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一维弱噪声随机 Burgers 方程的奇摄动解*

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摘要: 讨论了一类有界区域上具有有色噪声干扰的随机 Burgers 方程奇摄动解,其波动率服从弱噪声 Ornstein-Uhlenbeck(O-U)过程.由波运动的转移概率密度函数满足的后向 Kolmogorov 方程,得到随机 Burgers 的期望所满足的后向 Kolmogorov 方程.由于期望满足的后向 Kolmogorov 方程的初边值问题条件涉及到一类确定性 Burgers 方程的解,因此该问题实际上是 Burgers 方程和 Kolmogorov 方程的联立形式.首先,应用奇摄动方法,对一类确定性 Burgers 方程进行了正则渐近展开,由 Schauder 估计、Ascoli-Arzela 定理证明了非线性抛物方程渐近解的有界性与存在性,由 Lax-Milgram 定理证明了线性抛物方程渐近解的有界性与存在性,得到波速率的形式渐近解.其次,由奇摄动理论,对期望满足的方程进行了奇摄动渐近展开和边界层矫正,由二阶线性偏微分方程理论,得到边界层函数渐近解存在且有界.应用极值原理、De-Giorgi 迭代技术分别证明了波速率和波期望渐近解的余项有界,得到渐近解的一致有效性.

关 键 词: 奇摄动; 随机 Burgers 方程; 平均速率; Ornstein-Uhlenbeck(O-U)过程; 一致有效估计

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引 言

确定性 Burgers 方程用于流体的有限平面波问题的研究已有很长的历史,得到了大量的结果.对于有界区域上的 Burgers 方程,Kreiss 等[1],Laforgue、O' Malley[2-5] 进行了详细的讨论.但是具有噪声干扰的随机 Burgers 方程实际上更具有意义.Villarroel[6] 研究了在白噪声与有色噪声影响下的 Burgers 方程的解,得到了孤波特征、冲击波位置与抵达时间等,从而得到了波的平均速率.对于有界区域上的 Burgers 方程,Xiu 等[7] 讨论了随机边界扰动下的情况.对于随机影响出现在黏性系数扰动的情况,Le Maitre 等[8] 讨论了随机干扰服从 Gauss(高斯)分布的情形,分析了其对流体流动的影响.高飞[9] 应用格子 Boltzmann 模拟,讨论了其数值解.付新刚[10] 则应用数值分析,讨论了边界扰动的情形.近年来人们也将上述分析方法应用于 Kadomtsev-Petviashvili(KP)方程,如 Villarroel[11] 讨论了在白色噪声和有色噪声干扰下的 KP 方程,Xue[12] 在有界球几何和横向扰动共同作用下, 利用奇摄动方法讨论了 SKP 的特解, Yermakou 等[13], Ghanmi 等[14] 讨论了随机 KP 方程的数值解, Chakravarty, Kodama[15] 讨论了随机 KP 方程的线孤子解.

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本文讨论了一类黏性系数受到有色噪声干扰的随机 Burgers 方程,有色噪声服从一个弱噪声 Ornstein-Uhlenbeck 过程.本文的目的在于讨论相应波运动的平均速率,其平均速率服从后向的 Kolmogorov 方程,其初边值条件由一个确定性的 Burgers 方程得到.由于弱噪声的影响,应用奇摄动展开的方法确定了波运动平均速率的渐近解,得到了其一致有效性.

1　模型建立

现在作如下的假设:

[H1]　$Q_T=[0,T]\times[0,d]$,$\partial_P Q_T$ 为 Q_T 的边界区域,其中 $t\in[0,T]$,$[0,d]\subset \mathbf{R}$ 是有界区域.

[H2]　$a(t,x)$,$b(t,x)$ 是已知任意阶连续可微函数,其中 $b(t,x)\neq 0$,$a(t,x)>0$.

[H3]　Ψ 在 $[0,d]$ 上连续可微.

[H4]　$\alpha(t)$,$\beta(t)$ 是与 ε 无关的函数且在 $[0,d]$ 上连续可微.

考虑在吸收边界 $[0,d]$ 上的波运动轨迹 $u(t,x)$ 满足的随机偏微分方程:

$$\begin{cases}u_t+(u-\xi')u_z-\dfrac{\alpha^2}{2}u_{zz}=0, & u(t=0,z)=\Psi(z+\xi_0),\\ \mathrm{d}\xi=a(t,\xi)\mathrm{d}t+\sqrt{\varepsilon}\,b(t,\xi)\mathrm{d}\omega(t), & \xi(t=0)=\xi_0,\end{cases}\tag{1}$$

其中,α 是常数,ε 是大于零的小参数.

通过函数变换 $u(t,z)=g(t,z+\xi(t))$,可得 $g(t,z)$ 满足

$$\begin{cases}g_t+gg_z-\dfrac{(\alpha^2-\varepsilon\, b^2(t,z))}{2}g_{zz}=0,\\ g(0,z)=\Psi(z),\ g(t,0)=\alpha(t),\ g(t,d)=\beta(t).\end{cases}\tag{2}$$

设随机微分方程 $\mathrm{d}\xi=a(t,\xi)\mathrm{d}t+\sqrt{\varepsilon}\,b(t,\xi)\mathrm{d}\omega(t)$ 的转移概率密度为 $p(t,x,s,y)$,$p(t,x,s,y)$ 满足后向 Kolmogorov 方程,即

$$\begin{cases}\dfrac{\partial p(t,x,s,y)}{\partial t}+a(t,x)p_x(t,x,s,y)+\dfrac{\varepsilon b^2(t,x)}{2}\dfrac{\partial^2 p(t,x,s,y)}{\partial x^2}=0,\\ \lim\limits_{t\to s}p(t,x,s,y)=\sigma(x-y).\end{cases}\tag{3}$$

由

$$v(t,x)=E_{x,t}g(T,x+\xi(t))=\int g(T,y)p(t,x,T,y)\mathrm{d}y,$$

可得 $v(t,x)$ 满足的后向 Kolmogorov 方程:

$$\begin{cases}\dfrac{\partial v(t,x)}{\partial t}+a(t,x)v_x(t,x,s,y)+\dfrac{\varepsilon b^2(t,x)}{2}\dfrac{\partial^2 v(t,x)}{\partial x^2}=0,\\ v(T,x)=g(T,x),\ v(t,0)=\alpha(t),\ v(t,d)=\beta(t).\end{cases}\tag{4}$$

2　形式展开

2.1　波速率的形式渐近展开

首先对式(2)做形式渐近展开:

$$\begin{cases}g(t,z,\varepsilon)=g_0(t,z)+\varepsilon g_1(t,z)+\varepsilon^2 g_2(t,z)+\cdots,\\ g(0,z)=\Psi(z),\ g(t,0)=\alpha(t),\ g(t,d)=\beta(t).\end{cases}\tag{5}$$

关于 ε 做摄动展开,并比较 ε 的同次幂系数,可得

$$\sum_{k=0}^{N}\varepsilon^k g_{kt}+\left(\sum_{k=0}^{N}\varepsilon^k g_k\right)\left(\sum_{k=0}^{N}\varepsilon^k g_{kz}\right)-\frac{\alpha^2-\varepsilon b^2(t,z)}{2}\left(\sum_{k=0}^{N}\varepsilon^k g_{kzz}\right)=0,$$

$$\begin{cases}g_{0t}+g_0g_{0z}-\dfrac{\alpha^2}{2}g_{0zz}=0,\\ g_0(0,z)=\Psi(z),\ g_0(t,0)=\alpha(t),\ g_0(t,d)=\beta(t),\end{cases}\tag{6}$$

$$\begin{cases}g_{1t}+g_0g_{1z}+g_1g_{0z}-\dfrac{\alpha^2}{2}g_{1zz}+\dfrac{b^2(t,z)}{2}g_{0zz}=0,\\ g_1(0,z)=0,\ g_1(t,0)=0,\ g_1(t,d)=0,\end{cases}\tag{7}$$

$$\vdots$$

$$\begin{cases}g_{nt}+\displaystyle\sum_{i=1}^{n}g_ig_{(n-i)z}-\dfrac{\alpha^2}{2}g_{nzz}+\dfrac{b^2(t,z)}{2}g_{(n-1)zz}=0,\\ g_n(0,z)=0,\ g_n(t,0)=0,\ g_n(t,d)=0.\end{cases}\tag{8}$$

引理 1 式(6)解的存在唯一且有界.

证 令

$$\varphi=(g_0-k)^+=\begin{cases}g_0-k, & g_0\geqslant k,\\ 0, & g_0<k,\end{cases}\tag{9}$$

其中 k 为常数,$k>l(l=\sup_{\partial_PQ_T}|g_0|)$.

在式(6)两边同乘 φ, 可得

$$\iint_{Q_T}\left(\varphi_t\varphi+(\varphi+k)\varphi_z\varphi-\frac{\alpha^2}{2}\varphi_{zz}\varphi\right)\mathrm{d}t\mathrm{d}z=$$
$$\iint_{Q_T}\left(\frac{1}{2}(\varphi^2)_t+\frac{1}{3}(\varphi^3)_z+\frac{k}{2}(\varphi^2)_z-\frac{\alpha^2}{2}(\varphi\varphi_z)_z+\frac{\alpha^2}{2}(\varphi_z)^2\right)\mathrm{d}t\mathrm{d}z=0.$$

考虑到边界 ∂_PQ_T 上 $\varphi=0$,可得$\iint_{Q_T}\left(\frac{\alpha^2}{2}(\varphi_z)^2\right)\mathrm{d}t\mathrm{d}z=0$.从而有 $\varphi_z=0$,由边界条件可知 $\varphi=c=0$.则有 $|g_0|\leqslant k$,所以 g_0 有界.

由一般椭圆形方程的 Schauder 估计,同样可得 $g_{0z}(t,z)$ 的有界性.

因此, $g_{0z}(t,z)$ 是等度连续的,由 Ascoli-Arzela 定理,可得 $g_{0z}(t,z)$ 的存在唯一性.

引理 2 式(7)解的存在唯一且有界.

证 首先在 $W_2^{1,1}(Q_T)\times W_2^{1,1}(Q_T)$ 上构造一个泛函:

$$a(g_1,q)=\iint_{Q_T}\left(g_{1t}q_t+g_0g_{1t}q_t+g_{0z}g_1q_t+\frac{\alpha^2}{2}g_{1z}q_{tz}\right)\mathrm{e}^{-\theta t}\mathrm{d}t\mathrm{d}z,\tag{10}$$

显然 $a(g_1,q)$ 是双线性的,有界的.事实上,由 $g_0(t,z)$,$g_{0z}(t,z)$ 的有界性,可以证明 $a(g_1,q)$ 的强制性,即对于任意给定的 $q\in W_2^{1,1}(Q_T)$,有 $a(q,q)\geqslant\delta\|q\|^2_{W_2^{1,1}(Q_T)}$.

综合上述的结果,由 Lax-Milgram 定理,可以得到对 $W_2^{1,1}(Q_T)$ 上的任一有界线性泛函

$$F(q)=\iint_{Q_T}-\frac{b^2(t,z)}{2}g_{0zz}q_t\mathrm{e}^{-\theta t}\mathrm{d}t\mathrm{d}z,$$

恒存在唯一的 $g_1\in H$, 使得

$$F(q)=a(g_1,q),$$

其中对任意的 $q\in W_2^{1,1}(Q_T)$.所以式(7) 的解存在唯一,即 g_1 可解.

类似地,可得 $g_2,g_3,\cdots,g_N$ 的存在唯一性,从而得到式(2)的形式渐近解.

2.2 平均速率的形式渐近展开

对式(4)作形式渐近展开:

$$v(t,x)=\sum_{k=0}^{N}\varepsilon^k v_k(t,x)+\sum_{k=0}^{N}\varepsilon^k \Pi_k(t,\tau),\tag{11}$$

其中 $\tau=x/\varepsilon$.$\bar{v}(t,x)=\sum_{k=0}^{N}\varepsilon^k v_k(t,x)$ 是方程的正则部分.

$$\sum_{k=0}^{N}\varepsilon^k\frac{\partial v_k(t,x)}{\partial t}+\sum_{k=0}^{N}\varepsilon^k a(t,x)\frac{\partial v_k(t,x)}{\partial x}+\frac{\varepsilon}{2}\sum_{k=0}^{N}\varepsilon^k b^2(t,x)\frac{\partial^2 v_k(t,x)}{\partial x^2}=0,$$

比较 ε 的同次幂系数,可得

$$\begin{cases}\dfrac{\partial v_0(t,x)}{\partial t}+a(t,x)\dfrac{\partial v_0(t,x)}{\partial x}=0,\\ v_0(T,z)=g_0(T,z),\ v_0(t,d)=\beta(t),\end{cases}\tag{12}$$

$$\begin{cases}\dfrac{\partial v_1(t,x)}{\partial t}+a(t,x)\dfrac{\partial v_1(t,x)}{\partial x}+\dfrac{1}{2}b^2(t,x)\dfrac{\partial^2 v_0(t,x)}{\partial x^2}=0,\\ v_1(T,z)=g_1(T,z),\ v_1(t,d)=0,\end{cases}\tag{13}$$

$$\vdots$$

$$\begin{cases}\dfrac{\partial v_N(t,x)}{\partial t}+a(t,x)\dfrac{\partial v_N(t,x)}{\partial x}+\dfrac{1}{2}b^2(t,x)\dfrac{\partial^2 v_{N-1}(t,x)}{\partial x^2}=0,\\ v_N(T,z)=g_N(T,z),\ v_N(t,d)=0.\end{cases}\tag{14}$$

由于式(12)~(14)均为一阶齐次线性偏微分方程,因此可得 $v_0(t,x),v_1(t,x),\cdots,v_N(t,x)$, 即正则部分可解.

将式(11)代入式(4),可得

$$\sum_{k=0}^{N}\varepsilon^k\frac{\partial v_k(t,x)}{\partial t}+\sum_{k=0}^{N}\varepsilon^k a(t,x)\frac{\partial v_k(t,x)}{\partial x}+\frac{\varepsilon}{2}\sum_{k=0}^{N}\varepsilon^k b^2(t,x)\frac{\partial^2 v_k(t,x)}{\partial x^2}+$$
$$\sum_{k=0}^{N}\varepsilon^k\frac{\partial \Pi_k(t,\tau)}{\partial t}+\sum_{k=0}^{N}\varepsilon^{k-1}a(t,\varepsilon\tau)\frac{\partial \Pi_k(t,\tau)}{\partial \tau}+$$
$$\frac{1}{2}\sum_{k=0}^{N}\varepsilon^{k-1}b^2(t,\varepsilon\tau)\frac{\partial^2\Pi_k(t,\tau)}{\partial\tau^2}=0,$$

其中 $v_i(t,x),i=1,2,\cdots,N$ 为已知函数,且

$$\sum_{k=0}^{N}\varepsilon^k v_k(t,0)+\sum_{k=0}^{N}\varepsilon^k\Pi_k(t,0)=\sum_{k=0}^{N}\varepsilon^k(v_k(t,0)+\Pi_k(t,0))=\alpha(t).$$

按 ε 的幂指数形式分别对 $a(t,\varepsilon\tau),b(t,\varepsilon\tau)$ 进行 Taylor(泰勒)展开,即

$$a(t,\varepsilon\tau)\approx a_0(t,0)+\varepsilon a_1(t,\tau)+\cdots+\varepsilon^N a_N(t,\tau)=\sum_{k=0}^{N}\varepsilon^k a_k,$$

$$b(t,\varepsilon\tau)\approx b_0(t,0)+\varepsilon b_1(t,\tau)+\cdots+\varepsilon^N b_N(t,\tau)=\sum_{k=0}^{N}\varepsilon^k b_k,$$

其中 $a_i,b_i,i=1,2,\cdots,N$ 为已知函数.

对比 ε 的同次幂系数,可得

$$a_0(t,0)\frac{\partial\Pi_0}{\partial\tau}+\frac{1}{2}b_0^2(t,0)\frac{\partial^2\Pi_0}{\partial\tau^2}=0,$$

$$a_0(t,0)\frac{\partial\Pi_1}{\partial\tau}+\frac{1}{2}b_0^2\frac{\partial^2\Pi_1}{\partial\tau^2}=H_1(t,\tau),$$

$$\vdots$$

$$a_0(t,0)\frac{\partial\Pi_N}{\partial\tau}+\frac{1}{2}b_0^2\frac{\partial^2\Pi_N}{\partial\tau^2}=H_N(t,\tau),$$

其中 $H_i(t,\tau),i=1,2,\cdots,N$ 是关于 $a_j(t,\tau),b_j(t,\tau)$, $j\leqslant i$, $j=1,2,\cdots,N$ 的已知函数.所以

$$\begin{cases}a_0(t,0)\dfrac{\partial\Pi_0}{\partial\tau}+\dfrac{1}{2}b_0^2(t,0)\dfrac{\partial^2\Pi_0}{\partial\tau^2}=0,\\ \Pi_0(t,0)=\alpha(t)-v_0(t,0),\ \Pi_0(t,+\infty)=0,\end{cases}\tag{15}$$

$$\begin{cases}a_0(t,0)\dfrac{\partial\Pi_1}{\partial\tau}+\dfrac{1}{2}b_0^2\dfrac{\partial^2\Pi_1}{\partial\tau^2}=H_1(t,\tau),\\ \Pi_1(t,0)=0,\ \Pi_1(t,+\infty)=0,\end{cases}\tag{16}$$

$$\vdots$$

$$\begin{cases}a_0(t,0)\dfrac{\partial\Pi_N}{\partial\tau}+\dfrac{1}{2}b_0^2\dfrac{\partial^2\Pi_N}{\partial\tau^2}=H_N(t,\tau),\\ \Pi_N(t,0)=0,\ \Pi_N(t,+\infty)=0.\end{cases}\tag{17}$$

考虑到式(15)是退化的二阶线性偏微分方程,可解得

$$\Pi_0(t,\tau)=(\alpha(t)-v_0(t,0))\mathrm{e}^{-[2a_0(t,0)/b_0^2(t,0)]\tau}.$$

同理,由式(16)和式(17)是非齐次的二阶线性微分方程可知:$\Pi_1(t,\tau),\Pi_2(t,\tau),\cdots,\Pi_N(t,\tau)$ 可解.

3 余项估计

3.1 式(2)的余项估计

定理 1 式(5)的渐近解的余项 R_1 满足

$$|R_1|\leqslant \mathrm{e}^{\lambda T}(l+C|\Omega|^{1-2/p}2^{(3p-4)/(2p-4)}\|F\|_{L^\infty(Q_T)}).$$

证 对于上述的形式渐近解

$$g(t,x)=\bar{g}(t,x)+\varepsilon^{N+1}R_1,\tag{18}$$

其中 $\bar{g}(t,x)=\sum_{k=0}^{N}\varepsilon^k g_k(t,x)$.此时将式(18)代入式(2)中,可得 $R_1(t,x)$ 满足

$$\frac{\partial R_1}{\partial t}+\bar{g}(t,x)\frac{\partial R_1}{\partial x}+\frac{\partial\bar{g}(t,x)}{\partial x}R_1+\varepsilon^{N+1}R_1\frac{\partial R_1}{\partial x}-\frac{\alpha^2-\varepsilon b^2(t,x)}{2}\frac{\partial^2R_1}{\partial x^2}=H(t,x),$$

其中 $H(t,x)$ 为已知函数.

令 $R_1=\mathrm{e}^{\lambda t}P$,其中 $\lambda>0$, 可得

$$\frac{\partial P}{\partial t}+\bar{g}(t,x)\frac{\partial P}{\partial x}+\left(\frac{\partial\bar{g}(t,x)}{\partial x}+\lambda\right)P+\varepsilon^{N+1}\mathrm{e}^{\lambda t}P\frac{\partial P}{\partial x}-\frac{\alpha^2-\varepsilon b^2(t,x)}{2}\frac{\partial^2P}{\partial x^2}=\mathrm{e}^{-\lambda t}H(t,x).\tag{19}$$

令

$$\varphi=(P-k_1)_+\chi_{[t_1,t_2]}=\begin{cases}P-k_1, & P\geqslant k_1,\\ 0, & P<k_1,\end{cases}$$

其中 k_1 是常数,$k_1>l$,$l=\sup_{\partial_P Q_T}|P|$,$\chi_{[t_1,t_2]}(t)$ 为区间$[t_1,t_2]$的特征函数.

在式(19)两边同时乘上检验函数 φ,且取积分,可得

$$\iint_{Q_T}\varphi_t\varphi\mathrm{d}x\mathrm{d}t+\iint_{Q_T}\bar g\varphi_x\varphi\mathrm{d}x\mathrm{d}t+\iint_{Q_T}(\bar g_x+\lambda)\varphi(\varphi+k_1)\mathrm{d}x\mathrm{d}t+$$
$$\iint_{Q_T}\varepsilon^{N+1}(\varphi+k_1)\varphi\varphi_x\mathrm{e}^{\lambda t}\mathrm{d}x\mathrm{d}t-\iint_{Q_T}\frac{\alpha^2-\varepsilon b^2}{2}\varphi_{xx}\varphi\mathrm{d}x\mathrm{d}t=$$
$$\iint_{Q_T}H(t,x)\varphi\mathrm{e}^{-\lambda t}\mathrm{d}x\mathrm{d}t. \tag{20}$$

令

$$I_k(t)=\int_\Omega(P-k_1)_+^2\,\mathrm{d}x,$$

则 $I_k(t)$ 于$[0,T]$上绝对连续.

设 σ 为 $I_k(t)$ 在$[0,T]$上的最大值点.由于 $I_k(t=0)=0$,$I_k(t)\geqslant0$.不妨设 $\sigma>0$,对于充分小的 $\tau>0$,取 $t_1=\sigma-t$,$t_2=\sigma$,则

$$\frac{1}{2\tau}\int_{\sigma-\tau}^{\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(P-k_1)_+^2\,\mathrm{d}x\mathrm{d}t=\frac{1}{2\tau}(I_k(\sigma)-I_k(\sigma-\tau))\geqslant0,$$

$$\iint_{Q_T}\bar g\varphi_x\varphi\mathrm{d}x\mathrm{d}t=\frac12\iint_{Q_T}(\bar g\varphi^2)_x\mathrm{d}x\mathrm{d}t-\frac12\iint_{Q_T}\varphi^2\bar g_x\mathrm{d}x\mathrm{d}t=-\frac12\iint_{Q_T}\varphi^2\bar g_x\mathrm{d}x\mathrm{d}t,$$

$$\iint_{Q_T}(\bar g_x+\lambda)\varphi(\varphi+k_1)\mathrm{d}x\mathrm{d}t=\iint_{Q_T}(\bar g_x+\lambda)\varphi^2\mathrm{d}x\mathrm{d}t+\iint_{Q_T}(\bar g_x+\lambda)k_1\varphi\mathrm{d}x\mathrm{d}t,$$

$$\iint_{Q_T}\varepsilon^{N+1}(\varphi+k_1)\varphi\varphi_x\mathrm{e}^{\lambda t}\mathrm{d}x\mathrm{d}t=\iint_{Q_T}\varepsilon^{N+1}\varphi_x\varphi^2\,\mathrm{e}^{\lambda t}\mathrm{d}x\mathrm{d}t+\iint_{Q_T}\varepsilon^{N+1}k_1\varphi_x\varphi\mathrm{e}^{\lambda t}\mathrm{d}x\mathrm{d}t=0,$$

$$-\iint_{Q_T}\frac{\alpha^2-\varepsilon b^2}{2}\varphi_{xx}\varphi\mathrm{d}x\mathrm{d}t=$$
$$-\iint_{Q_T}\left(\frac{\alpha^2-\varepsilon b^2}{2}\varphi_x\varphi\right)_x\mathrm{d}x\mathrm{d}t+\iint_{Q_T}\varphi_x\left(\frac{\alpha^2-\varepsilon b^2}{2}\varphi\right)_x\mathrm{d}x\mathrm{d}t=$$
$$\iint_{Q_T}\varphi_x\left(\frac{\alpha^2-\varepsilon b^2}{2}\varphi\right)_x\mathrm{d}x\mathrm{d}t=$$
$$\iint_{Q_T}(\varphi_x)^2\frac{\alpha^2-\varepsilon b^2}{2}\mathrm{d}x\mathrm{d}t+\iint_{Q_T}\varphi_x\varphi\left(\frac{\alpha^2-\varepsilon b^2}{2}\varphi\right)_x\mathrm{d}x\mathrm{d}t=$$
$$\iint_{Q_T}(\varphi_x)^2\frac{\alpha^2-\varepsilon b^2}{2}\mathrm{d}x\mathrm{d}t-\iint_{Q_T}\frac12(\varphi)^2\left(\frac{\alpha^2-\varepsilon b^2}{2}\right)_{xx}\mathrm{d}x\mathrm{d}t.$$

所以式(20)化简为

$$\iint_{Q_T}\varphi_t\varphi\mathrm{d}x\mathrm{d}t+\iint_{Q_T}\varphi^2\left(-\frac12\bar g_x+\bar g_x+\lambda-\frac12\left(\frac{\alpha^2-\varepsilon b^2}{2}\right)_{xx}\right)\mathrm{d}x\mathrm{d}t+$$
$$\iint_{Q_T}(\bar g_x+\lambda)k_1\varphi\mathrm{d}x\mathrm{d}t+\iint_{Q_T}\varphi_x{}^2\left(\frac{\alpha^2-\varepsilon b^2}{2}\right)\mathrm{d}x\mathrm{d}t=\iint_{Q_T}H(t,x)\varphi\mathrm{e}^{-\lambda t}\mathrm{d}x\mathrm{d}t.$$

令 $F=H(t,x)\mathrm{e}^{-\lambda t}-(\bar g_x+\lambda)k_1$,考虑到 ε 为小参数,存在与 ε 无关的正常数 m,选取合适的 λ 使得

$$\frac{1}{2}\bar{g}_x + \lambda + \frac{\varepsilon}{4}(b^2)_{xx} \geqslant m \geqslant 0,$$

则可得

$$\frac{\alpha^2}{4}\iint_{Q_T}\varphi_x^2 \mathrm{d}x\mathrm{d}t + m\iint_{Q_T}\varphi^2 \mathrm{d}x\mathrm{d}t \leqslant$$

$$\iint_{Q_T}(|H(t,x)\mathrm{e}^{-\lambda t} - (\bar{g}_x + \lambda)k_1|)|\varphi| \mathrm{d}x\mathrm{d}t \leqslant \iint_{Q_T}|F\varphi| \mathrm{d}x\mathrm{d}t.$$

由嵌入定理与 Hölder 不等式,可得

$$\left(\int_{A_{k_1}(\sigma)}(P-k_1)_+^p \mathrm{d}x\right)^{1/p} \leqslant C\left(\int_{A_{k_1}(\sigma)}|F|^q \mathrm{d}x\right)^{1/q} \leqslant$$

$$C\|F\|_{L^\infty(Q_T)}|A_{k_1}(\sigma)|^{1/q} \leqslant C\|F\|_{L^\infty(Q_T)}|\mu_{k_1}|^{1/q},$$

其中

$$2 < p < \begin{cases} +\infty, & n = 1,2, \\ \dfrac{2n}{n-2}, & n > 2, \end{cases}$$

$$A_{k_1}(t) = \{x; R_1(t,x) > k_1\},\ \mu_{k_1} = \sup_{0<t<T}|A_{k_1}(t)|,\ \frac{1}{p} + \frac{1}{q} = 1.$$

对于任何 $h > k_1, t \in [0,T]$,有 $A_h(t) \subset A_{k_1}(t)$.所以

$$I_{k_1}(t) \geqslant \int_{A_h(t)}(P-k_1)_+^2 \mathrm{d}x \geqslant (h-k_1)^2|A_h(t)|.$$

注意到

$$I_{k_1}(t) \leqslant I_{k_1}(\sigma) \leqslant \left(\int_{A_{k_1}(\sigma)}(P-k_1)_+^p \mathrm{d}x\right)^{2/p}|A_{k_1}(\sigma)|^{(p-2)/p} \leqslant$$

$$(C\|F\|_{L^\infty}(Q_T))^2|\mu_{k_1}|^{(3p-4)/p},$$

由于 $p > 2$,故 $(3p-4)/p \geqslant 1$.

由文献[16]中的引理可知,$\mu_{l+d} = \sup_{0<t<T}|A_{l+d}| = 0$,其中

$$d = C\|F\|_{L^\infty(Q_T)}\mu_l^{1-2/p}2^{(3p-4)/(2p-4)} \leqslant C|\Omega|^{1-2/p}2^{(3p-4)/(2p-4)}\|F\|_{L^\infty(Q_T)}.$$

所以,$P \leqslant l + C|\Omega|^{1-2/p}2^{(3p-4)/(2p-4)}\|F\|_{L^\infty(Q_T)}$,可得 R_1 有界,即

$$|R_1| \leqslant \mathrm{e}^{\lambda T}(l + C|\Omega|^{1-2/p}2^{(3p-4)/(2p-4)}\|F\|_{L^\infty(Q_T)}).$$

因此,可得波速率 $g(t,z)$ 的渐近解一致有效.

3.2 平均速率的余项估计

定理 2 式(11)的渐近解的余项 $|r_1| \leqslant k\mathrm{e}^{\lambda T}$.

证 对于 $v(t,x)$ 的余项估计:

$$v(t,x) = \sum_{k=0}^{N}\varepsilon^k v_k(t,x) + \sum_{k=0}^{N}\varepsilon^k \Pi_k(t,\tau) + \varepsilon^{N+1}r_1(t,x), \tag{21}$$

其中 r_1 为边界层矫正函数的余项.将式(21)代入式(4),可得余项 $r_1(t,x)$ 满足

$$\begin{cases} \dfrac{\partial r_1(t,x)}{\partial t} + a(t,x)\dfrac{\partial r_1(t,x)}{\partial x} + \dfrac{\varepsilon b^2(t,x)}{2}\dfrac{\partial^2 r_1(t,x)}{\partial x^2} = f(t,\tau) - \dfrac{b^2}{2}\dfrac{\partial^2 v_N}{\partial x^2}, \\ r_1(0,x) = 0,\ r_1(t,0) = 0,\ r_1(t,d) = 0, \end{cases} \tag{22}$$

令 $s = T - t$,其中 $t \in [0,T]$,由式(22)可得

$$\frac{\partial r_1(s,x)}{\partial s}-a(s,x)\frac{\partial r_1(s,x)}{\partial x}-\frac{\varepsilon b^2(s,x)}{2}\frac{\partial^2 r_1(s,x)}{\partial x^2}=-f(s,\tau)+\frac{b^2}{2}\frac{\partial^2 v_N}{\partial x^2}=H(s,x),$$

其中 $H(s,x)$ 与 ε 有关.令 $r_1=\phi e^{\lambda s}$,其中 $\lambda>0$, 可得

$$\frac{\partial\phi(s,x)}{\partial s}-a(s,x)\frac{\partial\phi(s,x)}{\partial x}-\frac{\varepsilon b^2(s,x)}{2}\frac{\partial^2\phi(s,x)}{\partial x^2}+\lambda\phi=H(s,x)e^{-\lambda s}. \tag{23}$$

令

$$\varphi=(\phi-k)_+\chi_{[s_1,s_2]}=\begin{cases}\phi-k, & \phi\geqslant k,\\ 0, & \phi<k,\end{cases}$$

其中 k 是常数,$k>l$,$l=\sup_{\partial_P Q_T}|\phi|$,$\chi_{[s_1,s_2]}(s)$ 为区间$[s_1,s_2]$的特征函数.

在式(23)两边同时乘上检验函数 φ, 且取积分,可得

$$\iint_{Q_T}\varphi_s\varphi\,dxds-\iint_{Q_T}a(s,x)\varphi_x\varphi\,dxds-\iint_{Q_T}\frac{\varepsilon b^2}{2}\varphi_{xx}\varphi\,dxds+\iint_{Q_T}\lambda(\varphi+k)\varphi\,dxds=\iint_{Q_T}H(s,x)\varphi e^{-\lambda s}dxds.$$

令

$$I_k(s)=\int_\Omega(\phi-k)_+^2\,dx,$$

则 $I_k(s)$ 于$[0,T]$上绝对连续.

设 σ 为 $I_k(s)$ 在$[0,T]$上的最大值点.由于 $I_k(s=0)=0$,$I_k(s)\geqslant0$.不妨设 $\sigma>0$,对于充分小的 $\tau>0$,取 $s_1=\sigma-\tau$,$s_2=\sigma$, 则

$$\frac{1}{2\tau}\int_{\sigma-\tau}^{\sigma}\frac{d}{ds}\int_\Omega(\phi-k)_+^2\,dxds=\frac{1}{2\tau}(I_k(\sigma)-I_k(\sigma-\tau))\geqslant0,$$

$$-\iint_{Q_T}\frac{\varepsilon b^2}{2}\varphi_{xx}\varphi\,dxds=-\iint_{Q_T}\frac{\varepsilon}{2}(b^2\varphi_x\varphi)_x dxds+\iint_{Q_T}\frac{\varepsilon}{2}(b^2\varphi)_x\varphi_x dxds=\iint_{Q_T}\frac{\varepsilon b^2}{2}(\varphi_x)^2dxds-\iint_{Q_T}\frac{\varepsilon}{4}\varphi^2(b^2)_{xx}dxds,$$

$$-\iint_{Q_T}a(s,x)\varphi_x\varphi\,dxds=\iint_{Q_T}-\frac{1}{2}(a\varphi^2)_x dxds+\iint_{Q_T}\frac{1}{2}\varphi^2a_x dxds=\iint_{Q_T}\frac{1}{2}\varphi^2a_x dxds.$$

所以,上式可化简得

$$\iint_{Q_T}\varphi_s\varphi\,dxds+\iint_{Q_T}\frac{\varepsilon b^2}{2}(\varphi_x)^2dxds+\iint_{Q_T}\varphi^2\left(\frac{1}{2}a_x-\frac{\varepsilon}{4}\frac{\partial^2 b^2}{\partial x^2}+\lambda\right)dxds=\iint_{Q_T}(H\varphi e^{-\lambda s}-k\lambda\varphi)dxds.$$

选取合适 λ,存在与 ε 无关的正常数 m, 使得

$$\frac{1}{2}a_x-\frac{\varepsilon}{4}(b^2)_{xx}+\lambda\geqslant m>0,$$

则可得

$$\iint_{Q_T} m\varphi^2 \mathrm{d}x\mathrm{d}s \leqslant \iint_{Q_T} (He^{-\lambda s} - k\lambda)\varphi \mathrm{d}x\mathrm{d}s \leqslant 0.$$

所以几乎处处 $\varphi = 0$.则 $\phi \leqslant k$,即 $|r_1| \leqslant ke^{\lambda T}$.由 r_1 的有界性,可得到渐近解 $v(t,x)$ 的有效性.

4 结束语

本文讨论了一类在有界区域上的具有有色噪声影响的随机 Burgers 方程,该有色噪声服从弱噪声 O-U 过程.考虑到波运动的初边值条件和平均速率受到弱噪声的影响,分别采用正则展开和奇摄动展开,得到相应形式渐近解.应用极值原理、Schauder 不动点定理、Lax-Milgram 定理和 De-Giorgi 迭代技术证明渐近解的存在性、有界性和一致有效性.

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Singular Perturbation Solutions to 1D Stochastic Burgers Equations Under Weak Noises

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Abstract: The singular perturbation solutions to a class of bounded stochastic Burgers equations under colored noises were discussed, of which the volatility followed the weak noise Ornstein-Uhlenbeck (O-U) process. With the Kolmogorov equation satisfied by the probability density function of wave motion, the Kolmogorov equation satisfied by the expectation of the random Burgers equation was obtained. Since the initial boundary conditions for the Kolmogorov equation relate to a class of deterministic solutions to the Burgers equation, this problem is actually a simultaneous form of the Burgers equation and the Kolmogorov equation. Firstly, the regular asymptotic expansion of a class of deterministic Burgers equations was given. Based on the Schauder estimates and the Ascoli-Arzela theorem, boundedness and existence of the asymptotic solutions to the nonlinear parabolic equations were proved; moreover, according to the Lax-Milgram theorem, boundedness and existence of the asymptotic solutions to the linear parabolic equations were proved. The formal asymptotic solution of wave expectation was obtained. Secondly, with the singular perturbation theory, the asymptotic expansion of singular perturbation and the boundary layer correction of a class of expected equations were got. The existence and boundedness of the asymptotic solutions to the boundary layer functions were obtained according to the theory of linear partial differential equations. By means of the extremum principle and the De-Giorgi iterative techniques, the boundedness of the remainder terms of the asymtotic solutions of wave velocity and wave expectation was proved respectively, and the uniformly valid estimate for the asymptotic solution of the system was obtained.

Key words: singular perturbation; random Burgers equation; average velocity; Ornstein-Uhlenbeck (O-U) process; uniformly valid estimate

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