}_1\|_V^{1/2}\|\mathrm{P}_n\boldsymbol{W}\|^{1/2}\|\mathrm{P}_n\boldsymbol{W}\|_{1,2}^{1/2}\|\mathrm{Q}_n\boldsymbol{W}\|_{1,2}+\\&2c_4(1+\|\boldsymbol{u}_2\|_V)\|\mathrm{P}_n\boldsymbol{U}\|\|\mathrm{Q}_n\boldsymbol{U}\|_V+2\|\boldsymbol{F}\|_{V^*}\|\mathrm{Q}_n\boldsymbol{U}\|_V+4\nu_r\|\mathrm{P}_n\boldsymbol{W}\|\|\mathrm{Q}_n\boldsymbol{U}\|_V+\\&2c_3\|\mathrm{P}_n\boldsymbol{U}\|^{1/2}\|\mathrm{P}_n\boldsymbol{U}\|_V^{1/2}\|\boldsymbol{\omega}_2\|_{1,2}\|\mathrm{Q}_n\boldsymbol{W}\|^{1/2}\|\mathrm{Q}_n\boldsymbol{W}\|_{1,2}^{1/2}+4\nu_r\|\mathrm{P}_n\boldsymbol{U}\|\|\mathrm{Q}_n\boldsymbol{W}\|_{1,2}+\\&2\|\boldsymbol{F}\|_{V^*}\|\mathrm{Q}_n\boldsymbol{U}\|_V+2\|\tilde{\boldsymbol{F}}\|_{H^{-1}}\|\mathrm{Q}_n\boldsymbol{W}\|_{1,2}]\mathrm{d}\theta=0.\end{aligned} \tag{40}$$

进而结合式(39)和(40)，可知 $\varrho(t)$ 满足条件(6).

第三步：证明问题(3)具有有限的确定模个数.运用引理 3 和式(24)，可推得

$$\lim_{t\to+\infty}\zeta(t)=\lim_{t\to+\infty}(\|\mathrm{Q}_n\boldsymbol{U}(t)\|^2+\|\mathrm{Q}_n\boldsymbol{W}(t)\|^2)=0,$$

即式(11)成立，定理 1 得证.

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