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一类分数阶 Langevin 方程 block-by-block 算法的数值分析*

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摘要： 分数阶 Langevin 方程有重要的科学意义和工程应用价值，基于经典 block-by-block 算法，求解了一类含有 Caputo 导数的分数阶 Langevin 方程的数值解。Block-by-block 算法通过引入二次 Lagrange 基函数插值，构造出逐块收敛的非线性方程组，通过在每一块耦合求得分数阶 Langevin 方程的数值解。在 $0 < \alpha < 1$ 条件下，应用随机 Taylor 展开证明 block-by-block 算法是 $3 + \alpha$ 阶收敛的，数值试验表明在不同 α 和时间步长 h 取值下，block-by-block 算法具有稳定性和收敛性，克服了现有方法求解分数阶 Langevin 方程速度慢精度低的缺点，表明 block-by-block 算法求解分数阶 Langevin 方程是高效的。

关　键　词： 分数阶 Langevin 方程；　block-by-block 算法；　稳定性；　收敛性；　数值试验

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Numerical Analysis of a Class of Fractional Langevin Equations With the Block-by-Block Method

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Abstract: The fractional Langevin equation is of great scientific significance and engineering application value. Based on the classical block-by-block method, the numerical solution of a class of fractional Langevin equations with Caputo derivatives was obtained. Through introduction of the quadratic Lagrange basis function interpolation, the block-by-block convergent nonlinear equations were constructed, and the numerical solution of the Langevin equation was obtained by coupling in each block. Under the condition of $0 < \alpha < 1$, the stochastic Taylor expansion was used to prove that the block-by-block method is $(3 + \alpha)$-order convergent. Numerical experiments show that, the block-by-block method is stable and convergent under different values of α and time step h, and overcomes the existing methods' disadvantages of slow speed and poor accuracy for solving fractional Langevin equations.

Key words: fractional Langevin equation; block-by-block method; stability; convergence; numerical experiment

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引　　言

分数阶 Langevin 方程作为一类非线性随机分数阶微分方程,在数学、物理、生物化学以及工程领域得到广泛的应用[1-2].如用于单分子扩散、疏散过程及黏弹性力学等的建模[3-5].近年来,随着分数阶统计动力学研究的深入进行,发现分数阶 Langevin 方程能描述反常扩散现象[6-7].对于分数阶 Langevin 方程的数值求解,由于方程中存在随机变量,给方程数值求解算法的构造,以及通过方程模拟实际问题带来一定的困难,寻找高效的数值解法逐渐成为研究的热点问题[8-9].Bhrawy 和 Alghamdi[10] 提出移位 Jacobi-Gauss-Lobatto 搭配技术方法来数值求解非线性 Langevin 方程,通过适当选择搭配点得到数值结果,结果表明 Jacobi-Gauss-Lobatto 方法操作简单,数值算例验证了该方法的有效性.Guo 等[11] 对不同外力影响下的分数阶 Langevin 方程进行离散求得数值解,发现分数阶 Langevin 方程能够反映出反常扩散现象,数值模拟表明此数值解法是高效可行的.孙春艳和徐伟[12]引入两个线性非耦合的随机模拟方程,利用它们求解一类随机分数阶微分方程,采用 Laplace 变换及逆变换,得到方程解的积分表达式,同时建立两个模拟方程之间的联系,得到求解随机微分方程初值问题的数值迭代算法,数值算例表明了此方法的稳定性和有效性.这些数值算法虽然填补了数值求解的空缺,但其中一些求解难题仍未得到彻底解决,如这些数值格式多采用显式格式,其计算效率和精度等仍需提高.发展新的数值算法,特别是在保证计算可靠性和高精度的前提下,提高计算效率,解决分数阶微分方程数值算法计算量和存储量过大的问题,成为数值方法研究的重要任务.

Block-by-block 算法应用到分数阶微分方程数值求解时[13],能综合各类数值算法的优点,使得数值解很好地逼近解析解.Huang 等[14]结合一种非奇点核的 Volterra 积分方程 block-by-block 算法,对一类分数阶微分方程数值求解.此格式通过在每一块耦合求得数值解,克服了显式 Euler 法求解时遇到的数值不稳定性等问题,首次严格验证了该方法的收敛阶至少是 3 阶.Cao 和 Xu[15] 提出了一种修正的 block-by-block 算法求解非线性 Volterra 积分方程,得到一个新的高阶数值格式,其优势在于可以单独求解未知量,避免在每一步耦合求解,给出收敛性分析.Esmaeili[16] 提出了一种分段非多项式搭配方法,将整个区域划分为几个小的子域,在每个子域上用 block-by-block 方法构造多项式块,应用该方法求解线性和非线性分数阶微分方程,数值算例验证了该方法的有效性.

经典的 block-by-block 算法是求解积分方程的一种高效数值算法,已经成功运用于求解分数阶非线性微分方程.本文将此算法应用到一类分数阶 Langevin 方程的求解中,即采用 Lagrange 基函数插值来构造数值格式,应用随机 Taylor 展开进行收敛性和稳定性分析[17-18],得到最优收敛阶为 $3+\alpha$.数值试验表明 block-by-block 数值格式是高效的,在计算精度和计算复杂度方面具有明显优势.

1　分数阶 Langevin 方程的 block-by-block 数值格式

1.1　分数阶 Langevin 方程

分数阶 Langevin 方程的一般形式为[1]

$$ {}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha} v(t) = -\lambda^{\alpha} v(t) + v_0 \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \xi(t), \qquad 0<\alpha<1, \tag{1} $$

初始条件为

$$ v^{(k)}(0) = v_0^{(k)}, \qquad k = 0,1,\cdots,n-1, \tag{2} $$

其中 λ 为一个耗散参量;$\xi(t)$ 是随机力,服从 Gauss 分布,其一般可用 $\mathrm{d}W(t)/\mathrm{d}t$ 代替,$W(t)$ 表示经典 Brown 运动; n 是正整数;α 是分数阶导数的阶数;$v^{(k)}(t)$ 是 v 的 k 阶导数;${}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha} v(t)$ 为 α 阶 Caputo 分数阶导数,其定义为

$$ {}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha} v(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\tau)^{m-\alpha-1} v^{(m)}(\tau)\,\mathrm{d}\tau, & m-1<\alpha<m, \\ v^{(m)}(t), & \alpha = m, \end{cases} $$

其中 $\Gamma(\cdot)$ 表示 Gamma 函数.

1.2 Block-by-block 格式构造

考虑初值问题(1)、(2)，假设解是连续的，将其转化为 Volterra 积分方程，文献[19-21]已经证明它和下面的 Volterra 积分方程是等价的：

$$v(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\left[-\lambda^\alpha v(\tau)+v_0\frac{\tau^{-\alpha}}{\Gamma(1-\alpha)}+\xi(\tau)\right]\mathrm{d}\tau.$$

为方便叙述，记方程(1)等号右端项为 F，即 $F=f(t,v(t))+\xi(t)$. 有

$$v(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau,v(\tau))\mathrm{d}\tau+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\xi(\tau)\mathrm{d}\tau.$$

将区间 $[0,T]$ 分成 $2N$ 个等分的子区间，定义 $t_j=j\cdot h,j=0,1,\cdots,2N$，时间步长 $h=T/2N$，v_j 表示方程(1)在点 t_j 上的数值解，$v(t)$ 表示方程(1)的解析解. 假设已经构造出 $v_j,j=0,1,\cdots,2m$，为构造出 block-by-block 数值格式，令其逼近 $v(t_{2m+1})$ 和 $v(t_{2m+2})$，$m=1,2,\cdots,N-1$，则有如下逼近式：

$$\begin{aligned}
v(t_{2m+1})={}&\frac{1}{\Gamma(\alpha)}\int_0^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}f(\tau,v(\tau))\mathrm{d}\tau+\frac{1}{\Gamma(\alpha)}\int_0^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}\xi(\tau)\mathrm{d}\tau={}\\
&\frac{1}{\Gamma(\alpha)}\left[\sum_{k=0}^{m-1}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}f(\tau,v(\tau))\mathrm{d}\tau+\int_{t_{2m}}^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}f(\tau,v(\tau))\mathrm{d}\tau\right]+{}\\
&\frac{1}{\Gamma(\alpha)}\int_0^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}\xi(\tau)\mathrm{d}\tau\approx{}\\
&\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\int_{t_{2k}}^{t_{2k+1}}(t_{2m+1}-\tau)^{\alpha-1}[\psi_{0,k}(\tau)f_{2k}+\psi_{1,k}(\tau)f_{2k+1}+\psi_{2,k}(\tau)f_{2k+2}]\mathrm{d}\tau+{}\\
&\frac{1}{\Gamma(\alpha)}\int_{t_{2m}}^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}[\psi_{0,m}(\tau)f_{2m}+\psi_{1,m}(\tau)f_{2m+1/2}+\psi_{2,m}(\tau)f_{2m+1}]\mathrm{d}\tau+{}\\
&\frac{1}{\Gamma(\alpha)}\int_0^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}\xi(\tau)\mathrm{d}\tau,
\end{aligned}\tag{3}$$

其中 $\psi_{i,k}(t)\,(i=0,1,2;k=0,1,\cdots,m-1)$ 和 $\psi_{i,m}(t)\,(i=0,1,2)$ 分别为点 t_{2k},t_{2k+1},t_{2k+2} 和 $t_{2m},t_{2m+1/2},t_{2m+1}$ 上的二次 Lagrange 基函数，其定义如下：

$$\psi_{0,k}(t)=\frac{(t-t_{2k+1})(t-t_{2k+2})}{2h^2},\ \psi_{1,k}(t)=\frac{(t-t_{2k})(t-t_{2k+2})}{-h^2},$$

$$\psi_{2,k}(t)=\frac{(t-t_{2k})(t-t_{2k+1})}{2h^2},\ \psi_{0,m}(t)=\frac{2(t-t_{2m+1/2})(t-t_{2m+1})}{h^2},$$

$$\psi_{1,m}(t)=\frac{-4(t-t_{2m})(t-t_{2m+1})}{h^2},\ \psi_{2,m}(t)=\frac{2(t-t_{2m})(t-t_{2m+1/2})}{h^2}.$$

利用二次 Lagrange 插值，$f_{2m+1/2}$ 的逼近格式如下：

$$f_{2m+1/2}\approx\frac{3}{8}f_{2m}+\frac{3}{4}f_{2m+1}-\frac{1}{8}f_{2m+2}.\tag{4}$$

将式(4)代入式(3)，得到以下格式：

$$\begin{aligned}
v_{2m+1}={}&\sum_{k=0}^{m-1}(A_{2m+1}^{0,k}f_{2k}+A_{2m+1}^{1,k}f_{2k+1}+A_{2m+1}^{2,k}f_{2k+2})+{}\\
&A_{2m+1}^{0,m}f_{2m}+A_{2m+1}^{1,m}f_{2m+1}+A_{2m+1}^{2,m}f_{2m+2}+\frac{1}{\Gamma(\alpha)}\int_0^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}\xi(\tau)\mathrm{d}\tau,
\end{aligned}\tag{5}$$

其中

$$A_{2m+1}^{i,k}=\frac{1}{\Gamma(\alpha)}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}\psi_{i,k}(\tau)\mathrm{d}\tau,\qquad i=0,1,2;\ k=0,1,\cdots,m-1,$$

$$A_{2m+1}^{0,m}=\omega_{2m+1}^{0,m}+\frac{3}{8}\omega_{2m+1}^{1,m},\ A_{2m+1}^{1,m}=\frac{3}{4}\omega_{2m+1}^{1,m}+\omega_{2m+1}^{2,m},\ A_{2m+1}^{2,m}=-\frac{1}{8}\omega_{2m+1}^{1,m},$$

$$\omega_{2m+1}^{i,m} = \frac{1}{\Gamma(\alpha)}\int_{t_{2m}}^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}\psi_{i,m}(\tau)\mathrm{d}\tau, \qquad i=0,1,2.$$

下面采用如下逼近来计算 $v(t_{2m+2})$:

$$\begin{aligned}
v(t_{2m+2}) &= \frac{1}{\Gamma(\alpha)}\int_0^{t_{2m+2}}(t_{2m+2}-\tau)^{\alpha-1}f(\tau,v(\tau))\mathrm{d}\tau + \frac{1}{\Gamma(\alpha)}\int_0^{t_{2m+2}}(t_{2m+2}-\tau)^{\alpha-1}\xi(\tau)\mathrm{d}\tau = \\
&\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+2}-\tau)^{\alpha-1}f(\tau,v(\tau))\mathrm{d}\tau + \frac{1}{\Gamma(\alpha)}\int_0^{t_{2m+2}}(t_{2m+2}-\tau)^{\alpha-1}\xi(\tau)\mathrm{d}\tau \approx \\
&\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+2}-\tau)^{\alpha-1}[\psi_{0,k}(\tau)f_{2k}+\psi_{1,k}(\tau)f_{2k+1}+\psi_{2,k}(\tau)f_{2k+2}]\mathrm{d}\tau + \\
&\frac{1}{\Gamma(\alpha)}\int_0^{t_{2m+2}}(t_{2m+2}-\tau)^{\alpha-1}\xi(\tau)\mathrm{d}\tau.
\end{aligned}$$

得到第 $2m+2$ 步的数值格式:

$$v_{2m+2} = \sum_{k=0}^{m}(A_{2m+2}^{0,k}f_{2k}+A_{2m+2}^{1,k}f_{2k+1}+A_{2m+2}^{2,k}f_{2k+2}) + \frac{1}{\Gamma(\alpha)}\int_0^{t_{2m+2}}(t_{2m+2}-\tau)^{\alpha-1}\xi(\tau)\mathrm{d}\tau, \tag{6}$$

其中

$$A_{2m+2}^{i,k} = \frac{1}{\Gamma(\alpha)}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+2}-\tau)^{\alpha-1}\psi_{i,k}(\tau)\mathrm{d}\tau, \qquad i=0,1,2;\ k=0,1,\cdots,m.$$

结合式(5)、(6),得到如下数值格式:

$$\begin{cases}
v_{2m+1} = \sum_{k=0}^{m-1}(A_{2m+1}^{0,k}f_{2k}+A_{2m+1}^{1,k}f_{2k+1}+A_{2m+1}^{2,k}f_{2k+2}) + A_{2m+1}^{0,m}f_{2m}+A_{2m+1}^{1,m}f_{2m+1} + \\
\quad A_{2m+1}^{2,m}f_{2m+2} + \frac{1}{\Gamma(\alpha)}\int_0^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}\xi(\tau)\mathrm{d}\tau, \\
v_{2m+2} = \sum_{k=0}^{m}(A_{2m+2}^{0,k}f_{2k}+A_{2m+2}^{1,k}f_{2k+1}+A_{2m+2}^{2,k}f_{2k+2}) + \\
\quad \frac{1}{\Gamma(\alpha)}\int_0^{t_{2m+2}}(t_{2m+2}-\tau)^{\alpha-1}\xi(\tau)\mathrm{d}\tau.
\end{cases} \tag{7}$$

2　Block-by-block 格式的数值分析

2.1　Block-by-block 格式的截断误差

对格式(7)的截断误差进行精细估计.首先估计奇数层的局部截断误差,定义 $2m+1$ 层的截断误差为

$$r_{2m+1}(h) := v(t_{2m+1}) - \hat{v}_{2m+1}, \tag{8}$$

其中 $\hat{v}_{2m+1}$ 是 $v(t_{2m+1})$ 的一个逼近.将解析解代入式(5),得到

$$\begin{aligned}
\hat{v}_{2m+1} = &\sum_{k=0}^{m-1}[A_{2m+1}^{0,k}f(t_{2k},v(t_{2k}))+A_{2m+1}^{1,k}f(t_{2k+1},v(t_{2k+1}))+A_{2m+1}^{2,k}f(t_{2k+2},v(t_{2k+2}))] + \\
&A_{2m+1}^{0,m}f(t_{2m},v(t_{2m}))+A_{2m+1}^{1,m}f(t_{2m+1},v(t_{2m+1}))+A_{2m+1}^{2,m}f(t_{2m+2},v(t_{2m+2})) + \\
&\frac{1}{\Gamma(\alpha)}\int_0^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}\xi(\tau)\mathrm{d}\tau.
\end{aligned} \tag{9}$$

对于 $r_{2m+1}(h)$, 有如下估计.

引理 1　设 $r_{2m+1}(h)$ 是式(8)中定义的截断误差,设 $f(\cdot,v(\cdot)) \in C^4[0,T]$,当 $0<\alpha<1$ 时,有

$$|r_{2m+1}(h)| \leqslant Ch^{3+\alpha}.$$

证明　结合式(3)、(5)和(9),有

$$\begin{aligned}
r_{2m+1}(h) = &v(t_{2m+1}) - \sum_{k=0}^{m-1}\sum_{i=0}^{2}A_{2m+1}^{i,k}f(t_{2k+i},v(t_{2k+i})) - f(t_{2m},v(t_{2m}))\omega_{2m+1}^{0,m} - \\
&\left[\frac{3}{8}f(t_{2m},v(t_{2m}))+\frac{3}{4}f(t_{2m+1},v(t_{2m+1}))-\frac{1}{8}f(t_{2m+2},v(t_{2m+2}))\right]\omega_{2m+1}^{1,m} -
\end{aligned}$$

$$
\begin{aligned}
&f(t_{2m+1},v(t_{2m+1}))\omega_{2m+1}^{2,m}-\frac{1}{\Gamma(\alpha)}\int_0^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}\xi(\tau)\mathrm{d}\tau=\\
&\frac{1}{\Gamma(\alpha)}\left[\sum_{k=0}^{m-1}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}f(\tau,v(\tau))\mathrm{d}\tau+\int_{t_{2m}}^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}f(\tau,v(\tau))\mathrm{d}\tau\right]-\\
&\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\sum_{i=0}^{2}f(t_{2k+i},v(t_{2k+i}))\times\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}\psi_{i,k}(\tau)\mathrm{d}\tau+\\
&\frac{1}{\Gamma(\alpha)}\Bigg[f(t_{2m},v(t_{2m}))\times\int_{t_{2m}}^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}\psi_{0,m}(\tau)\mathrm{d}\tau+\\
&\left[\frac{3}{8}f(t_{2m},v(t_{2m}))+\frac{3}{4}f(t_{2m+1},v(t_{2m+1}))-\frac{1}{8}f(t_{2m+2},v(t_{2m+2}))\right]\times\\
&\int_{t_{2m}}^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}\psi_{1,m}(\tau)\mathrm{d}\tau+f(t_{2m+1},v(t_{2m+1}))\int_{t_{2m}}^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}\psi_{2,m}(\tau)\mathrm{d}\tau\Bigg]=\\
&\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}\left[f(\tau,v(\tau)-\sum_{i=0}^{2}f(t_{2k+i},v(t_{2k+i}))\psi_{i,k}(\tau)\right]\mathrm{d}\tau+\\
&\frac{1}{\Gamma(\alpha)}\int_{t_{2m}}^{t_{2m+1}}\Bigg\{(t_{2m+1}-\tau)^{\alpha-1}[f(\tau,v(\tau)-[f(t_{2m},v(t_{2m}))\psi_{0,m}(\tau)+\\
&f(t_{2m+1/2},v(t_{2m+1/2}))\psi_{1,m}(\tau)+f(t_{2m+1},v(t_{2m+1}))\psi_{2,m}(\tau)]+\\
&\int_{t_{2m}}^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}\Bigg[f(t_{2m+1/2},v(t_{2m+1/2}))-\Bigg(\frac{3}{8}f(t_{2m},v(t_{2m}))+\\
&\frac{3}{4}f(t_{2m+1},v(t_{2m+1}))-\frac{1}{8}f(t_{2m+2},v(t_{2m+2}))\Bigg)\Bigg]\psi_{1,m}(\tau)\Bigg\}\mathrm{d}\tau=\\
&\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}R_{2k}(\tau)\mathrm{d}\tau+\\
&\frac{1}{\Gamma(\alpha)}\int_{t_{2m}}^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}(R_{2m-1}(\tau)+R_{2m}(\tau))\psi_{1,m}(\tau)\mathrm{d}\tau.
\end{aligned}
$$

根据 Taylor 定理,所有的 $\tau\in[t_{2k},t_{2k+2}]$,存在 $\eta_k(\tau)\in[t_{2k},t_{2k+2}]$, 使得

$$R_{2k}(\tau)=\frac{f^{(3)}(\eta_k(\tau),v(\eta_k(\tau)))}{3!}\prod_{i=0}^{2}(\tau-t_{2k+i}),\qquad\forall\tau\in[t_{2k},t_{2k+2}]$$

和所有的 $\tau\in[t_{2m},t_{2m+1}]$,存在 $\eta_1(\tau),\eta(\tau)\in[t_{2m},t_{2m+1}]$, 使得

$$R_{2m-1}(\tau)=\frac{f^{(3)}(\eta_1(\tau),v(\eta_1(\tau)))}{3!}(\tau-t_{2m})(\tau-t_{2m+1/2})(\tau-t_{2m+1}),$$

$$R_{2m}(\tau)=\frac{1}{16}h^3f^{(3)}(\eta(\tau),v(\eta(\tau))).$$

则可得到

$$
\begin{aligned}
r_{2m+1}(h)=&\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}\frac{f^{(3)}(\eta_k(\tau),v(\eta_k(\tau)))}{3!}\prod_{i=0}^{2}(\tau-t_{2k+i})\mathrm{d}\tau+\\
&\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\int_{t_{2m}}^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}\frac{f^{(3)}(\eta_1(\tau),v(\eta_1(\tau)))}{3!}\prod_{i=0}^{2}(\tau-t_{2m+1/2})+\\
&\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\int_{t_{2m}}^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}\frac{1}{16}h^3f^{(3)}(\eta(\tau),v(\eta(\tau)))\psi_{1,m}\mathrm{d}\tau.
\end{aligned}\tag{10}
$$

逐一估计式(10)等号右端的三项,分别记为 R_1,R_2,R_3.对于第一项 R_1, 得到如下估计:

$$|R_1|\leqslant\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\left|\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}\frac{f^{(3)}(\hat\eta_k(\tau),v(\hat\eta_k(\tau)))}{3!}\prod_{i=0}^{2}(\tau-t_{2k+1})\mathrm{d}\tau\right|+$$

$$\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\left|\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}\frac{f^{(3)}(\eta_k(\tau),v(\eta_k(\tau)))-f^{(3)}(\hat{\eta}_k,\ v(\hat{\eta}_k))}{3!}\prod_{i=0}^{2}(\tau-t_{2k+1/2})\right|,\tag{11}$$

其中 $\hat{\eta}_k=t_{2k+1}$.由式(11)右端第一项,得到

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\left|\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}\frac{f^{(3)}(\hat{\eta}_k(\tau),v(\hat{\eta}_k(\tau)))}{3!}\prod_{i=0}^{2}(\tau-t_{2k+i})\mathrm{d}\tau\right|\leqslant\\
&\frac{M_1}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\left|\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}\prod_{i=0}^{2}(\tau-t_{2k+i})\mathrm{d}\tau\right|\leqslant\\
&\frac{M_1}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\left|\int_{t_{2k}}^{t_{2k+1}}(t_{2m+1}-\tau)^{\alpha-1}\prod_{i=0}^{2}(\tau-t_{2k+i})\mathrm{d}\tau+\int_{t_{2k+1}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}\prod_{i=0}^{2}(\tau-t_{2k+i})\mathrm{d}\tau\right|=\\
&\frac{M_1}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\left|(t_{2m+1}-\hat{\tau}_k)^{\alpha-1}\int_{t_{2k}}^{t_{2k+1}}\prod_{i=0}^{2}(\tau-t_{2k+i})\mathrm{d}\tau+(t_{2m+1}-\bar{\tau}_k)^{\alpha-1}\int_{t_{2k+1}}^{t_{2k+2}}\prod_{i=0}^{2}(\tau-t_{2k+i})\mathrm{d}\tau\right|=\\
&\frac{M_1h^4}{4\Gamma(\alpha)}\sum_{k=0}^{m-1}|(t_{2m+1}-\hat{\tau}_k)^{\alpha-1}+(t_{2m+1}-\bar{\tau}_k)^{\alpha-1}|=\\
&\frac{M_1h^4}{4\Gamma(\alpha)}\sum_{k=0}^{m-1}|(\alpha-1)(t_{2m+1}-\tilde{\tau}_k)^{\alpha-2}(\bar{\tau}_k-\hat{\tau}_k)|\leqslant\\
&\frac{M_1h^4}{4\Gamma(\alpha)}|\alpha-1|\sum_{k=0}^{m-1}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}\mathrm{d}\tau\leqslant\\
&\frac{M_1h^4}{4\Gamma(\alpha)}|\alpha-1|\left|\int_{t_0}^{t_{2m}}(t_{2m+1}-\tau)^{\alpha-2}\mathrm{d}\tau\right|=\\
&\frac{M_1h^4}{4\Gamma(\alpha)}|\alpha-1|\left|-\frac{1}{\alpha-1}(t_{2m+1}-\tau)^{\alpha-1}|_{t_0}^{t_{2m}}\right|\leqslant\\
&\frac{M_1h^4}{4\Gamma(\alpha)}(|(t_{2m+1}-t_{2m})^{\alpha-1}|+|(t_{2m+1}-t_0)^{\alpha-1}|)=\\
&\frac{M_1h^4}{4\Gamma(\alpha)}(h^{\alpha-1}+t_{2m+1}^{\alpha-1})=\frac{M_1h^{3+\alpha}}{4\Gamma(\alpha)}+\frac{M_1h^4}{4\Gamma(\alpha)}t_{2m+1}^{\alpha-1},
\end{aligned}\tag{12}$$

其中　$\hat{\tau}_k\leqslant\tilde{\tau}_k\leqslant\bar{\tau}_k,\ t_{2k}\leqslant\hat{\tau}_k\leqslant t_{2k+1},\ t_{2k+1}\leqslant\bar{\tau}_k\leqslant t_{2k+2},\ M_1=\sup_{t\in[0,T]}|f^{(3)}(t,v(t))|$.

由式(11)右端第二项,得到

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\left|\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}\frac{f^{(3)}(\eta_k(\tau),v(\eta_k(\tau)))-f^{(3)}(\hat{\eta}_k,\ v(\hat{\eta}_k))}{3!}\prod_{i=0}^{2}(\tau-t_{2k+i})\mathrm{d}\tau\right|\leqslant\\
&\frac{M_2\,h}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\left|\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}(\tau-t_{2k})(\tau-t_{2k+1})(\tau-t_{2k+2})\mathrm{d}\tau\right|\leqslant\\
&\frac{M_2h^4}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}\mathrm{d}\tau\leqslant\frac{M_2h^4}{\Gamma(\alpha)}\int_{t_0}^{t_{2m}}(t_{2m+1}-\tau)^{\alpha-1}\mathrm{d}\tau\leqslant\\
&\frac{M_2h^4}{\Gamma(\alpha)}\left|-\frac{1}{\alpha}(t_{2m+1}-\tau)|_{t_0}^{t_{2m}}\right|\leqslant\frac{M_2h^4}{\alpha\Gamma(\alpha)}(|(t_{2m+1}-t_{2m})^{\alpha}|+|(t_{2m+1}-t_0)^{\alpha}|)=\\
&\frac{M_2h^4}{\alpha\Gamma(\alpha)}(h^{\alpha}+t_{2m+1}^{\alpha})\leqslant\frac{M_2h^4}{\alpha\Gamma(\alpha)}(h^{\alpha}+T^{\alpha}),
\end{aligned}\tag{13}$$

其中　$M_2=\sup_{t\in[0,T]}|f^{(4)}(t,v(t))|$.

在上式估计中,用到如下不等式：

$$\left|\frac{f^{(3)}(\eta_k(\tau),v(\eta_k(\tau)))-f^{(3)}(\hat{\eta}_k,\ v(\hat{\eta}_k))}{3!}\right|\leqslant M_2h,\qquad\hat{\eta}_k=t_{2k+1},\ \forall\tau\in[t_{2k},t_{2k+2}].$$

将式(12)、(13)代入式(11),得到

$$|R_1| \leqslant \frac{M_1 h^{3+\alpha}}{4\Gamma(\alpha)} + \frac{M_1 h^4}{4\Gamma(\alpha)} t_{2m+1}^{\alpha-1} + \frac{M_2 h^{\alpha+4}}{\alpha\Gamma(\alpha)} + \frac{M_2 h^4}{\alpha\Gamma(\alpha)} T^{\alpha}. \tag{14}$$

对于式(10)右端第二项 R_2,得到如下估计:

$$|R_2| \leqslant \frac{1}{\Gamma(\alpha)}\int_{t_{2m}}^{t_{2m+1}} (t_{2m+1}-\tau)^{\alpha-1} \left| \frac{f^{(3)}(\eta_1(\tau), v(\eta_1(\tau)))}{3!} \prod_{i=0}^{2} (\tau - t_{2m+i/2}) \right| \mathrm{d}\tau \leqslant$$
$$\frac{M_1 h^3}{4\Gamma(\alpha)}\int_{t_{2m}}^{t_{2m+1}} (t_{2m+1}-\tau)^{\alpha-1}\mathrm{d}\tau = \frac{M_1 h^{3+\alpha}}{\alpha\Gamma(\alpha)}. \tag{15}$$

对于式(10)右端第三项 R_3,得到如下估计:

$$|R_3| \leqslant \frac{M_1 h^3}{16\Gamma(\alpha)} \left| \int_{t_{2m}}^{t_{2m+1}} (t_{2m+1}-\tau)^{\alpha-1} \frac{-4(\tau - t_{2m})(\tau - t_{2m+1})}{h^2} \right| \leqslant$$
$$\frac{M_1 h^3}{\Gamma(\alpha)}\int_{t_{2m}}^{t_{2m+1}} (t_{2m+1}-\tau)^{\alpha-1}\mathrm{d}\tau \leqslant \frac{M_1 h^{3+\alpha}}{\alpha\Gamma(\alpha)}. \tag{16}$$

联立式(14)~(16),得到

$$|r_{2m+1}(h)| \leqslant \frac{M_1 h^{3+\alpha}}{4\Gamma(\alpha)} + \frac{M_1 h^4}{4\Gamma(\alpha)} t_{2m+1}^{\alpha-1} + \frac{M_2 h^{\alpha+4}}{\alpha\Gamma(\alpha)} + \frac{M_2 h^4}{\alpha\Gamma(\alpha)} T^{\alpha} + \frac{2M_1 h^{3+\alpha}}{\alpha\Gamma(\alpha)} =$$
$$\frac{M_1 h^{3+\alpha}}{4\Gamma(\alpha)} + \frac{M_1 h^4}{4\Gamma(\alpha)} [(2m+1)h]^{\alpha-1} + \frac{M_2 h^{\alpha+4}}{\alpha\Gamma(\alpha)} + \frac{M_2 h^4}{\alpha\Gamma(\alpha)} T^{\alpha} + \frac{2M_1 h^{3+\alpha}}{\alpha\Gamma(\alpha)} =$$
$$\frac{M_1 h^{3+\alpha}}{4\Gamma(\alpha)} + \frac{M_1 (2m+1)^{\alpha+1}}{4\Gamma(\alpha)} h^{3+\alpha} + \frac{M_2 h^{\alpha+4}}{\alpha\Gamma(\alpha)} + \frac{M_2 h^4}{\alpha\Gamma(\alpha)} T^{\alpha} + \frac{2M_1 h^{3+\alpha}}{\alpha\Gamma(\alpha)} \leqslant$$
$$Ch^{3+\alpha},$$

这里 C 仅依赖于 M_1, M_2, α,和 T.引理 1 证明完毕.

类似于奇数层的截断误差,定义偶数层的截断误差:

$$r_{2m+2}(h) := v(t_{2m+2}) - \hat{v}_{2m+2}, \tag{17}$$

这里 $\hat{v}_{2m+2}$ 是 $v(t_{2m+2})$ 的一个逼近.

将解析解代入式(6)即可得到如下 $r_{2m+2}(h)$ 的估计.

引理 2 设 $r_{2m+2}(h)$ 是式(17)定义的截断误差,设 $f(\cdot, v(\cdot)) \in C^4[0,T]$,当 $0 < \alpha < 1$ 时,有

$$|r_{2m+2}(h)| \leqslant Ch^{3+\alpha}.$$

证明 其证明过程与引理 1 类似.根据引理 1 和引理 2,得到所构造 block-by-block 格式(7)的截断误差是 $3+\alpha$ 阶($0 < \alpha < 1$).

2.2 Block-by-block 格式的稳定性

考虑到叙述的简单性原则,通过以下系数来重写格式:

$$\begin{cases} \bar{B}_0 = \dfrac{A_{2m+1}^{0,0}}{h^{\alpha}},\ \bar{B}_{2k+1} = \dfrac{A_{2m+1}^{1,k}}{h^{\alpha}}, & k = 0,1,\cdots,m, \\ \bar{B}_{2k} = \dfrac{A_{2m+1}^{2,k-1} + A_{2m+1}^{0,k}}{h^{\alpha}},\ \bar{B}_{2m+2} = \dfrac{A_{2m+1}^{2,m}}{h^{\alpha}}, & k = 1,2,\cdots,m, \\ B_0 = \dfrac{A_{2m+2}^{0,0}}{h^{\alpha}},\ B_{2k+1} = \dfrac{A_{2m+2}^{1,k}}{h^{\alpha}}, & k = 0,1,\cdots,m, \\ B_{2k} = \dfrac{A_{2m+2}^{2,k-1} + A_{2m+2}^{0,k}}{h^{\alpha}},\ B_{2m+2} = \dfrac{A_{2m+2}^{2,m}}{h^{\alpha}}, & k = 1,2,\cdots,m. \end{cases} \tag{18}$$

格式(7)的等价形式如下:

$$\begin{cases} v_{2m+1} = h^{\alpha}\sum_{j=0}^{2m+1}\bar{B}_j f_j + \dfrac{1}{\Gamma(\alpha)}\int_0^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}\xi(\tau)\,\mathrm{d}\tau, & m=1,2,\cdots,N-1, \\ v_{2m+2} = h^{\alpha}\sum_{j=0}^{2m+2}B_j f_j + \dfrac{1}{\Gamma(\alpha)}\int_0^{t_{2m+2}}(t_{2m+2}-\tau)^{\alpha-1}\xi(\tau)\,\mathrm{d}\tau, & m=1,2,\cdots,N-1. \end{cases} \tag{19}$$

引理 3[15]　系数 $\bar{B}_j$ 和 B_j 如式(18)所定义,则

$$\begin{cases} |\bar{B}_j| \le C(2m+3-j)^{\alpha-1}, & j=0,1,\cdots,2m+2, \\ |B_j| \le C(2m+3-j)^{\alpha-1}, & j=0,1,\cdots,2m+2. \end{cases} \tag{20}$$

下面研究当

$$f(t,v(t)) := \gamma v(t) \tag{21}$$

时格式(19)的稳定性,其中 γ 是一个实数.

定理 1　记

$$M_0 = \max\{|v_0^{(0)}|,|v_0^{(1)}|,|v_0^{(2)}|,\cdots,|v_0^{(k)}|,\cdots,|v_0^{(n-1)}|\}.$$

则当 f 具有式(21)的形式时,格式(19)是按照初值稳定的,且其稳定性条件为

$$|\gamma||\hat{B}_{2m+1}|h^{\alpha} < 1, \tag{22}$$

其中

$$|\hat{B}_{2m+1}| = \max\{|B_{2m+1}|+|B_{2m+2}|,|\bar{B}_{2m+1}|+|\bar{B}_{2m+2}|\}.$$

即若式(22)满足,则有

$$|v_j| \le CM_0, \qquad j=1,2,\cdots,2N,$$

这里 C 仅依赖于 γ,α 和 T.

证明　不妨设

$$|\hat{v}_{2i+1}| = |\hat{v}_{2i+2}| = \max\{|v_{2i+1}|,|v_{2i+2}|\}, \qquad i=0,1,\cdots,m.$$

根据格式(19)和式(21),得到

$$\begin{cases} v_{2m+1} = h^{\alpha}\sum_{j=0}^{2m+1}\gamma\bar{B}_j v_j + \dfrac{1}{\Gamma(\alpha)}\int_0^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}\xi(\tau)\,\mathrm{d}\tau, \\ v_{2m+2} = h^{\alpha}\sum_{j=0}^{2m+2}\gamma B_j v_j + \dfrac{1}{\Gamma(\alpha)}\int_0^{t_{2m+2}}(t_{2m+2}-\tau)^{\alpha-1}\xi(\tau)\,\mathrm{d}\tau. \end{cases} \tag{23}$$

令 $\Gamma(t)=\int_t^{t+h}\xi(\tau)\,\mathrm{d}\tau$,由文献[17]知随机力冲量的 $\langle\Gamma(t)\rangle=0$ 和 $\langle\Gamma^2(t)\rangle=2Dh$.因此,将随机力冲量模拟为 $\Gamma(t)=\sqrt{2Dh}\,\omega$,其中 $D=\gamma K_B T$,K_B 是扩散系数,取阻尼系数 $\gamma=1$,ω 是一个标准 Gauss 随机数.则

$$\frac{1}{\Gamma(\alpha)}\int_0^{t_{2m+i}}(t_{2m+i}-\tau)^{\alpha-1}\xi(\tau)\,\mathrm{d}\tau =$$

$$\frac{1}{\Gamma(\alpha)}\int_0^{t_{2m+i}}(t_{2m+i}-\tau)^{\alpha-1}\mathrm{d}\tau\sqrt{2Dh}\,\omega \le 2Nh\sqrt{2Dh}\,\omega \le C_1, \qquad i=1,2,$$

即

$$|\hat{v}_{2m+1}| \le C_2|\gamma|h^{\alpha}\sum_{j=0}^{2m}(2m+3-j)^{\alpha-1}|\hat{v}_j| + |\gamma|h^{\alpha}|\hat{B}_{2m+1}||\hat{v}_{2m+1}| + C_1. \tag{24}$$

由稳定性条件(22),可以将式(24)改写成

$$|\hat{v}_{2m+1}| \le C_2|\gamma|h^{\alpha}\sum_{j=0}^{2m}(2m+3-j)^{\alpha-1}|\hat{v}_j| + C_1 \le$$

$$C_2|\gamma|h^{\alpha}\sum_{j=0}^{2m}(2m+1-j)^{\alpha-1}|\hat{v}_j| + C_1 \le$$

$$C_2|\gamma| + h^{\alpha}\sum_{j=0}^{2m}(2m+1-j)^{\alpha-1}|\hat{v}_j|(C_2|\gamma|-1) + C_1 \le$$

$$C_2 | \gamma | + h^\alpha \sum_{j=0}^{2m} (2m+1-j)^{\alpha-1} | \hat{v}_j | + C_1 \leqslant$$

$$C | \gamma + 1 | + h^\alpha \sum_{j=0}^{2m} (2m+1-j)^{\alpha-1} | \hat{v}_j |, \qquad C = \max\{C_1, C_2\}. \tag{25}$$

对式(25)应用离散的 Gronwall 定理[22],得到

$$| \hat{v}_{2m+1} | \leqslant C | \gamma + 1 | E_\alpha(C | \gamma + 1 | \Gamma(\alpha)((2m+1)h)^\alpha) \leqslant$$

$$CE_\alpha(C | \gamma + 1 | \Gamma(\alpha) T^\alpha) | \gamma + 1 | \leqslant CM_0.$$

定理 1 证明完毕.

2.3 Block-by-block 格式的收敛性

考虑一般的 $f(t,v)$, 假设它对第二个变量满足 Lipschitz 条件,即存在一个常数 L, 使得

$$| f(t, v_1) - f(t, v_2) | \leqslant L | v_1 - v_2 |, \qquad \forall v_1, v_2 \in \mathbf{R}. \tag{26}$$

定理 2 设 v 为式(1)、(2)的解析解, $\{v_j\}_{j=0}^{2N}$ 为格式(23)的数值解,假设时间步长 h 满足

$$| \hat{B}_{2m+1} | h^\alpha L < 1, \tag{27}$$

其中

$$| \hat{B}_{2m+1} | = \max\{ | B_{2m+1} | + | B_{2m+2} |, | \bar{B}_{2m+1} | + | \bar{B}_{2m+2} | \},$$

有如下误差估计成立:

$$| v(t_j) - v_j | \leqslant Ch^{3+\alpha}, \qquad j = 1, 2, \cdots, 2N,\ 0 < \alpha < 1,$$

这里 C 仅依赖于 f, α, T 和 L.

证明 记 $e_j = v(t_j) - v_j$, $j = 0, 1, 2, \cdots, 2N$, 易知 $e_0 = 0$, 当 $j \geqslant 1$ 时, e_j 满足

$$\begin{cases} e_{2m+1} = h^\alpha \sum_{j=0}^{2m+2} \bar{B}_j [f(t_j, v(t_j)) - f(t_j, v_j)] + r_{2m+1}(h), \\ e_{2m+2} = h^\alpha \sum_{j=0}^{2m+2} B_j [f(t_j, v(t_j)) - f(t_j, v_j)] + r_{2m+2}(h). \end{cases}$$

这里系数 $\bar{B}_j, B_j$ 由式(18)给出,根据式(20)和假设条件(26),有

$$\begin{cases} e_{2m+1} \leqslant LCh^\alpha \sum_{j=0}^{2m} (2m+3-j)^{\alpha-1} | e_j | + Lh^\alpha | \bar{B}_{2m+1} | | e_{2m+1} | + \\ \qquad Lh^\alpha | \bar{B}_{2m+2} | | e_{2m+2} | + | r_{2m+1}(h) |, \\ e_{2m+2} \leqslant LCh^\alpha \sum_{j=0}^{2m} (2m+3-j)^{\alpha-1} | e_j | + Lh^\alpha | B_{2m+1} | | e_{2m+1} | + \\ \qquad Lh^\alpha | B_{2m+2} | | e_{2m+2} | + | r_{2m+1}(h) |. \end{cases}$$

记

$$| \varepsilon_{2i+1} | = | \varepsilon_{2i+2} | = \max\{ | e_{2i+1} |, | e_{2i+2} | \},$$

则

$$| \varepsilon_{2m+1} | \leqslant LCh^\alpha \sum_{j=0}^{2m} (2m+3-j)^{\alpha-1} | \varepsilon_j | + Lh^\alpha | \bar{B}_{2m+2} | | \varepsilon_{2m+1} | + | r(h) |,$$

其中

$$| r(h) | = \max\{ | r_{2m+1}(h) |, | r_{2m+2}(h) | \},$$

得到

$$(1 - Lh^\alpha | \hat{B}_{2m+1} |) | \varepsilon_{2m+1} | \leqslant LCh^\alpha \sum_{j=0}^{2m} (2m+3-j)^{\alpha-1} | \varepsilon_j | + | r(h) |.$$

根据条件(27)得到

$$| \varepsilon_{2m+1} | \leqslant LCh^\alpha \sum_{j=0}^{2m} (2m+3-j)^{\alpha-1} | \varepsilon_j | + C | r(h) | \leqslant$$

$$LCh^{\alpha}\sum_{j=0}^{2m}(2m+1-j)^{\alpha-1}|\varepsilon_j| + C|r(h)|. \tag{28}$$

将离散的 Gronwall 定理应用到式(28)得到

$$|\varepsilon_{2m+1}| \leqslant C|r(h)|E_{\alpha}(LC\Gamma(\alpha)(jh)^{\alpha}) \leqslant |r(h)|E_{\alpha}(LC\Gamma(\alpha)T^{\alpha}).$$

对于上式的估计,结合引理 1 和引理 2 有

$$|\varepsilon_{2m+1}| \leqslant Ch^{3+\alpha}.$$

定理 2 证明完毕.

3 数值试验

对于如下分数阶 Langevin 方程[23]：

$$\begin{cases} {}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha}v(t) = -\lambda^{\alpha}v(t) + v_0\dfrac{t^{-\alpha}}{\Gamma(1-\alpha)} + \xi(t), & 0<\alpha<1,\ 0<t<T, \\ v^{(k)}(0) = v_0^{(k)}, & k=0,1,\cdots,n-1. \end{cases} \tag{29}$$

利用方程的逆 Laplace 变换,得到分数阶 Langevin 方程的一般解：

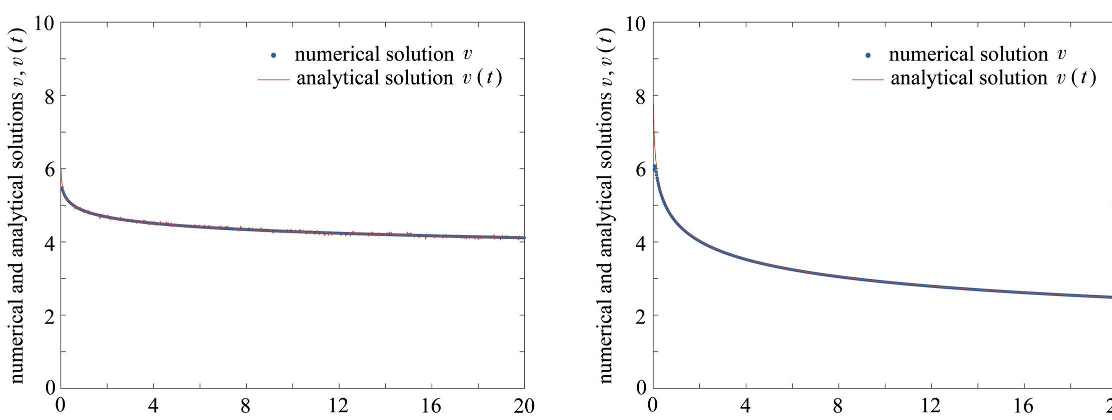
$$v(t) = E_{\alpha}[-(t)^{\alpha}] + \int_0^t \frac{E_{\alpha,\alpha}[-(t-t')^{\alpha}]}{(t-t')^{1-\alpha}}\xi(t')\mathrm{d}t', \tag{30}$$

其中 Mittag-Laffler 函数[24]表示为

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)}, \qquad \alpha>0,\ \beta>0, \tag{31}$$

$E_{\alpha}(z)$ 是 $\beta=1$ 时的特殊情况.

为验证 block-by-block 数值格式的有效性与准确性,分别给出不同 α 和时间步长 h 取值下,数值解与解析解的比较.从比较曲线图(图 1、2)可以看出,数值解很好地逼近解析解.其中 $\xi(t)$ 是一个 Wiener 过程,它对有限时间间隔内的积分作为一个随机变量,遵守一个 Gauss 分布[1,17],令 $T=20$, $v_0=10$.



(a) 数值解和解析解的比较($\alpha=0.1$)

(a) Comparison of numerical and analytical solutions ($\alpha=0.1$)

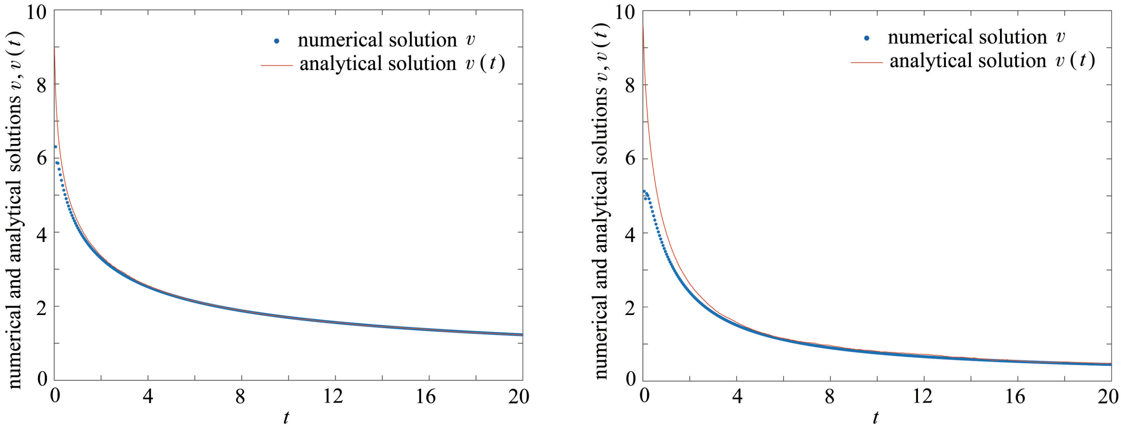
(b) 数值解和解析解的比较($\alpha=0.3$)

(b) Comparison of numerical and analytical solutions ($\alpha=0.3$)

图 1 Block-by-block 算法数值解和解析解比较($h=0.025$)

Fig. 1 Comparison of numerical and analytical solutions based on the block-by-block method ($h=0.025$)

数值算例得出不同 α 和时间步长 h 取值下解析解与数值解的误差,对每一时刻产生的误差求和后取均值,得出平均误差(mean error).为削弱方程中随机项对数值解的影响,得到的数值解是由 50 条样本轨道取均值来表现的,解析解是随机得到的某一条轨道.在不同时刻和 α 取值下,用最小二乘法拟合线性方程 $\lg|v(t_i)-v_i| = \lg a + R\lg h$ 的系数 R, 得到对应的收敛阶.误差及收敛阶的值具体见表 1.



(a) 数值解和解析解的比较(α = 0.5)

(a) Comparison of numerical and analytical solutions (α = 0.5)

(b) 数值解和解析解的比较(α = 0.7)

(b) Comparison of numerical and analytical solutions (α = 0.7)

图 2 Block-by-block 算法数值解和解析解比较(h = 0.05)

Fig. 2 Comparison of numerical and analytical solutions based on the block-by-block method (h = 0.05)

表 1 Block-by-block 算法数值解和解析解的平均误差及收敛阶

Table 1 Mean errors and convergence orders of numerical and analytic solutions based on the block-by-block method

α	h	$\lvert v(t_i) - v_i \rvert_{\text{mean}}$	rate R	α	h	$\lvert v(t_i) - v_i \rvert_{\text{mean}}$	rate R
0.1	1/10	2.125 4E−2	3.092 3	0.3	1/10	5.423 4E−2	3.260 7
	1/20	2.944 1E−2	3.096 2		1/20	5.017 6E−2	3.264 2
	1/40	2.925 0E−2	3.100 5		1/40	4.691 4E−2	3.294 5
	1/80	3.762 8E−2	3.105 2		1/80	4.252 4E−2	3.305 2
	1/160	4.751 2E−2	3.118 0		1/160	3.985 4E−2	3.308 0
0.5	1/10	1.257 2E−1	3.445 2	0.7	1/10	2.146 3E−1	3.662 5
	1/20	1.262 8E−1	3.448 7		1/20	2.141 3E−1	3.668 1
	1/40	1.268 9E−1	3.450 2		1/40	2.342 5E−1	3.676 9
	1/80	1.271 0E−1	3.454 2		1/80	2.345 4E−1	3.705 2
	1/160	1.278 8E−1	3.458 0		1/160	2.321 4E−1	3.706 4

固定时间步长 $h = 0.05$,作出不同 α 下的数值解,如图 3 所示.当 α 和时间 t 较小时,数值解稍有起伏,后面逐渐趋于稳定;当 α 趋近 1,时间 t 越长时,数值解一直都保持平稳状态.与预估校正算法求得的数值解[25]相比,block-by-block 算法数值解更加平缓稳定,且更趋近于解析解.

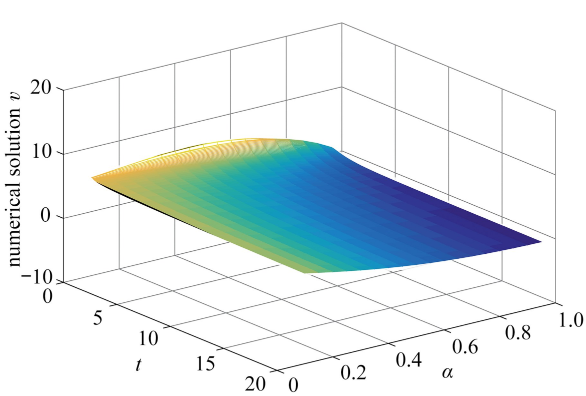


图 3 不同 α 取值下的 block-by-block 算法数值解(h = 0.05)

Fig. 3 The numerical solution based on the block-by-block method for different values of α (h = 0.05)

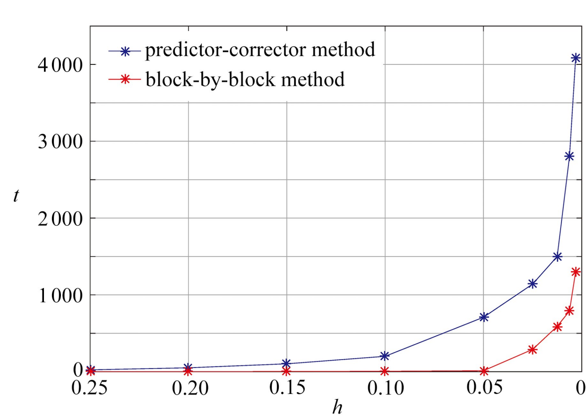


图 4 两种格式运算时间随时间步长 h 的变化趋势(α = 0.3)

Fig. 4 The operation time vs. time step h for both schemes (α = 0.3)

从图 4 可以明显得到,在时间步长 h 大于 0.1 时,block-by-block 算法运算时间略低于预估校正算法;在时间步长 h 小于 0.1 时,block-by-block 运算时间明显低于预估校正算法.从曲线趋势可以看出,时间步长 h 取值较小时,block-by-block 算法运算时间低于预估校正算法,表明 block-by-block 算法计算效率上具有明显优势.

4 结　　论

本文将 block-by-block 算法应用于一类分数阶 Langevin 方程的数值求解.理论分析了所构造 block-by-block 格式的稳定性和收敛性.数值试验部分将 block-by-block 算法的数值解与解析解进行误差分析.结果表明 block-by-block 算法的数值解与解析解趋势相同,且很好地逼近解析解,同时降低了计算复杂度,明显缩短了运算时间,体现出 block-by-block 算法在计算精度和计算效率上的明显优势.

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