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直角坐标系下黏弹性层状地基动力响应分析*

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摘要: 基于直角坐标系下黏弹性力学的基本控制方程,运用 Fourier-Laplace 积分变换、解耦变换、微分方程组理论和矩阵理论,推导轴对称动荷载及非轴对称动荷载作用时黏弹性地基三维空间问题积分变换域内的解析单元刚度矩阵;根据边界条件和层间连续条件集成总刚度矩阵;求解含有总刚度矩阵方程的代数方程,得到积分变换域内相应问题的解;利用 Fourier-Laplace 积分逆变换得到真实物理域内的解.编制相应程序计算黏弹性层状地基动力响应与已有解答进行对比,验证了提出方法的正确性.

关 键 词: 直角坐标系; 轴对称动荷载; 黏弹性层状地基; 解析刚度矩阵; 动力响应

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引 言

车辆、地铁等动荷载引起天然地基的力学响应成为软土地区交通基础设施建设中亟待解决的问题.天然地基在沉积形成中呈现出层状特征,层内土体性质较均匀,层间土体性质则差异较大.同时,荷载作用下天然地基具有明显的流变特性.因此,采用流变层状模型描述天然地基更加符合实际工况.

现有针对层状地基受荷作用的研究多集中在求解极坐标系下三维空间和平面应变弹性静力问题[1-10],而较少关注表征地基流变性质的黏弹性问题[11-12].此外,天然地基局部承受外部荷载作用在实际工程中时常发生.因此,分析直角坐标系下黏弹性层状地基的动力响应更具有理论及实际意义.

目前层状结构的解析求解方法主要有传递矩阵法[3-4]和刚度矩阵法[5-8,11-13]等.由于刚度矩阵的对称性及矩阵元素只存在负指数,从而避免传递矩阵法中正指数存在导致的计算溢出问题[11-12].而钟阳等[13]推导得到的弹性三维空间问题的单元刚度矩阵为 6 行 6 列,且矩阵元素表达式复杂不便实际应用.本文从直角坐标系下动荷载作用的黏弹性力学的基本控制方程入手,推导动荷载作用时黏弹性层状地基三维空间问题的解析单元刚度矩阵为 4 行 4 列,并建立了完备的解析刚度矩阵法求解直角坐标系下黏弹性层状地基动力响应的理论体系,为实际工程中的地基动载响应问题提供了理论基础.

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1 推导动载作用时黏弹性地基的单元刚度矩阵

1.1 黏弹性三维空间问题的单元刚度矩阵

假定天然地基土体为黏弹性体,土体的应力和应变以压缩为正.直角坐标系下用位移表示动荷载作用的黏弹性三维空间问题的控制方程表达式为

$$\mathrm{d}G(t) * \nabla^2 u(x,y,z,t) - \mathrm{d}[\lambda(t) + G(t)] * \frac{\partial \varepsilon(x,y,z,t)}{\partial x} - \rho \frac{\partial^2 u(x,y,z,t)}{\partial t^2} = 0, \tag{1a}$$

$$\mathrm{d}G(t) * \nabla^2 v(x,y,z,t) - \mathrm{d}[\lambda(t) + G(t)] * \frac{\partial \varepsilon(x,y,z,t)}{\partial y} - \rho \frac{\partial^2 v(x,y,z,t)}{\partial t^2} = 0, \tag{1b}$$

$$\mathrm{d}G(t) * \nabla^2 w(x,y,z,t) - \mathrm{d}[\lambda(t) + G(t)] * \frac{\partial \varepsilon(x,y,z,t)}{\partial z} - \rho \frac{\partial^2 w(x,y,z,t)}{\partial t^2} = 0, \tag{1c}$$

式中

$$\varepsilon = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}, \tag{1d}$$

ε 为体积应变; $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ 为 Laplace 算子; $u(x,y,z,t)$,$v(x,y,z,t)$,$w(x,y,z,t)$ 为 x, y,z 方向的位移;$\lambda(t) = K(t) - (2/3)G(t)$, $G(t)$ 为剪切松弛函数, $K(t)$ 为体积松弛函数; $*$ 为广义 Stieltjes 卷积; $\mathrm{d}f(t)$ 为 $f(t)$ 对时间 t 的导数;ρ 为密度;z 坐标向下为正.

直角坐标系下用位移表示的黏弹性三维空间问题的物理方程为

$$\sigma_z(x,y,z,t) = \mathrm{d}\lambda(t) * \varepsilon - 2\mathrm{d}G(t) * \frac{\partial w}{\partial z}, \tag{2a}$$

$$\tau_{xz}(x,y,z,t) = -\mathrm{d}G(t) * \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \tag{2b}$$

$$\tau_{yz}(x,y,z,t) = -\mathrm{d}G(t) * \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \tag{2c}$$

$$\sigma_x(x,y,z,t) = \mathrm{d}\lambda(t) * \varepsilon - 2\mathrm{d}G(t) * \frac{\partial u}{\partial x}, \tag{2d}$$

$$\sigma_y(x,y,z,t) = \mathrm{d}\lambda(t) * \varepsilon - 2\mathrm{d}G(t) * \frac{\partial v}{\partial y}, \tag{2e}$$

$$\tau_{xy}(x,y,z,t) = -\mathrm{d}G(t) * \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right). \tag{2f}$$

对式(1a)~(1d)进行变量 x,y 的双重 Fourier 变换和时间 t 的 Laplace 变换,得到

$$\frac{\mathrm{d}^2 \tilde{\tilde{u}}}{\mathrm{d}z^2} - \xi^2 \tilde{\tilde{u}} - \frac{\mathrm{i}(\tilde{\lambda}^* + \tilde{G}^*)\xi_x}{\tilde{G}^*} \tilde{\tilde{\varepsilon}} - \frac{\rho s^2}{\tilde{G}^*} \tilde{\tilde{u}} = 0, \tag{3a}$$

$$\frac{\mathrm{d}^2\tilde{\tilde{v}}}{\mathrm{d}z^2}-\xi^2\tilde{\tilde{v}}-\frac{\mathrm{i}(\tilde{\lambda}^*+\tilde{G}^*)\xi_y}{\tilde{G}^*}\tilde{\tilde{\varepsilon}}-\frac{\rho s^2}{\tilde{G}^*}\tilde{\tilde{v}}=0,\tag{3b}$$

$$\frac{\mathrm{d}^2\tilde{\tilde{w}}}{\mathrm{d}z^2}-\xi^2\tilde{\tilde{w}}-\frac{(\tilde{\lambda}^*+\tilde{G}^*)}{\tilde{G}^*}\frac{\mathrm{d}\tilde{\tilde{\varepsilon}}}{\mathrm{d}z}-\frac{\rho s^2}{\tilde{G}^*}\tilde{\tilde{w}}=0,\tag{3c}$$

$$\tilde{\tilde{\varepsilon}}=-\mathrm{i}\xi_x\tilde{\tilde{u}}-\mathrm{i}\xi_y\tilde{\tilde{v}}-\frac{\mathrm{d}\tilde{\tilde{w}}}{\mathrm{d}z},\tag{3d}$$

式中，i 为虚数单位，

$$\tilde{\lambda}^*=s\tilde{\lambda}(s),\ \tilde{G}^*=s\tilde{G}(s),$$

$$\xi^2=\xi_x^2+\xi_y^2,\ \tilde{\tilde{f}}(\xi_x,\xi_y,z,s)=\int_0^{\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(x,y,z,t)\mathrm{e}^{-\mathrm{i}\xi_x x}\mathrm{e}^{-\mathrm{i}\xi_y y}\mathrm{e}^{-st}\mathrm{d}x\mathrm{d}y\mathrm{d}t.$$

将式(3d)代入式(3a)~(3c)，得到

$$\frac{\mathrm{d}^2\tilde{\tilde{u}}}{\mathrm{d}z^2}-(\xi^2+\psi\xi_x^2)\tilde{\tilde{u}}-\psi\xi_x\xi_y\tilde{\tilde{v}}+\mathrm{i}\psi\xi_x\frac{\mathrm{d}\tilde{\tilde{w}}}{\mathrm{d}z}+\theta s^2\tilde{\tilde{u}}=0,\tag{4a}$$

$$\frac{\mathrm{d}^2\tilde{\tilde{v}}}{\mathrm{d}z^2}-(\xi^2+\psi\xi_y^2)\tilde{\tilde{v}}-\psi\xi_x\xi_y\tilde{\tilde{u}}+\mathrm{i}\psi\xi_y\frac{\mathrm{d}\tilde{\tilde{w}}}{\mathrm{d}z}+\theta s^2\tilde{\tilde{v}}=0,\tag{4b}$$

$$(1+\psi)\frac{\mathrm{d}^2\tilde{\tilde{w}}}{\mathrm{d}z^2}+\mathrm{i}\psi\left(\xi_x\frac{\mathrm{d}\tilde{\tilde{u}}}{\mathrm{d}z}+\xi_y\frac{\mathrm{d}\tilde{\tilde{v}}}{\mathrm{d}z}\right)-\xi^2\tilde{\tilde{w}}+\theta s^2\tilde{\tilde{w}}=0,\tag{4c}$$

式中

$$\psi=\frac{(\tilde{\lambda}^*+\tilde{G}^*)}{\tilde{G}^*},\ \theta=\rho/\tilde{G}^*.$$

为了解除式中出现积分变换后位移分量之间的耦合，令

$$M=\frac{\xi_x\tilde{\tilde{u}}+\xi_y\tilde{\tilde{v}}}{\xi},\ N=\frac{\xi_y\bar{u}-\xi_x\bar{v}}{\xi},\ W=\tilde{\tilde{w}},$$

得到

$$\tilde{\tilde{u}}=\frac{\xi_x M+\xi_y N}{\xi},\ \tilde{\tilde{v}}=\frac{\xi_y M-\xi_x N}{\xi}.\tag{5}$$

将式(5)代入式(4a)~(4c)，得到

$$\frac{\mathrm{i}\mathrm{d}^2M}{\mathrm{d}z^2}-\xi\psi\frac{\mathrm{d}W}{\mathrm{d}z}-\mathrm{i}\xi^2(1+\psi)M-\mathrm{i}\theta s^2M=0,\tag{6a}$$

$$(1+\psi)\frac{\mathrm{d}^2W}{\mathrm{d}z^2}+\xi\psi\frac{\mathrm{i}\mathrm{d}M}{\mathrm{d}z}-\xi^2W-\theta s^2W=0,\tag{6b}$$

$$\frac{\mathrm{d}^2N}{\mathrm{d}z^2}-\xi^2N-\theta s^2N=0.\tag{6c}$$

将式(6a)和(6b)转换为常微分方程组的矩阵形式[11-12,14]，即

$$\begin{bmatrix}1&0\\0&1+\psi\end{bmatrix}\begin{bmatrix}\dfrac{\mathrm{i}\mathrm{d}^2M}{\mathrm{d}z^2}\\\dfrac{\mathrm{d}^2W}{\mathrm{d}z^2}\end{bmatrix}+\begin{bmatrix}0&-\xi\psi\\\xi\psi&0\end{bmatrix}\begin{bmatrix}\dfrac{\mathrm{i}\mathrm{d}M}{\mathrm{d}z}\\\dfrac{\mathrm{d}W}{\mathrm{d}z}\end{bmatrix}-$$

$$\begin{bmatrix}s^2\theta+\xi^2(1+\psi)&0\\0&s^2\theta+\xi^2\end{bmatrix}\begin{bmatrix}\mathrm{i}M\\W\end{bmatrix}=\mathbf{0}.\tag{7}$$

根据文献[11-12]提出的求解方法,得到矩阵方程(7)的解为

$$\begin{bmatrix} \mathrm{i}M \\ W \end{bmatrix} = C_1 \mathrm{e}^{\lambda_1 z} \begin{bmatrix} -1/\xi \\ 1/\lambda_1 \end{bmatrix} + C_2 \mathrm{e}^{-\lambda_1 z} \begin{bmatrix} -1/\xi \\ -1/\lambda_1 \end{bmatrix} + C_3 \mathrm{e}^{\lambda_3 z} \begin{bmatrix} -\xi/\lambda_3^2 \\ 1/\lambda_3 \end{bmatrix} + C_4 \mathrm{e}^{-\lambda_3 z} \begin{bmatrix} -\xi/\lambda_3^2 \\ -1/\lambda_3 \end{bmatrix}, \tag{8}$$

式中　$\lambda_1 = \sqrt{\xi^2 + \theta s^2}$, $\lambda_3 = \sqrt{\xi^2 + \chi s^2}$, $\chi = \rho/(\tilde{\lambda}^* + 2\tilde{G}^*)$.

对式(2a)~(2c)进行变量 x,y 的双重 Fourier 变换和时间 t 的 Laplace 变换,得到

$$\tilde{\tilde{\sigma}}_z = \tilde{\lambda}^* \tilde{\tilde{\varepsilon}} - 2\tilde{G}^* \frac{\mathrm{d}\tilde{\tilde{w}}}{\mathrm{d}z} + \tilde{\tilde{p}}, \tag{9a}$$

$$\tilde{\tilde{\tau}}_{xz} = -\tilde{G}^* \left(\frac{\mathrm{d}\tilde{\tilde{u}}}{\mathrm{d}z} + \mathrm{i}\xi_x \tilde{\tilde{w}} \right), \tag{9b}$$

$$\tilde{\tilde{\tau}}_{yz} = -\tilde{G}^* \left(\frac{\mathrm{d}\tilde{\tilde{v}}}{\mathrm{d}z} + \mathrm{i}\xi_y \tilde{\tilde{w}} \right). \tag{9c}$$

令

$$X = \frac{\xi_x \tilde{\tilde{\tau}}_{xz} + \xi_y \tilde{\tilde{\tau}}_{yz}}{\xi}, \; Y = \frac{\xi_y \tilde{\tilde{\tau}}_{xz} - \xi_x \tilde{\tilde{\tau}}_{yz}}{\xi}, \; Z = \tilde{\tilde{\sigma}}_z, \tag{10}$$

由式(5)、(8)和(9)得到

$$\begin{aligned} X = &-\tilde{G}^* \left(\frac{\mathrm{d}M}{\mathrm{d}z} + \mathrm{i}\xi W \right) = \\ &\tilde{G}^* [-\mathrm{i}C_1 \mathrm{e}^{\lambda_1 z} (2\xi^2 + \theta s^2)/(\xi\lambda_1) + \mathrm{i}C_2 \mathrm{e}^{-\lambda_1 z} (2\xi^2 + \theta s^2)/(\xi\lambda_1) - \\ &2\mathrm{i}C_3 \xi \mathrm{e}^{\lambda_3 z}/\lambda_3 + 2\mathrm{i}C_4 \xi \mathrm{e}^{-\lambda_3 z}/\lambda_3], \end{aligned} \tag{11a}$$

$$\begin{aligned} Z = &-\mathrm{i}\tilde{\lambda}^* \xi M - (\tilde{\lambda}^* + 2\tilde{G}^*) \frac{\mathrm{d}W}{\mathrm{d}z} = \\ &-2\tilde{G}^* C_1 \mathrm{e}^{\lambda_1 z} - 2\tilde{G}^* C_2 \mathrm{e}^{-\lambda_1 z} - 2\tilde{G}^* C_3 \mathrm{e}^{\lambda_3 z} (2\xi^2 + \theta s^2)/(2\lambda_3^2) - \\ &2\tilde{G}^* C_4 \mathrm{e}^{-\lambda_3 z} (2\xi^2 + \theta s^2)/(2\lambda_3^2), \end{aligned} \tag{11b}$$

$$Y = \frac{\xi_y \tilde{\tilde{\tau}}_{xz} - \xi_x \tilde{\tilde{\tau}}_{yz}}{\xi} = -\tilde{G}^* \frac{\mathrm{d}N}{\mathrm{d}z}. \tag{11c}$$

由式(8)得到位移分量与待定常数 $C_i(i=1,2,3,4)$ 之间的矩阵关系式,即

$$\boldsymbol{U} = [W(\xi,s,0), \mathrm{i}M(\xi,s,0), W(\xi,s,z), \mathrm{i}M(\xi,s,z)]^{\mathrm{T}} = \boldsymbol{R} [C_1, C_2, C_3, C_4]^{\mathrm{T}}. \tag{12}$$

由式(11a)、(11b)得到应力与待定常数 $C_i(i=1,2,3,4)$ 之间的矩阵关系式,即

$$\begin{aligned} \boldsymbol{Q} = &[-Z(\xi,s,0), -\mathrm{i}X(\xi,s,0), Z(\xi,s,z), \mathrm{i}X(\xi,s,z)]^{\mathrm{T}} = \\ &\boldsymbol{L} [C_1, C_2, C_3, C_4]^{\mathrm{T}}. \end{aligned} \tag{13}$$

可知, $\boldsymbol{U}$ 和 $\boldsymbol{Q}$ 均可以由待定常数 $C_i(i=1,2,3,4)$ 表达,由两者之间的关系 $\boldsymbol{Q} = \boldsymbol{KU}$ 得到 $\boldsymbol{K} = \boldsymbol{LR}^{-1}$, 即为直角坐标系下动荷载作用时黏弹性三维空间问题在积分变换域内的单元刚度矩阵.

1.2　轴对称动荷载作用的黏弹性三维空间问题的单元刚度矩阵

引入正交辅助坐标系 $\alpha O\beta$, 其与坐标系 xOy 夹角为 θ, 如图 1 所示,两坐标系转换关系为

$$\alpha = x\cos\theta + y\sin\theta, \; \beta = -x\sin\theta + y\cos\theta.$$

将双重 Fourier 逆变换变量 ξ_x,ξ_y 用 θ 和引入变量 ρ 表达为

$$\xi_x = \rho\cos\theta, \; \xi_y = \rho\sin\theta. \tag{14}$$

则函数 $f(x,y,z)$ 的 Fourier 逆变换变为

$$f(x,y,z)=\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_0^{2\pi}\tilde{f}(\rho\cos\theta,\rho\sin\theta,z)\mathrm{e}^{\mathrm{i}\rho\alpha}\rho\mathrm{d}\theta\mathrm{d}\rho.\tag{15}$$

可知,函数 $f(x,y,z)$ 在 $\alpha O\beta$ 坐标系下与坐标轴 β 无关.对于直角坐标系下层状地基承受轴对称荷载作用时,可引入基于荷载对称轴的正交坐标系 $\alpha O\beta$, 则控制方程式(1a)~(1c)变为

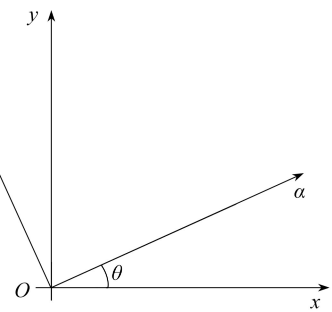

图 1 辅助坐标系

Fig. 1 auxiliary coordinate system

$$\mathrm{d}G(t)*\nabla^2u_\alpha(\alpha,\beta,z,t)-\mathrm{d}[\lambda(t)+G(t)]*\frac{\partial\varepsilon(\alpha,\beta,z,t)}{\partial\alpha}-\rho\frac{\partial^2u_\alpha(x,y,z,t)}{\partial t^2}=0,\tag{16a}$$

$$\mathrm{d}G(t)*\nabla^2w_z(\alpha,\beta,z,t)-\mathrm{d}[\lambda(t)+G(t)]*\frac{\partial\varepsilon(\alpha,\beta,z,t)}{\partial z}-\rho\frac{\partial^2w_z(x,y,z,t)}{\partial t^2}=0,\tag{16b}$$

$$\mathrm{d}G(t)*\nabla^2u_\beta(\alpha,\beta,z,t)-\rho\frac{\partial^2u_\beta(x,y,z,t)}{\partial t^2}=0,\tag{16c}$$

式中

$$\varepsilon=-\frac{\partial u_\alpha}{\partial\alpha}-\frac{\partial w_z}{\partial z}.\tag{16d}$$

物理方程变化为

$$\sigma_z(\alpha,\beta,z,t)=\mathrm{d}\lambda(t)*\varepsilon-2\mathrm{d}G(t)*\frac{\partial w_z}{\partial z},\tag{17a}$$

$$\tau_{\alpha z}(\alpha,\beta,z,t)=-\mathrm{d}G(t)*\left(\frac{\partial u_\alpha}{\partial z}+\frac{\partial w_z}{\partial\alpha}\right),\tag{17b}$$

$$\tau_{\beta z}(\alpha,\beta,z,t)=-\mathrm{d}G(t)*\frac{\partial u_\beta}{\partial z}.\tag{17c}$$

对式(16a)、(16b)、(16d)、(17a)和(17b)进行变量 α 的 Fourier 变换和时间 t 的 Laplace 变换,通过 1.1 小节的方法得到轴对称动荷载作用时黏弹性三维空间问题的单元刚度矩阵.

2 解析刚度矩阵法求解黏弹性层状地基的动力响应

根据天然地基土层及计算点将地基划分为 n 个计算层,H_i^- 和 H_i^+ 分别为第 i 层顶面和底面距离地表的距离,$\Delta h_i=H_i^+-H_i^-$ 为第 i 计算层的厚度($i=1,2,\cdots,n$).地基表面受动荷载 $q_i(x,y,z,t)$ 作用.

假定地基底部是固定,即 $z=H_n$ 处,有

$$u=v=w=0.$$

对于层间连续条件,假定相邻层间完全接触,连续条件为

$$M(\xi,s,H_i^+)=M(\xi,s,H_{i+1}^-),\ W(\xi,s,H_i^+)=W(\xi,s,H_{i+1}^-),$$
$$N(\xi,s,H_i^+)=N(\xi,s,H_{i+1}^-),\ X(\xi,s,H_i^+)=X(\xi,s,H_{i+1}^-),$$
$$Y(\xi,s,H_i^+)=Y(\xi,s,H_{i+1}^-),\ Z(\xi,s,H_i^+)=Z(\xi,s,H_{i+1}^-).$$

对于划分为 n 个计算层的天然地基,将解析单元刚度矩阵应用于各计算层,结合计算层间的连续条件集成整个天然地基的总体刚度矩阵并建立代数方程,得到

$$\begin{bmatrix} -Z(\xi,s,0) \\ -\mathrm{i}X(\xi,s,0) \\ \vdots \\ 0 \\ 0 \\ \vdots \\ Z(\xi,s,H_n) \\ \mathrm{i}X(\xi,s,H_n) \end{bmatrix} = \begin{bmatrix} \boldsymbol{K}^{(1)} & & & & \\ & \ddots & & & \\ & & \boldsymbol{K}^{(i)} & & \\ & & & \ddots & \\ & & & & \boldsymbol{K}^{(n)} \end{bmatrix} \begin{bmatrix} W(\xi,s,0) \\ \mathrm{i}M(\xi,s,0) \\ \vdots \\ W(\xi,s,H_i) \\ \mathrm{i}M(\xi,s,H_i) \\ \vdots \\ W(\xi,s,H_n) \\ \mathrm{i}M(\xi,s,H_n) \end{bmatrix}, \tag{23}$$

式中, $\boldsymbol{K}^{(i)}$ 为第 i 层刚度矩阵.

结合边界条件求解矩阵方程(23),得到在积分变化域内的解答;采用截断分段计算 Fourier 逆变换的方法[11-12,15]和 Takbot 提出的 Laplace 逆变换的方法[11-12,16]进行 Fourier-Laplace 逆变换即可得到真实物理域内的解.

对于直角坐标系下轴对称动荷载作用时黏弹性层状地基的三维问题,首先按 1.2 小节得到解析单元刚度矩阵,求解得到变换域内的解.当求得 $\tilde{u}_\alpha$,$\tilde{u}_\beta$ 后,通过坐标轴旋转变换得到

$$\begin{bmatrix} \tilde{\tilde{u}}_x \\ \tilde{\tilde{u}}_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \tilde{\tilde{u}}_\alpha \\ \tilde{\tilde{u}}_\beta \end{bmatrix} = \frac{1}{\rho} \begin{bmatrix} \xi_x & -\xi_y \\ \xi_y & \xi_x \end{bmatrix} \begin{bmatrix} \tilde{\tilde{u}}_\alpha \\ \tilde{\tilde{u}}_\beta \end{bmatrix}. \tag{24}$$

最后,进行 Fourier-Laplace 逆变换即可得到真实物理域内的解.

3 算 例

某 3 层路面地基,各层土体参数见表 1.地基表面作用动荷载为 $p(t)$, 其表达式为

$$p(t)=\begin{cases} p\sin(\omega t), & 0<r\leqslant R, \\ 0, & r>R, \end{cases}$$

式中 $p=1\ \mathrm{MPa},\ \omega=\pi/10,\ R=0.15\ \mathrm{m}$.

设第 1 层土体为黏弹性材料,体积符合弹性变形,剪切变形符合 Burgers 模型[13],即

$$\tilde{Y}^{(1)}(s)=\frac{q_2}{p_2}\frac{s+q_1/q_2}{s^2+(p_1/p_2)s+1/p_1},\ \tilde{Y}^{(2)}(s)=3K/s,$$

式中

$$p_1=\eta_{11}/E_{11}+(\eta_{11}+\eta_{12})/E_{12},\ p_2=\eta_{11}\eta_{12}/(E_{11}E_{12}),$$
$$q_1=2\eta_{11},\ q_2=2\eta_{11}\eta_{12}/E_{12},$$

得到

$$\tilde{G}^*=s\tilde{G}(s)=s\tilde{Y}^{(1)}(s)/2,\ \tilde{K}(s)=\tilde{Y}^{(2)}(s)/3,\ \tilde{\lambda}^*=s\tilde{\lambda}(s)=s[\tilde{K}(s)-2\tilde{G}(s)/3].$$

设第 2、3 层土体为弹性材料[13],则

$$\tilde{G}^*=G=\frac{E}{2(1+\mu)},\ \tilde{\lambda}^*=\lambda=\frac{E\mu}{(1+\mu)(1-2\mu)},$$

式中，E 为弹性模量，μ 为 Poisson（泊松）比.

表 1　层状地基的土体参数

Table 1　Soil parameters of multi-layered foundation

foundation	E /MPa	η /(MPa·s)	h /m	μ	γ /(kg/m^3)
the 1st layer	$E_{11} = E_{12} = 10^3$	$\eta_{11} = \eta_{12} = 1.12 \times 10^5$	0.15	–	3.2×10^3
the 2nd layer	300	–	0.2	0.25	2.1×10^3
the 3rd layer	95	–	0.2	0.35	1.5×10^3

分别计算在 $t = 0 \sim 10$ s 内动荷载为 $p(t)$ 作用下 $z = 0, y = 0, x = 0, 20, 30, 40, 65, 80, 100$ cm 处的竖向变形，如图 2 所示.然后将 $t = 1$ s 的计算结果与文献[13]结果对比，如图 3 所示.

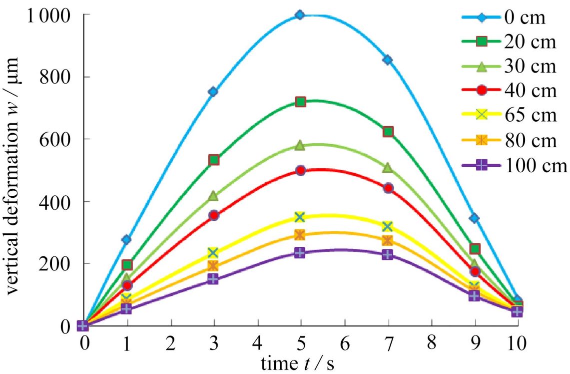

图 2　动荷载作用下黏弹性层状地基的竖向变形

Fig. 2　Vertical deformation of viscoelastic multilayered foundation under dynamic loading

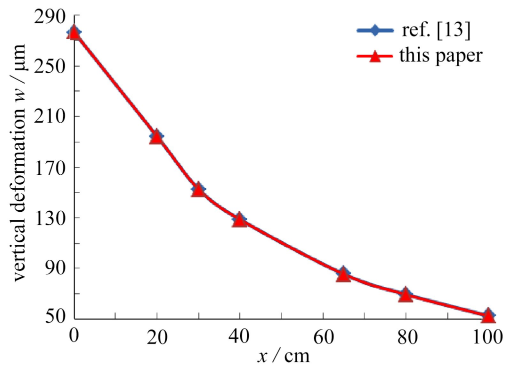

图 3　本文与文献[13]计算结果对比

Fig. 3　Results of this paper compared with those of ref. [13]

从图 3 可知，采用本文的方法计算轴对称荷载作用地基的变形与已有文献相当吻合，从而验证本文提出计算方法的正确性和准确性.从图 2 可知，在 $t = 0 \sim 5$ s 内，随着时间的增加，路面地基各计算点处的竖向变形在不断增大；在 $t = 5 \sim 10$ s 内，随着时间的增加，地基各计算点处的竖向变形在不断减小，由于地基土体具有黏性特征，竖向变形不能完全恢复；与荷载中心点的距离越小，地基的竖向变形越显著.

4　结　　语

基于直角坐标系下黏弹性力学的基本控制方程，运用 Fourier-Laplace 积分变换、解耦变换、微分方程组理论和矩阵理论，推导轴对称动荷载及非轴对称动荷载作用时黏弹性地基三维空间问题积分变换域内的解析单元刚度矩阵；根据边界条件和层间连续条件集成总刚度矩阵；求解含有总刚度矩阵方程的代数方程，得到积分变换域内相应问题的解；利用 Fourier-Laplace 积分逆变换得到真实物理域内的解，建立完备的解析刚度矩阵法求解黏弹性层状地基动力响应的理论体系.

对于黏弹性层状天然地基，土体的蠕变特性越明显，天然地基的变形响应值越显著.

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Dynamic Response Analysis of Viscoelastic Multilayered Foundation in the Cartesian Coordinate System

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Abstract: Based on the governing viscoelastic equations in the Cartesian coordinate system, the analytical element stiffness matrices in the integral transform domain of 3D problems of viscoelastic foundation under axisymmetric and non-axisymmetric dynamic loads were derived with the Fourier-Laplace transform, the decoupling transformation, the differential equations theory and the matrix theory. The global stiffness matrix was assembled in view of the boundary conditions and the continuity between adjacent layers, the solution in the integral transform domain was obtained from the algebraic equations involving the global stiffness matrix, and the solution in the physical domain was acquired through the inverse Fourier-Laplace transform. Through a corresponding computer program, comparison of the dynamic responses of viscoelastic multilayered foundation between the proposed algorithm and the existing method proves the validity of the former.

Key words: Cartesian coordinate system; axisymmetric dynamic loading; viscoelastic layered foundation; analytical stiffness matrix; dynamic response

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