应用数学和力学
APPLIED MATHEMATICS AND MECHANICS

**时滞Lagrange系统的Lie对称性与守恒量研究**

郑明亮

**Lie Symmetries and Conserved Quantities of Lagrangian Systems With Time Delays**

ZHENG Mingliang

在线阅读 View online: https://doi.org/10.21656/1000-0887.410184 http://www.applmathmech.cn

# 时滞 Lagrange 系统的 Lie 对称性与守恒量研究*

郑明亮

( 无锡太湖学院 机电工程学院，江苏 无锡 214064 )

**摘要：** 研究了位形间中含单时滞参数的非保守力学系统的 Lie 对称性和守恒量.首先，利用含时滞的动力学 Hamilton 原理，建立了含时滞的非保守系统的分段 Lagrange 运动方程；其次，利用微分方程容许 Lie 群理论，得到系统的 Lie 对称确定方程；然后，根据对称性与守恒量之间的关系，通过构造结构方程，得到含时滞的非保守系统的 Lie 定理；最后，给出了两个具体的算例说明了方法的应用.结果表明：时滞参数的存在使非保守系统的 Lagrange 方程呈现分段特性，相应的 Lie 对称性确定方程的个数应是自由度数目的 2 倍，这对生成元函数提出了更高的限制，同时，守恒量呈现依赖速度项的分段表达.

**关　键　词：** 时滞；　非保守系统；　Lie 对称；　守恒量

**中图分类号：** O316; O322　　**文献标志码：** A　　DOI：10.21656/1000-0887.410184

# Lie Symmetries and Conserved Quantities of Lagrangian Systems With Time Delays

ZHENG Mingliang

( *School of Mechanical and Electrical Engineering*, *Wuxi Taihu University*, *Wuxi*, *Jiangsu* 214064, *P.R.China* )

**Abstract:** The Lie symmetries and conserved quantities of non-conservative mechanical systems with time delays in configuration space were studied. Firstly, the piecewise Lagrangian equations for non-conservative systems with time delays were established according to the Hamiltonian principle of dynamics with time delay. Secondly, the determining equations of the Lie symmetry were obtained by means of the permissible Lie group theory for differential equations. Then, according to the relationship between symmetries and conserved quantities, the Lie theorem of non-conservative systems with time delays was obtained through construction of structural equations. Finally, 2 examples were given to illustrate the application of the method. The results show that, the time delay makes the Lagrangian equations of non-conservative systems piecewise, and the number of determining equations for Lie symmetry is twice of the number of degrees of freedom, which imposes higher restrictions on the generator functions. Meanwhile, the conserved quantity is also in a piecewise expression depending on the velocity term.

**Key words:** time delay; nonconservative system; Lie symmetry; conserved quantity

## 引　　言

现代数学和物理力学的一个重要交叉部分就是动力学系统的对称性与守恒量[1].当力学系统出现后效、空

* **收稿日期：** 2020-06-23; **修订日期：** 2020-12-23
**基金项目：** 江苏省高等学校自然科学基金(20KJD460001)
**作者简介：** 郑明亮 (1988—), 男, 副教授, 博士 (E-mail: zhmlwxcstu@163.com).
**引用格式：** 郑明亮. 时滞 Lagrange 系统的 Lie 对称性与守恒量研究[J]. 应用数学和力学, 2021, **42**(11): 1161-1168.

载或继承特性时，它的运动方程会出现一类偏差参数，属于无限维泛函微分方程，统称为时滞力学系统[2-3]。在数学物理和工程技术等领域中时滞现象普遍存在。马丽等[4]最近研究了一类随机泛函微分方程带随机步长的 EM 逼近的渐近稳定，给出了时滞系统一种高效的数值解法。在分析力学领域，时滞系统的对称性和守恒量相关研究一直受到有关学者的注意，并取得了一些进展。Rosenblueth[5]和 Chan 等[6]分别给出了含时滞的力学变分问题的充分条件；Frederico 等[7]首次初步研究了含时滞的变分和最优控制问题的 Noether 对称性；Zhang 等[8-10]研究了含时滞的非保守系统动力学的 Noether 理论；祖启航等[11]在相空间中研究了含时滞的非保守力学系统的 Noether 守恒律；贺东海[12]在时间尺度上首先研究了时滞系统的 Noether 对称性和最优控制问题。但是目前的研究大都是离散力学系统且未考虑含积分项[13]的运动，而实际中众多的时滞系统通常具有分布参数特性以及包含局部项，因此时滞系统的动力学特性和控制问题依旧还是一个开放的课题。

需要指出的是，关于时滞动力学系统的对称性与守恒量的研究尚刚刚起步，大都是针对 Noether 对称性，且仅考虑无限小变换为不依赖于广义速度的对称变换。而时滞系统由于时滞参数偏差性，它的 Lie 对称性和守恒量对分析力学同样有着重要意义，关于约束力学系统的 Lie 对称性和守恒量研究最新的成果见文献 [14]。基于上述文献研究，本文利用泛函微分方程容许 Lie 群[15]来克服经典流形上 Lie 群的困难，主要的手段就是将时滞参数看作一个时间参数，从而时间运动区间可分段化，进一步按照非保守系统 Hamilton 原理和微分方程群不变性，具体推导了时滞 Lagrange 系统的变分运算和 Lie 对称性理论，并给出了结果的说明应用。

## 1 时滞 Lagrange 系统的运动方程

设系统的 $n$ 个广义坐标为$q_s(s=1,2,\cdots,n)$，考虑系统具有时滞特性，它的 Lagrange 函数为$L(t,\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{q}(t-\tau),\dot{\boldsymbol{q}}(t-\tau))$，$\tau$是连续型时滞参数，非势广义力为$Q_s(t,\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{q}(t-\tau),\dot{\boldsymbol{q}}(t-\tau))$。

时滞非保守系统的 Hamilton 原理为

$$\delta I(\boldsymbol{q}(\cdot),\boldsymbol{q}_\tau(\cdot))=\int_{t_1}^{t_2}(\delta L+Q_s\delta q_s+Q_s\delta q_{s\tau})\mathrm{d}t=0,\ q_s(t)=\delta_s(t),\qquad t\in[t_1-\tau,t_1],\ \tau<t_2-t_1,\tag{1}$$

其中$\delta_s(t)$为逐段连续光滑函数。对式 (1) 进一步化简得

$$\delta I(\boldsymbol{q}(\cdot),\boldsymbol{q}_\tau(\cdot))=\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_s}\delta q_s+\frac{\partial L}{\partial\dot{q}_s}\delta\dot{q}_s+Q_s\delta q_s\right)\mathrm{d}t+\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_{s\tau}}\delta q_{s\tau}+\frac{\partial L}{\partial\dot{q}_{s\tau}}\delta\dot{q}_{s\tau}+Q_s\delta q_{s\tau}\right)\mathrm{d}t=0.\tag{2}$$

利用边界条件和分部积分法，式 (2) 可变为

$$\begin{aligned}&\int_{t_1}^{t_2-\tau}\left[\left(\frac{\partial L}{\partial q_s}(t)\delta q_s+\frac{\partial L}{\partial q_{s\tau}}(t+\tau)\delta q_s+Q_s(t)+Q_s(t+\tau)\right)\delta q_s+\left(\frac{\partial L}{\partial\dot{q}_s}(t)+\frac{\partial L}{\partial\dot{q}_{s\tau}}(t+\tau)\right)\delta\dot{q}_s\right]\mathrm{d}t+\\&\quad\int_{t_2-\tau}^{t_2}\left[\left(\frac{\partial L}{\partial q_s}(t)+Q_s(t)\right)\delta q_s+\left(\frac{\partial L}{\partial\dot{q}_s}(t)\right)\delta\dot{q}_s\right]\mathrm{d}t=\\&\quad\int_{t_1}^{t_2-\tau}\delta\dot{q}_s\left[\int_t^{t_2-\tau}\left(\frac{\partial L}{\partial q_s}(\theta)\delta q_s+\frac{\partial L}{\partial q_{s\tau}}(\theta+\tau)\delta q_s+Q_s(\theta)+Q_s(\theta+\tau)\right)\mathrm{d}\theta\right]\mathrm{d}t+\\&\quad\int_{t_1}^{t_2-\tau}\left(\frac{\partial L}{\partial\dot{q}_s}(t)+\frac{\partial L}{\partial\dot{q}_{s\tau}}(t+\tau)\right)\delta\dot{q}_s\mathrm{d}t-\int_{t_2-\tau}^{t_2}\delta\dot{q}_s\left[\int_{t_2-\tau}^{t}\left(\frac{\partial L}{\partial q_s}(\theta)+Q_s(\theta)\right)\mathrm{d}\theta\right]\mathrm{d}t+\int_{t_2-\tau}^{t_2}\left(\frac{\partial L}{\partial\dot{q}_s}(t)\right)\delta\dot{q}_s\mathrm{d}t=\\&\quad\int_{t_1}^{t_2-\tau}\delta\dot{q}_s\left[\frac{\partial L}{\partial\dot{q}_s}(t)+\frac{\partial L}{\partial\dot{q}_{s\tau}}(t+\tau)+\int_t^{t_2-\tau}\left(\frac{\partial L}{\partial q_s}(\theta)\delta q_s+\frac{\partial L}{\partial q_{s\tau}}(\theta+\tau)\delta q_s+Q_s(\theta)+Q_s(\theta+\tau)\right)\mathrm{d}\theta\right]\mathrm{d}t+\\&\quad\int_{t_2-\tau}^{t_2}\delta\dot{q}_s\left[\frac{\partial L}{\partial\dot{q}_s}(t)-\int_{t_2-\tau}^{t}\left(\frac{\partial L}{\partial q_s}(\theta)+Q_s(\theta)\right)\mathrm{d}\theta\right]\mathrm{d}t=0.\end{aligned}\tag{3}$$

考虑积分区间的任意性以及$\delta\dot{q}_s$的独立性，由式 (3) 可得到

$$\begin{cases}\dfrac{\partial L}{\partial\dot{q}_s}(t)+\dfrac{\partial L}{\partial\dot{q}_{s\tau}}(t+\tau)+\displaystyle\int_t^{t_2-\tau}\left(\frac{\partial L}{\partial q_s}(\theta)\delta q_s+\frac{\partial L}{\partial q_{s\tau}}(\theta+\tau)\delta q_s+Q_s(\theta)+Q_s(\theta+\tau)\right)\mathrm{d}\theta=0,\\\left[\dfrac{\partial L}{\partial\dot{q}_s}(t)-\displaystyle\int_{t_2-\tau}^{t}\left(\frac{\partial L}{\partial q_s}(\theta)+Q_s(\theta)\right)\mathrm{d}\theta\right]\mathrm{d}t=0.\end{cases}\tag{4}$$

进一步将式 (4) 对时间 $t$ 求导，有

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial \dot{q}_s}(t)+\dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)-\dfrac{\partial L}{\partial q_s}(t)-\dfrac{\partial L}{\partial q_{s\tau}}(t+\tau)=Q_s(t)+Q_s(t+\tau), \quad t_1\leqslant t\leqslant t_2-\tau,\ s=1,2,\cdots,n,\\ \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial \dot{q}_s}(t)-\dfrac{\partial L}{\partial q_s}(t)=Q_s(t), \quad t_2-\tau<t\leqslant t_2,\ s=1,2,\cdots,n.\end{cases} \tag{5}$$

式 (5) 就是含时滞非保守系统的 Lagrange 方程, 它考虑了含时滞的广义坐标$\boldsymbol{q}_\tau$的虚功, 严格来说它属于常泛函微分方程类型. 当广义非势力$Q_s=0$时, 它就是时滞保守系统的 Lagrange 方程, 且若时滞量$\tau$=0, 它就是经典力学系统标准 Lagrange 方程. 同时, 对比非时滞力学系统的 Lagrange 方程, 我们发现, 时滞参数$\tau$使系统的运动方程呈现分段表达. 这也很好理解, 区别一般含小参数非线性动力学模型, 时滞参数是时间的整体移动, 因此它必然可以将时间区间分段化.

## 2 系统的 Lie 对称性

上面我们已经得到时滞 Lagrange 系统的运动方程, 它是一组分段二阶常泛函微分方程, 只是多含了时滞参数 (时间区间整体移动), 因此它的 Lie 对称性与经典动力学系统的 Lie 对称性方法是一致的. 进一步我们可以将式 (5) 记为

$$\begin{cases}F_i^1(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}},\boldsymbol{q}_\tau,\dot{\boldsymbol{q}}_\tau,\ddot{\boldsymbol{q}}_\tau)=0, \quad t_1\leqslant t\leqslant t_2-\tau,\ i=1,2,\cdots,n,\\ F_j^2(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}},\boldsymbol{q}_\tau,\dot{\boldsymbol{q}}_\tau,\ddot{\boldsymbol{q}}_\tau)=0, \quad t_2-\tau<t\leqslant t_2,\ j=1,2,\cdots,n.\end{cases} \tag{6}$$

引进单参 Lie 变换群为

$$\begin{cases}t^*=t+\varepsilon\xi_0(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\\ q_s^*(t^*)=q_s(t)+\varepsilon\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\end{cases} \tag{7}$$

它的生成元向量以及一阶和二阶扩展为

$$\begin{cases}X^{(0)}=\xi_0\dfrac{\partial}{\partial t}+\xi_s\dfrac{\partial}{\partial q_s},\ X^{(1)}=X^{(0)}+(\xi_s-\dot{q}_s\dot{\xi}_0)\dfrac{\partial}{\partial \dot{q}_s},\\ X^{(2)}=X^{(1)}+(\ddot{\xi}_s-2\ddot{q}_s\dot{\xi}_0-\dot{q}_s\ddot{\xi}_0)\dfrac{\partial}{\partial \ddot{q}_s}.\end{cases} \tag{8}$$

时滞动力学系统的 Lie 对称性是指式 (5) 在式 (7) 变换下保持不变性, 即

$$\begin{cases}X^{(2)}(F_i^1)\Big|_{F_i^1(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}},\boldsymbol{q}_\tau,\dot{\boldsymbol{q}}_\tau,\ddot{\boldsymbol{q}}_\tau)=0}=0, \quad t_1\leqslant t\leqslant t_2-\tau,\ i=1,2,\cdots,n,\\ X^{(2)}(F_j^2)\Big|_{F_j^2(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}},\boldsymbol{q}_\tau,\dot{\boldsymbol{q}}_\tau,\ddot{\boldsymbol{q}}_\tau)=0}=0, \quad t_2-\tau<t\leqslant t_2,\ j=1,2,\cdots,n.\end{cases} \tag{9}$$

方称组 (9) 就是系统的 Lie 对称确定方程组. 对比非时滞力学系统的 Lie 对称确定方程组, 明显地, 方程组 (9) 的数目成倍增加, 因此生成元函数$\xi_0,\xi_s$相应的光滑性要增强. 注意到$\boldsymbol{q}_\tau,\dot{\boldsymbol{q}}_\tau$将出现在$\xi_0,\xi_s$中, 但$\ddot{\boldsymbol{q}}_\tau,\ddot{\boldsymbol{q}}_\tau$并不出现, 因此确定方程组可以分解成几个方程, 而变为对未知函数$\xi_0$和$\xi_s$的超定二阶常微分方程组. 另外, 如果在微分算子空间$X^{(0)}$中定义换位子运算即 Lie 括号[, ], 则可得到相应 Lie 代数结构$\{X^{(0)}\}$:

$$[X_1^{(0)},X_2^{(0)}]=X_1^{(0)}X_2^{(0)}-X_2^{(0)}X_1^{(0)}. \tag{10}$$

## 3 系统的守恒量

连续变换的对称性都对应着一条守恒定律, 守恒量是动力学系统更深层次的规律[16], 可使系统达到降阶和约化. 时滞系统由于时滞参数的影响, 其运动方程呈现分段特性, 因此一般来说其守恒量也具有分段特性.

**Lie 定理** 若无限小生成元满足式 (9), 且同时存在规范函数$G=G(t,\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{q}_\tau,\dot{\boldsymbol{q}}_\tau)$满足下列结构方程:

$$\begin{cases}\dfrac{\partial L}{\partial t}(t)\xi_0+\left(\dfrac{\partial L}{\partial \dot{q}_s}(t)+\dfrac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)\left(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0\right)+\left(\dfrac{\partial L}{\partial q_s}(t)+\dfrac{\partial L}{\partial q_{s\tau}}(t+\tau)\right)\xi_s+\\ \qquad (Q_s(t)+Q_s(t+\tau))(\xi_s-\dot{q}_s\xi_0)+L\dot{\xi}_0+\dot{G}=0, \quad t_1\leqslant t\leqslant t_2-\tau,\\ \dfrac{\partial L}{\partial t}(t)\xi_0+\dfrac{\partial L}{\partial \dot{q}_s}(t)\xi_s+\dfrac{\partial L}{\partial \dot{q}_s}(t)\left(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0\right)+Q_s(t)(\xi_s-\dot{q}_s\xi_0)+L\dot{\xi}_0+\dot{G}=0, \quad t_2-\tau<t\leqslant t_2.\end{cases} \tag{11}$$

则时滞非保守 Lagrange 系统的 Lie 对称性可导致守恒量:

$$\begin{cases} I_N(t,t+\tau,\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{q}_\tau,\dot{\boldsymbol{q}}_\tau)=\left(\dfrac{\partial L}{\partial \dot{q}_s}(t)+\dfrac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)(\xi_s-\dot{q}_s\xi_0)+L\xi_0+G=0, & t_1\leqslant t\leqslant t_2-\tau, \\ I_N(t,t+\tau,\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{q}_\tau,\dot{\boldsymbol{q}}_\tau)=\dfrac{\partial L}{\partial \dot{q}_s}(t)(\xi_s-\dot{q}_s\xi_0)+L\xi_0+G=0, & t_2-\tau<t\leqslant t_2. \end{cases} \tag{12}$$

**证明** 对式 (12) 分段求导得

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}I_N(t,t+\tau,\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{q}_\tau,\dot{\boldsymbol{q}}_\tau)=\dfrac{\mathrm{d}}{\mathrm{d}t}\left[\left(\dfrac{\partial L}{\partial \dot{q}_s}(t)+\dfrac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)(\xi_s-\dot{q}_s\xi_0)\right]+ \\ \quad\left(\dfrac{\partial L}{\partial t}+\dfrac{\partial L}{\partial q_s}\dot{q}_s+\dfrac{\partial L}{\partial q_{s\tau}}\dot{q}_{s\tau}+\dfrac{\partial L}{\partial \dot{q}_s}\ddot{q}_s+\dfrac{\partial L}{\partial \dot{q}_{s\tau}}\ddot{q}_{s\tau}\right)\xi_0+L\dot{\xi}_0+\dot{G}, \quad t_1\leqslant t\leqslant t_2-\tau, \\ I_N(t,t+\tau,\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{q}_\tau,\dot{\boldsymbol{q}}_\tau)=\dfrac{\mathrm{d}}{\mathrm{d}t}\left[\dfrac{\partial L}{\partial \dot{q}_s}(t)(\xi_s-\dot{q}_s\xi_0)\right]+\left(\dfrac{\partial L}{\partial t}+\dfrac{\partial L}{\partial q_s}+\dfrac{\partial L}{\partial \dot{q}_s}\right)\xi_0+L\dot{\xi}_0+\dot{G}, \quad t_2-\tau<t\leqslant t_2. \end{cases} \tag{13}$$

利用结构方程 (11), 得到

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}I_N(t,t+\tau,\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{q}_\tau,\dot{\boldsymbol{q}}_\tau)= \\ &\quad\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{\partial L}{\partial \dot{q}_s}(t)+\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)(\xi_s-\dot{q}_s\xi_0)\right]+\left(\frac{\partial L}{\partial t}+\frac{\partial L}{\partial q_s}\dot{q}_s+\frac{\partial L}{\partial q_{s\tau}}\dot{q}_{s\tau}+\frac{\partial L}{\partial \dot{q}_s}\ddot{q}_s+\right. \\ &\quad\left.\frac{\partial L}{\partial \dot{q}_{s\tau}}\ddot{q}_{s\tau}\right)\xi_0-\left[\frac{\partial L}{\partial t}(t)\xi_0+\left(\frac{\partial L}{\partial \dot{q}_s}(t)+\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)\left(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0\right)+\left(\frac{\partial L}{\partial q_s}(t)+\right.\right. \\ &\quad\left.\left.\frac{\partial L}{\partial q_{s\tau}}(t+\tau)\right)\xi_s+(Q_s(t)+Q_s(t+\tau))(\xi_s-\dot{q}_s\xi_0)\right]= \\ &\quad\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}(t)+\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)(\xi_s-\dot{q}_s\xi_0)+\left(\frac{\partial L}{\partial \dot{q}_s}(t)+\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)\left(\dot{\xi}_s-\ddot{q}_s\xi_0-\dot{q}_s\dot{\xi}_0\right) \\ &\quad\left(\frac{\partial L}{\partial q_s}\dot{q}_s+\frac{\partial L}{\partial q_{s\tau}}\dot{q}_{s\tau}+\frac{\partial L}{\partial \dot{q}_s}\ddot{q}_s+\frac{\partial L}{\partial \dot{q}_{s\tau}}\ddot{q}_{s\tau}\right)\xi_0-\left(\frac{\partial L}{\partial \dot{q}_s}(t)+\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)\left(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0\right)- \\ &\quad\left(\frac{\partial L}{\partial q_s}(t)+\frac{\partial L}{\partial q_{s\tau}}(t+\tau)\right)\xi_s-(Q_s(t)+Q_s(t+\tau))(\xi_s-\dot{q}_s\xi_0)= \\ &\quad\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}(t)+\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)(\xi_s-\dot{q}_s\xi_0)-\left(\frac{\partial L}{\partial \dot{q}_s}(t)+\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)\ddot{q}_s\xi_0- \\ &\quad\left(\frac{\partial L}{\partial q_s}(t)+\frac{\partial L}{\partial q_{s\tau}}(t+\tau)\right)\xi_s+\left(\frac{\partial L}{\partial q_s}\dot{q}_s+\frac{\partial L}{\partial q_{s\tau}}\dot{q}_{s\tau}+\frac{\partial L}{\partial \dot{q}_s}\ddot{q}_s+\frac{\partial L}{\partial \dot{q}_{s\tau}}\ddot{q}_{s\tau}\right)\xi_0- \\ &\quad(Q_s(t)+Q_s(t+\tau))(\xi_s-\dot{q}_s\xi_0), \qquad t_1\leqslant t\leqslant t_2-\tau, \end{aligned} \tag{14a}$$

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}I_N(t,t+\tau,\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{q}_\tau,\dot{\boldsymbol{q}}_\tau)= \\ &\quad\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{q}_s}(t)(\xi_s-\dot{q}_s\xi_0)\right]+\left(\frac{\partial L}{\partial t}+\frac{\partial L}{\partial q_s}\dot{q}_s+\frac{\partial L}{\partial \dot{q}_s}\ddot{q}_s\right)\xi_0- \\ &\quad\left[\frac{\partial L}{\partial t}(t)\xi_0+\frac{\partial L}{\partial \dot{q}_s}(t)(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)+\frac{\partial L}{\partial q_s}(t)\xi_s+Q_s(t)(\xi_s-\dot{q}_s\xi_0)\right]= \\ &\quad\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}(t)\right)(\xi_s-\dot{q}_s\xi_0)+\frac{\partial L}{\partial \dot{q}_s}(t)(\dot{\xi}_s-\ddot{q}_s\xi_0-\dot{q}_s\dot{\xi}_0)+\left(\frac{\partial L}{\partial q_s}\dot{q}_s+\frac{\partial L}{\partial \dot{q}_s}\ddot{q}_s\right)\xi_0- \\ &\quad\frac{\partial L}{\partial q_s}(t)\xi_s-\frac{\partial L}{\partial \dot{q}_s}(t)(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)-Q_s(t)(\xi_s-\dot{q}_s\xi_0)= \\ &\quad\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}(t)\right)(\xi_s-\dot{q}_s\xi_0)-\frac{\partial L}{\partial \dot{q}_s}(t)\ddot{q}_s\xi_0+\left(\frac{\partial L}{\partial q_s}\dot{q}_s+\frac{\partial L}{\partial \dot{q}_s}\ddot{q}_s\right)\xi_0- \\ &\quad\frac{\partial L}{\partial q_s}(t)\xi_s-Q_s(t)(\xi_s-\dot{q}_s\xi_0), \qquad t_2-\tau<t\leqslant t_2. \end{aligned} \tag{14b}$$

再利用$\dot{q}_s=\dot{q}_{s\tau},\ddot{q}_s=\ddot{q}_{s\tau}$以及系统的 Lagrange 方程 (5), 继续化简式 (14) 得到

$$\begin{cases}\dfrac{\mathrm{d}I_N}{\mathrm{d}t}=\left[\dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial\dot{q}_s}(t)+\dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial\dot{q}_{s\tau}}(t+\tau)-Q_s(t)-Q_s(t+\tau)\right](\xi_s-\dot{q}_s\xi_0)-\\ \left(\dfrac{\partial L}{\partial q_s}(t)+\dfrac{\partial L}{\partial q_{s\tau}}(t+\tau)\right)\xi_s+\left(\dfrac{\partial L}{\partial q_s}\dot{q}_s+\dfrac{\partial L}{\partial q_{s\tau}}\dot{q}_{s\tau}\right)\xi_0=\\ \left[\dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial\dot{q}_s}(t)+\dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial\dot{q}_{s\tau}}(t+\tau)-Q_s(t)-Q_s(t+\tau)\right](\xi_s-\dot{q}_s\xi_0)-\\ \left(\dfrac{\partial L}{\partial q_s}(t)+\dfrac{\partial L}{\partial q_{s\tau}}(t+\tau)\right)(\xi_s-\dot{q}_s\xi_0)=\\ \left[\dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial\dot{q}_s}(t)+\dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial\dot{q}_{s\tau}}(t+\tau)-Q_s(t)-Q_s(t+\tau)-\dfrac{\partial L}{\partial q_s}(t)-\dfrac{\partial L}{\partial q_{s\tau}}(t+\tau)\right]\times\\ (\xi_s-\dot{q}_s\xi_0)=0\cdot(\xi_s-\dot{q}_s\xi_0)=0,\qquad t_1\leqslant t\leqslant t_2-\tau,\\ \dfrac{\mathrm{d}I_N}{\mathrm{d}t}=\left[\dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial\dot{q}_s}(t)-Q_s(t))\right](\xi_s-\dot{q}_s\xi_0)+\dfrac{\partial L}{\partial q_s}\dot{q}_s\xi_0-\dfrac{\partial L}{\partial q_s}(t)\xi_s=\\ \left[\dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial\dot{q}_s}(t)-Q_s(t)\right](\xi_s-\dot{q}_s\xi_0)-\dfrac{\partial L}{\partial q_s}(\xi_s-\dot{q}_s\xi_0)=\\ \left[\dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial\dot{q}_s}(t)-Q_s(t)-\dfrac{\partial L}{\partial q_s}\right](\xi_s-\dot{q}_s\xi_0)=0\cdot(\xi_s-\dot{q}_s\xi_0)=0,\qquad t_2-\tau<t\leqslant t_2.\end{cases}\tag{15}$$

证毕.

这里, 为进一步说明时滞动力学系统的 Lie 对称性和守恒量特点, 利用文献 [17] 给出的非时滞非保守 Lagrange 系统的 Lie 对称性和守恒量:

$$X^{(2)}\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}(t)-\frac{\partial L}{\partial q_s}(t)-Q_s(t)\right]\Bigg|_{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}(t)-\frac{\partial L}{\partial q_s}(t)-Q_s(t)=0}=0,\tag{16}$$

$$I_N(t,\boldsymbol{q},\dot{\boldsymbol{q}})=L\xi_0+\frac{\partial L}{\partial\dot{q}_s}(\xi_s-\dot{q}_s\xi_0)+G.\tag{17}$$

对比式 (16) 和式 (9) 可以发现, 时滞参数$\tau$的存在, 使得 Lie 对称的生成元函数$\xi_0,\xi_s$都会发生延后行为, 即有

$$\begin{cases}\xi_0^-=\xi_0(t,t+\tau,q_s(t),q_s(t-\tau),\dot{q}_s(t),\dot{q}_s(t-\tau)),\\ \xi_s^-=\xi_s(t,t+\tau,q_s(t),q_s(t-\tau),\dot{q}_s(t),\dot{q}_s(t-\tau)).\end{cases}\tag{18}$$

对比式 (17) 和式 (12) 可以发现, 时滞参数$\tau$的存在, 使得守恒量$I_N$也出现了延后行为, 即

$$I_N^-=I_N(t,t+\tau,q_s(t),q_s(t-\tau),\dot{q}_s(t),\dot{q}_s(t-\tau)).\tag{19}$$

这显然是正确的, 因为即使时间发生了延后, 力学系统的一个守恒量依旧还是恒定的. 因此, 非时滞非保守 Lagrange 系统的 Lie 对称性和守恒量应是时滞系统的一个特例.

## 4 算例说明

**例 1** 设时间区间$[t_1,t_2]$上时滞系统的 Lagrange 函数和非有势广义力为

$$\begin{cases}L=\dfrac{1}{2}[\dot{q}(t)-\dot{q}(t-\tau)]^2-[q(t)+q(t-\tau)],\\ Q=-\dot{q}(t-\tau),\end{cases}\tag{20}$$

其中$\tau<[t_1,t_2]$是固定常数, 试研究位形空间中研究系统的 Lie 对称性和守恒量.

由式 (5) 可得系统的运动方程为

$$\begin{cases}\ddot{q}(t)+\ddot{q}(t+\tau)+2=-\dot{q}(t-\tau),\qquad t_1\leqslant t\leqslant t_2-\tau,\\ \ddot{q}(t)+1=-\dot{q}(t-\tau),\qquad t_2-\tau<t\leqslant t_2,\end{cases}\tag{21}$$

可以看出, 时滞参数$\tau$使系统的 Lagrange 方程出现了分段形式.

根据 Lie 对称确定方程 (9), 系统无限小生成元$\xi_0,\xi_s$满足如下等式:

$$\begin{cases}\ddot{\xi}_1-2\ddot{q}\dot{\xi}_0-\dot{q}\ddot{\xi}_0=0, & t_1\leqslant t\leqslant t_2-\tau,\\ \ddot{\xi}_1-2\ddot{q}\dot{\xi}_0-\dot{q}\ddot{\xi}_0=0, & t_2-\tau<t\leqslant t_2.\end{cases} \tag{22}$$

相应的一组特殊无限小变换生成元为

$$\xi_0=0,\ \xi_1=1,\ t_1\leqslant t\leqslant t_2, \tag{23}$$

这里因为选取的无限小生成元函数$\xi_0,\xi_s$为常数, 所以它对时滞参数$\tau$不分段.

将式 (23) 代入结构方程 (11), 得

$$\begin{cases}[\dot{q}(t)+\dot{q}(t+\tau)]\dot{\xi}_1-2\xi_1-\dot{q}(t-\tau)\xi_1+\dot{G}=0\Rightarrow G=2t+q(t-\tau), & t_1\leqslant t\leqslant t_2-\tau,\\ \dot{q}(t)\dot{\xi}_1-\xi_1-\dot{q}(t-\tau)\xi_1+\dot{G}=0\Rightarrow G=t+q(t-\tau), & t_2-\tau<t\leqslant t_2.\end{cases} \tag{24}$$

将式 (24) 代入系统 Lie 对称导致的守恒量式 (12), 得到

$$I_N=\begin{cases}\dot{q}(t)+\dot{q}(t+\tau)+2t+q(t-\tau)=\mathrm{const}, & t_1\leqslant t\leqslant t_2-\tau,\\ \dot{q}(t)+t+q(t-\tau)=\mathrm{const}, & t_2-\tau<t\leqslant t_2.\end{cases} \tag{25}$$

由式 (25) 可以看出, 时滞参数使系统的守恒量发生了分段特性. 式 (25) 是单自由度时滞非保守 Lagrange 系统的 Noether 型守恒量, 显然它是正确的, 因为将运动方程 (21) 左右积分一次, 即可得到式 (25), 因此它是系统的首次积分.

上面的例子很好地说明了时滞系统中时滞参数$\tau$一般情况下会使系统的 Lagrange 方程出现分段表达, 但系统的特殊 Lie 对称性无限小生成元函数$\xi_0,\ \xi_s$依旧光滑连续, 且系统的守恒量形式也出现分段化, 因此可以认为时滞参数$\tau$会使系统的 Lie 对称性和守恒量在形式上发生变化. 下面的例子则可说明, 时滞参数$\tau$并没有改变对称性和守恒量的物理意义.

**例 2** 设时间区间$[t_1,t_2]$上带有时滞参数$\tau$的自由弹簧振子, 如图 1 所示.

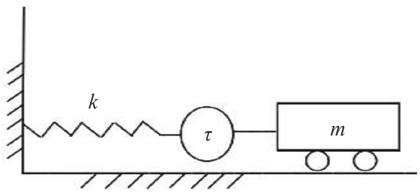


**图 1** 时滞弹簧振子模型

Fig. 1 The spring oscillator model with time delay

系统的 Lagrange 函数为

$$L=\frac{1}{2}m[\dot{q}(t)+\dot{q}(t-\tau)]^2-\frac{1}{2}k[q(t)+q(t-\tau)]^2. \tag{26}$$

由式 (5) 可得系统的运动方程为

$$\begin{cases}2m\ddot{q}(t)+m\ddot{q}(t+\tau)+m\ddot{q}(t-\tau)+2kq(t)+kq(t+\tau)+kq(t-\tau)=0, & t_1\leqslant t\leqslant t_2-\tau,\\ m\ddot{q}(t)+m\ddot{q}(t-\tau)+kq(t)+kq(t-\tau)=0, & t_2-\tau<t\leqslant t_2.\end{cases} \tag{27}$$

可以看出, 时滞参数$\tau$使系统的 Lagrange 方程出现了分段形式.

根据 Lie 对称确定方程 (9), 系统无限小生成元$\xi_0,\xi_s$满足如下等式:

$$\begin{cases}2m(\ddot{\xi}_1-2\ddot{q}\dot{\xi}_0-\dot{q}\ddot{\xi}_0)+2k\xi_1=0, & t_1\leqslant t\leqslant t_2-\tau,\\ m(\ddot{\xi}_1-2\ddot{q}\dot{\xi}_0-\dot{q}\ddot{\xi}_0)+k\xi_1=0, & t_2-\tau<t\leqslant t_2.\end{cases} \tag{28}$$

相应的一组特殊无限小变换生成元为

$$\begin{cases}\xi_0=0,\ \xi_1=2\dot{q}(t)+\dot{q}(t+\tau)+\dot{q}(t-\tau), & t_1\leqslant t\leqslant t_2-\tau,\\ \xi_0=0,\ \xi_1=\dot{q}(t)+\dot{q}(t-\tau), & t_2-\tau<t\leqslant t_2,\end{cases} \tag{29}$$

这里无限小生成元函数$\xi_0$为常数, $\xi_1$出现了分段.

将式 (29) 代入结构方程 (11), 得

$$\begin{cases} m[2\dot q(t)+\dot q(t+\tau)+\dot q(t-\tau)]\dot\xi_1-k[2q(t)+q(t+\tau)+q(t-\tau)]\xi_1+\dot G=0\Rightarrow \\ \qquad G=-\dfrac{1}{2}m[2\dot q(t)+\dot q(t+\tau)+\dot q(t-\tau)]^2+\dfrac{1}{2}k[2q(t)+q(t+\tau)+q(t-\tau)]^2, \qquad t_1\leqslant t\leqslant t_2-\tau, \\ m[\dot q(t)+\dot q(t-\tau)]\dot\xi_1-k[q(t)+q(t-\tau)]\xi_1+\dot G=0\Rightarrow \\ \qquad G=-\dfrac{1}{2}m[\dot q(t)+\dot q(t-\tau)]^2+\dfrac{1}{2}k[q(t)+q(t-\tau)]^2, \qquad t_2-\tau<t\leqslant t_2. \end{cases} \tag{30}$$

将式 (30) 代入系统 Lie 对称导致的守恒量式 (12), 得到

$$I_N=\begin{cases} \dfrac{1}{2}m[2\dot q(t)+\dot q(t+\tau)+\dot q(t-\tau)]^2+ \\ \qquad \dfrac{1}{2}k[2q(t)+q(t+\tau)+q(t-\tau)]^2=\text{const}, \qquad t_1\leqslant t\leqslant t_2-\tau, \\ \dfrac{1}{2}m[\dot q(t)+\dot q(t-\tau)]^2+\dfrac{1}{2}k[q(t)+q(t-\tau)]^2=\text{const}, \qquad t_2-\tau<t\leqslant t_2, \end{cases} \tag{31}$$

这里, 很明显, 时滞参数使系统的守恒量也发生了分段特性. 众所周知, 对于无时滞参数的弹簧振子系统, 系统存在较为明显的守恒量即机械能守恒:

$$I'_N=\frac{1}{2}m\dot q^2(t)+\frac{1}{2}kq^2(t). \tag{32}$$

对比式 (31) 和 (32), 我们可认为时滞弹簧振子的动能在$m[2\dot q(t)+\dot q(t+\tau)+\dot q(t-\tau)]^2/2$和$m[\dot q(t)+\dot q(t-\tau)]^2/2$之间切换, 而弹簧势能对应在$k[q(t)+q(t+\tau)+q(t-\tau)]^2/2$和$k[q(t)+q(t-\tau)]^2/2$之间切换. 但在任何运动时间区间上, 系统的机械能依旧守恒, 即物块的动能加弹簧的势能之和是固定不变的, 因此, 时滞参数$\tau$仅仅是改变了系统的对称性和守恒量形式, 但守恒量的本质物理意义依旧是不变的.

## 5　结 束 语

时滞动力学系统的对称性和守恒量在物理科学和工程中有着重要作用, 本文对含固定时滞参数的非保守 Lagrange 系统的 Lie 定理进行了研究. 主要工作有:

1) 通过对区间的分段处理, 建立了时滞非保守系统的 Hamilton 原理和 Lagrange 方程, 即式 (5);

2) 运用泛函微分方程的经典 Lie 变换群, 给出了时滞 Lagrange 系统的 Lie 对称性的确定方程, 即式 (9), 它是经典力学 Lie 对称性的一致延伸;

3) 通过构造规范函数, 给出了时滞系统 Lie 对称性导致的 Noether 型守恒量, 即式 (12), 并给出了证明, 且算例也说明了方法和结果的有效性.

同时, 本文也简单讨论了时滞参数对力学系统的 Lie 对称性和守恒量的变化影响. 应该看到, 由于时滞系统的广泛存在性, 本文的内容可以进一步推广到含时滞非完整约束力学系统的各类对称性如 Noether、Lie 和 Mei 以及相应的守恒量研究中.

### 参考文献（References）:


[1] 梅凤翔. 约束力学系统的对称性与守恒量[M]. 北京: 北京理工大学出版社, 2004.（MEI Fengxiang. *Symmetries and Conserved Quantities of Constrained Mechanical Systems*[M]. Beijing: Beijing University of Technology Press, 2004.(in Chinese)）

[2] RICHARD J P. Time-delay systems: an overview of some recent advances and open problems[J]. *Automatica*, 2003, **39**(10): 1667-1694.

[3] 胡海岩, 王在华. 非线性时滞动力学系统的研究进展[J]. 力学进展, 1999, **29**(4): 501-512.（HU Haiyan, WANG Zaihua. Review nonlinear dynamic systems involving time delays[J]. *Advances in Mechanics*, 1999, **29**(4): 501-512.(in Chinese)）

[4] 马丽, 马瑞楠. 一类随机泛函微分方程带随机步长的EM逼近的渐近稳定[J]. 应用数学和力学, 2019, **40**(1): 97-107.（MA Li, MA Ruinan. Almost sure asymptotic stability of the Euler-Maruyama method with random variable stepsizes for stochastic functional differential equations[J]. *Applied Mathematics and Mechanics*, 2019, **40**(1): 97-

107.(in Chinese))

[5] ROSENBLUETH J F. Systems with time delay in the calculus of variations: a variational approach[J]. *IMA Journal of Mathematical Control and Information*, 1988, **5**(2): 125-145.

[6] CHAN W L, YUNG S P. Sufficient conditions for variational problems with delayed argument[J]. *Journal of Optimization Theory and Applications*, 1993, **76**(1): 131-144.

[7] FREDERICO G S F, TOMES D F M. Noether's symmetry theorem for variational and optimal control problems with time delay[J]. *Numerical Algebra*, 2012, **2**(3): 619-630.

[8] ZHANG Y, JIN S X. Noether symmetries of dynamics for non-conservative systems with time delay[J]. *Acta Physica Sinica*, 2013, **62**(23): 234502.

[9] JIN S X, ZHANG Y. Noether symmetry and conserved quantity for a Hamilton system with time delay[J]. *Chinese Physics B*, 2014, **23**(5): 339-346.

[10] 金世欣, 张毅. 相空间中含时滞的非保守力学系统的Noether定理[J]. 中山大学学报(自然科学版), 2014, **53**(4): 57-62.（JIN Shixin, ZHANG Yi. Noether theorem for nonconservative mechanical system with time delay in phase space[J]. *Acta Scientiarum Naturalium Universitatis Sunyatseni*, 2014, **53**(4): 57-62.(in Chinese))

[11] 祖启航, 朱建青, 张毅. 基于微分变分原理的相空间中含时滞的非保守力学系统的守恒律[J]. 广西师范学院学报(自然科学版), 2015, **32**(4): 40-46.（ZU Qihang, ZHU Jianqing, ZHANG Yi. A study of conservative laws of nonconservative mechanical system with time delay in phase space based on differential variational principle[J]. *Journal of Guangxi Teachers Education University* (*Natural Science Edition*), 2015, **32**(4): 40-46.(in Chinese))

[12] 贺东海. 含时滞的约束力学系统的对称性与守恒量研究[D]. 硕士学位论文. 杭州: 浙江理工大学, 2014.（HE Donghai. Symmetries and conserved quantities of constrained mechanical systems with time delay[D]. Master Thesis. Hangzhou: Zhejiang Sci-Tech University, 2014.(in Chinese))

[13] LIN F B, MELESHKO S V, FLOOD A E. Symmetries of population balance equations for aggregation, and growth processes[J]. *Applied Mathematics and Computation*, 2017, **307**: 193-203.

[14] 周景润, 傅景礼. 约束Hamilton系统的Lie对称性及其在场论中的应用[J]. 应用数学和力学, 2019, **40**(7): 810-822.（ZHOU Jingrun, FU Jingli. Lie symmetry of constrained Hamiltonian systems and its application in field theory[J]. *Applied Mathematics and Mechanics*, 2019, **40**(7): 810-822.(in Chinese))

[15] ANRIOPOULOS K, LEACH P G L. Symmetry and singularity properties of second-order ordinary differential equations of Lie's type Ⅲ[J]. *Journal of Mathematical Analysis and Applications*, 2007, **328**(2): 860-875.

[16] 梅凤翔. 分析力学(下卷)[M]. 北京: 北京理工大学出版社, 2013: 32-45.（MEI Fengxiang. *Analytical Mechanics* (*Volume* Ⅱ)[M]. Beijing: Beijing University of Technology Press, 2013: 32-45.(in Chinese))

[17] 方建会. 二阶非完整力学系统的Lie对称性与守恒量[J]. 应用数学和力学, 2002, **23**(9): 982-986.（FANG Jianhui. Lie symmetries and conserved quantities of second-order nonholonomic mechanical system[J]. *Applied Mathematics and Mechanics*, 2002, **23**(9): 982-986.(in Chinese))