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广义(3+1)维 KdV 方程的 lump 解、相互作用解和呼吸子解*

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摘要: 基于广义(3+1)维 KdV 方程的双线性形式,得到了方程的 lump 解、相互作用解及呼吸子解.证明了 lump 解在空间各个方向上都是有理局域的,并在 lump 波与线孤子的相互作用过程中观察到了“聚变”和“裂变”现象,最后得到了方程的呼吸子解.

关 键 词: 广义(3+1)维 KdV 方程; lump 解; 相互作用解; 呼吸子解

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Lump Solutions, Interaction Solutions and Breather Solutions of Generalized (3+1)-Dimensional KdV Equations

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Abstract: Based on the bilinear form of the generalized (3+1)-dimensional KdV equation, the lump solution, the interaction solution and the breather solution of the equation were obtained. The obtained lump solutions were proved to be rationally localized in all directions of the space, then the “fusion” and “fission” phenomena were observed during the interaction between the lump soliton wave and the one-stripe soliton. Finally, the breather solution of the equation was obtained.

Key words: generalized (3+1)-dimensional KdV equation; lump solution; interaction solution; breather solution

0 引 言

非线性演化方程在非线性科学中有着重要的应用,它为描述流体力学、光纤、等离子体物理等科学领域中的一些非线性现象提供了有效的模型[1-2],例如 Korteweg-de Vries(KdV)方程可用于模拟分层流体中的内波和等离子体中的离子声波[3],Kadomtsev-Petviashvili(KP)方程可用于模拟长横向扰动下流体中的弱色散波[4].在孤子理论中,寻找非线性演化方程的精确解一直以来都是众多学者关注的研究热点,如孤立波解、

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lump 解以及怪波解等[5-7].目前,人们已经发现了许多有效求解非线性演化方程精确解的方法,如反散射变换[1]、Bäckland 变换[8]、Darboux 变换[9]和双线性方法[10]等.

双线性方法是求解非线性演化方程精确解的一种重要方法,该方法的优点在于一旦将给定的非线性演化方程通过变量变换转化为双线性方程,则可通过直接求解双线性方程得到给定的非线性演化方程的精确解.随着双线性方法的广泛应用,众多学者在此基础上提出了各种直接求解方法.2015 年,Ma(马文秀)提出了一种直接构造双线性可积方程 lump 解的方法,得到了(2+1)维 KPI 方程的 lump 解[11].Zhang 和 Chen 等[12-13]进一步发展完善了该方法,并利用该方法在 KP 方程中首次发现了由共振孤子所激发产生的怪波,发现了线孤子和 lump 波相互作用产生的"聚变"和"裂变"现象.此后, 国内外学者在这一领域开展了广泛的工作,发现了越来越多的 lump-扭结解、lump-线孤子解等[14-30].

高维非线性演化方程由于可以为探究物理现象提供更多的信息而受到广泛关注[31-33].本文主要研究如下的广义(3+1)维 KdV 方程[34]:

$$v_t + v_{xxx} + \alpha(v\partial_y^{-1}v_x)_x + \beta(\partial_y^{-1}v_{xx}) + \gamma(\partial_y^{-1}v_{yy}) + \delta(\partial_y^{-1}v_{yz}) = 0, \tag{1}$$

v 是关于空间变量 x,y,z 和时间变量 t 的函数,α,β,γ 和 δ 均为非零参数,$\partial_y^{-1}v = \int v\mathrm{d}y$.方程(1)及其约化方程已被应用于模拟数学物理、非线性光学、流体力学和等离子体物理中的物理现象[34-36].

当 $\delta = 0$ 时,方程(1)可退化为如下(2+1)维 KdV 方程:

$$v_t + v_{xxx} + \alpha(v\partial_y^{-1}v_x)_x + \beta(\partial_y^{-1}v_{xx}) + \gamma(\partial_y^{-1}v_{yy}) = 0. \tag{2}$$

文献[34]证明了方程(2)是 Painlevé 可积的,并利用双线性方法得到了其多实孤子解和多复孤子解.文献[37]利用相容 tanh 展开法得到其共振解、孤子与椭圆余弦波的相互作用解,文献[38]基于双线性方法得到了其 lump 解、相互作用解和怪波解.

当 $\alpha = 3,\beta = 0$ 和 $\gamma = 0$ 时,方程(2)进一步转化为

$$v_t + v_{xxx} + 3(v\partial_y^{-1}v_x)_x = 0. \tag{3}$$

方程(3)由 Boiti 等[35]首次给出,可用于描述不可压缩流体的动力学行为并已被证明具有 Lax 对、无穷多守恒律.文献[39]利用双线性方法、tanh-coth 方法以及幂函数法得到了该方程的包括多孤子解在内的行波解.

当 $y = x$ 时,方程(3)即为经典的(1+1)维 KdV 方程[40]:

$$v_t + 6vv_x + v_{xxx} = 0. \tag{4}$$

文献[34]利用 Painlevé 分析和双线性方法研究了方程(1)可积性检验和多孤子解;文献[41]利用双线性方法和 Riemann-theta 函数研究了方程(1)的双线性形式、N-孤子解、呼吸子解、混合解和拟周期波解.本文利用方程(1)的双线性形式,通过拟设的方法直接构造其 lump 解、相互作用解和呼吸子解.本文的主要结构如下:第 1 节中,通过对双线性方程中的 f 函数取 3 种不同的形式,分别得到了方程(1)的 lump 解、相互作用解和呼吸子解,证明了 lump 解的局域性,发现了 lump 波与线孤子的相互作用过程中的"聚变"和"裂变"现象,并利用图形展示了解的动力学特征.第 2 节中,我们将给出一些结论.

1 Lump 解、相互作用解和呼吸子解

为得到方程的双线性形式,首先引入势变换:

$$v = u_y, \tag{5}$$

方程(1)变为

$$u_{ty} + u_{xxxy} + \alpha u_{xx}u_y + \alpha u_x u_{xy} + \beta u_{xx} + \gamma u_{yy} + \delta u_{zy} = 0. \tag{6}$$

再利用对数变换

$$u = \frac{6}{\alpha}(\ln f)_x, \tag{7}$$

可得方程(1)的双线性形式:

$$(D_yD_t + D_x^3D_y + \beta D_x^2 + \gamma D_y^2 + \delta D_zD_y)f\cdot f = 0, \tag{8}$$

其中 f 是关于空间变量 x,y,z 和时间变量 t 的函数,D 是 Hirota 双线性算子,其定义为[10]

$$D_x^nD_t^mF\cdot G = (\partial_x - \partial_{x'})^n(\partial_t - \partial_{t'})^mF(x,t)\times G(x',t')|_{x'=x,t'=t},$$

其中 n 和 m 均为非负整数.

为得到方程(1)的 lump 解、相互作用解和呼吸子解,关键是构造双线性方程(8)的解.由此,我们做如下假设:

$$f = g^2 + h^2 + k\mathrm{e}^{\eta} + l\mathrm{e}^{-\eta} + a_{11}, \tag{9}$$

其中 g,h 和 η 分别具有如下形式:

$$\begin{cases} g = a_1x + a_2y + a_3z + a_4t + a_5, \\ h = a_6x + a_7y + a_8z + a_9t + a_{10}, \\ \eta = k_1x + k_2y + k_3z + k_4t, \end{cases} \tag{10}$$

其中 $a_i(i = 1,2,\cdots,11)$,$k_i(i = 1,2,3,4)$,k 和 l 均为待定参数.

1.1　Lump 解

基于文献[11],在函数(9)中取 $k = 0,l = 0$ 并将其代入双线性方程(8)中,利用符号计算软件 MAPLE,收集变量 x,y,z,t 的不同幂次系数,求解相应的代数方程组,可得参数 $a_1 \sim a_{11}$ 之间具有如下关系:

$$a_4 = -\frac{A_1}{a_2^2 + a_7^2},\ a_8 = -\frac{A_2}{\delta(a_2^2 + a_7^2)},\ a_{11} = -\frac{A_3}{\beta(a_1a_7 - a_2a_6)^2}, \tag{11}$$

其中

$$(a_1a_7 - a_2a_6)\beta \neq 0,\ (a_2^2 + a_7^2)\delta \neq 0, \tag{12}$$

且

$$A_1 = \beta(a_2a_6^2 - 2a_1a_6a_7 - a_1^2a_2) - a_2\gamma(a_2^2 + a_7^2) + a_3\delta(a_2^2 - a_7^2),$$
$$A_2 = \beta(a_1^2a_7 - 2a_1a_2a_6 - a_6^2a_7) - a_7\gamma(a_2^2 + a_7^2) + a_9(a_2^2 + a_7^2),$$
$$A_3 = 3\alpha(a_2^2 + a_7^2)(a_1^2 + a_7^6)(a_1a_2 + a_6a_7).$$

由此,方程(1)的解可写为

$$v = \frac{6}{\alpha}(\ln f)_{xy} = \frac{12a_6[(g^2 - h^2 + a_{11})a_7 - 2gha_2] - 12a_1[2gha_7 - a_2(h^2 + a_{11})]}{\alpha f^2}, \tag{13}$$

其中

$$\begin{cases} f = g^2 + h^2 - \dfrac{A_3}{\beta(a_1a_7 - a_2a_6)^2}, \\ g = a_1x + a_2y + a_3z - \dfrac{A_1}{a_2^2 + a_7^2}t + a_5, \\ h = a_6x + a_7y - \dfrac{A_2}{\delta(a_2^2 + a_7^2)}z + a_9t + a_{10}, \end{cases} \tag{14}$$

且需满足如下约束条件:

$$A_3\beta < 0, \tag{15}$$

以确保函数 f 的正则性及解(13)的解析性.从式(14)中显然可知

$$\lim_{x^2+y^2+z^2\to\infty} f(x,y,z,t) = \infty,\quad \forall t \in \mathbb{R}, \tag{16}$$

由此可得解(13)满足

$$\lim_{x^2+y^2+z^2\to\infty} v(x,y,z,t) = 0,\quad \forall t \in \mathbb{R}. \tag{17}$$

基于以上分析,解(13)在条件(12)、(15)、(16)和(17)下在空间各个方向都是有理局域的,因此被定义为 lump 解[11].

为直观地了解解(13)的性质,不失一般性,进一步考虑解(13)在 $y=0$ 和 $t=0$ 情形下的动力学特征.取参数 $\alpha=\gamma=\delta=a_5=a_6=1,\beta=-1,a_1=a_2=a_7=a_9=2,a_3=5,a_{10}=18$, 则解(13)简化为

$$v=-\frac{384(48x^2+40xz-325z^2+608x+2\,120z-2\,416)}{5(16x^2+40xz+125z^2+128x-400z+1\,616)^2}. \tag{18}$$

图 1(a)表明解(18)有 3 个极值点(一个波峰,两个波谷且呈对称形式),利用符号计算软件 MAPLE,我们发现波峰和波谷分别位于

$$(x,z)=\left(-\frac{15}{2},\frac{14}{5}\right) \tag{19}$$

和

$$(x,z)=\left(\pm\frac{(21+6\sqrt{10})\sqrt{15\sqrt{10}-40}}{10}-\frac{15}{2},\ \frac{2}{5}\left(7\mp\frac{3\sqrt{15\sqrt{10}-40}}{5}\right)\right), \tag{20}$$

波峰高度为 $v_{\max}=2/5$,波谷高度为 $v_{\min}=|-1/(3(3+\sqrt{10}))|=1/(3(3+\sqrt{10}))$.显然,图 2 与图 1 中的 lump 波具有类似的动力学特性.

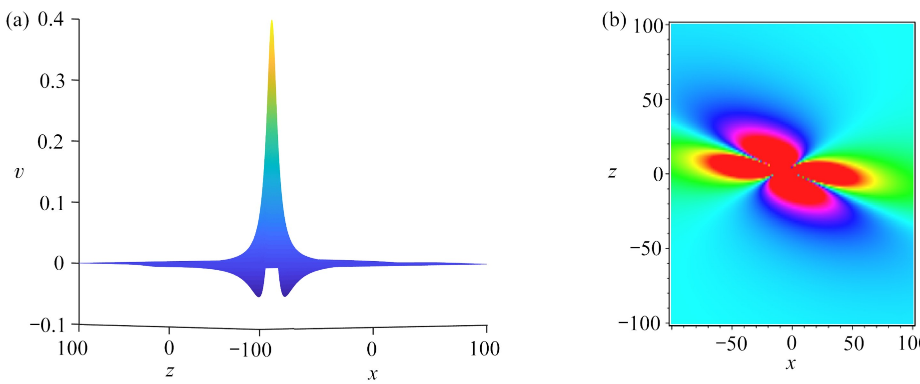


图 1 Lump 解(18)的三维图和密度图 ($y=0,\ t=0$)

Fig. 1 The 3D plot and the density plot with lump solutions (18) ($y=0,\ t=0$)

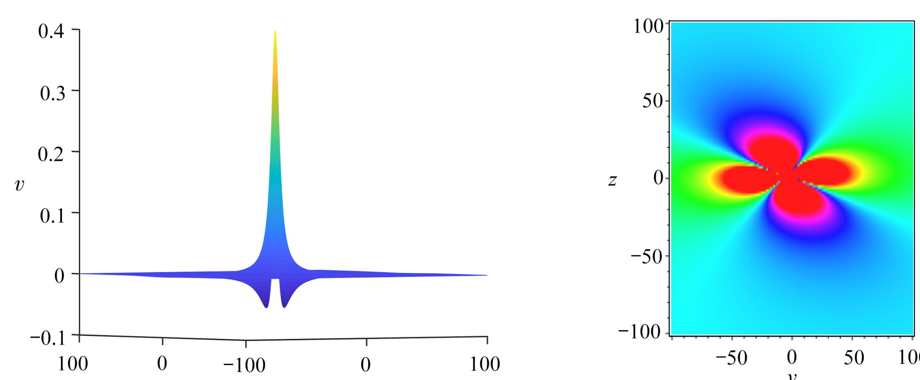


图 2 取参数 $\alpha=\gamma=\delta=a_5=a_6=1,\beta=-1,a_1=a_2=a_7=a_9=2,a_3=5,a_{10}=18$ 时, lump 解(13)的三维图和密度图 ($x=0,\ t=0$)

Fig. 2 The 3D plot and the density plot with lump solutions (13) under parameters selected as $\alpha=\gamma=\delta=a_5=a_6=1,\ \beta=-1,\ a_1=a_2=a_7=a_9=2,\ a_3=5,\ a_{10}=18$ ($x=0,\ t=0$)

1.2　Lump 波与线孤子的相互作用解

本小节主要构造方程(1)的相互作用解.基于文献[12-13],在式(9)中取 $l=0$, 可得

$$f=g^2+h^2+k\mathrm{e}^{\eta}+a_{11}, \tag{21}$$

其中 g, h, η 与式(10)相同.将拟设解(21)代入双线性方程(8) 中,收集变量 x,y,z,t 和 e^{η} 的不同幂次系数,求解相应的代数方程组,得到参数关系如下:

$$\begin{cases} a_2=0,\ \alpha=-\dfrac{2\beta a_7^2\sqrt{a_7^2-a_{11}k_2^2}}{3(a_{11}k_2^2-2a_7^2)^2k_2^2a_1},\ a_4=-\dfrac{4a_3a_7^2\delta(a_7^2-a_{11}k_2^2)^{3/2}+a_1^2a_{11}\beta k_2^2}{4(a_7^2-a_{11}k_2^2)^{3/2}a_7^2}, \\ a_6=\dfrac{a_1a_{11}k_2^2}{8a_7(a_7^2-a_{11}k_2^2)^{3/2}},\ a_9=-\dfrac{a_1a_{11}k_2^4\beta}{4a_7^3(a_7^2-a_{11}k_2^2)}+\dfrac{a_1^2\beta}{a_7}-a_7\gamma-a_8\delta, \\ k_1=-\dfrac{2\sqrt{a_7^2-a_{11}k_2^2}(a_{11}k_2^2-2a_7^2)k_2^2a_1}{a_7},\ k_4=\dfrac{a_1^2k_2\beta(a_{11}^2k_2^4-4a_7^4)}{12a_7^4(a_7^2-a_{11}k_2^2)}-k_2\gamma-k_3\delta, \end{cases} \tag{22}$$

且需满足如下条件:

$$a_{11}>0,\ a_7^2-a_{11}k_2^2>0,\ a_1a_7k_2(a_{11}k_2^2-2a_7^2)\neq 0. \tag{23}$$

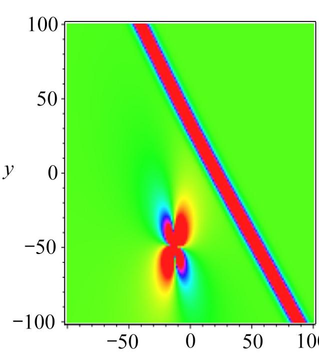
由此,得到方程(1)的解为

$$v=\frac{6(2a_1a_2+2a_6a_7+kk_1k_2\mathrm{e}^{\eta})}{\alpha f}-\frac{6(2ga_1+2ha_6+kk_1\mathrm{e}^{\eta})(2ga_2+2ha_7)+kk_2\mathrm{e}^{\eta}}{\alpha f^2}, \tag{24}$$

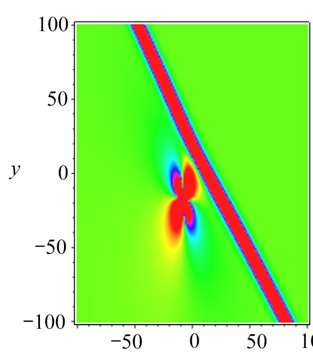
其中

$$\begin{cases} g=a_1x+a_3z-\dfrac{4a_3a_7^2\delta(a_7^2-a_{11}k_2^2)^{3/2}+a_1^2a_{11}\beta k_2^2}{4(a_7^2-a_{11}k_2^2)^{3/2}a_7^2}t+a_5, \\ h=\dfrac{a_1a_{11}k_2^2}{8a_7(a_7^2-a_{11}k_2^2)^{3/2}}x+a_7y+a_8z-\left[\dfrac{a_1a_{11}k_2^4\beta}{4a_7^3(a_7^2-a_{11}k_2^2)}-\dfrac{a_1^2\beta}{a_7}+a_7\gamma+a_8\delta\right]t+a_{10}, \\ \eta=-\dfrac{2\sqrt{a_7^2-a_{11}k_2^2}(a_{11}k_2^2-2a_7^2)k_2^2a_1}{a_7}x+k_2y++k_3z+ \\ \quad\left[\dfrac{a_1^2k_2\beta(a_{11}^2k_2^4-4a_7^4)}{12a_7^4(a_7^2-a_{11}k_2^2)}-k_2\gamma-k_3\delta\right]t. \end{cases} \tag{25}$$

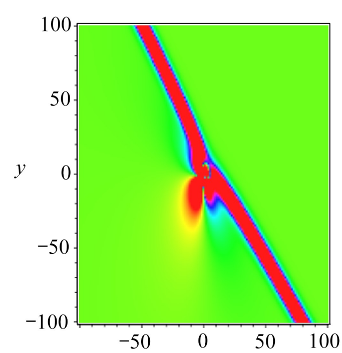
图 3 展示了解(24)中 lump 波与线孤子相互作用过程中产生的“聚变”现象[42-43]. 参数取值为 $\beta=-2$, $\gamma=-1$, $\delta=a_1=a_5=a_8=a_{10}=k_3=1$, $a_3=5$, $a_7=a_{11}=0.5$, $k=2$, $k_2=1/3$, $z=0$.“聚变”现象的产生主要由解(24)中的多项式函数和指数函数引起.当 $k_2>0$, $t\to\infty$ 时,解(24)受指数函数影响较大,反之 $t\to-\infty$ 时,解(24)受多项式函数影响较大.



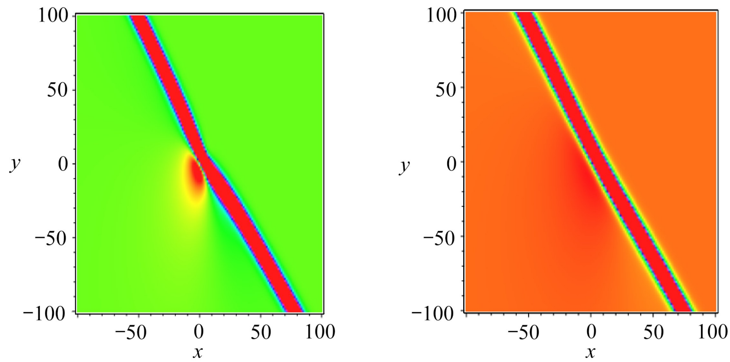
(a) $t=-5$



(b) $t=-2$



(c) $t=0$



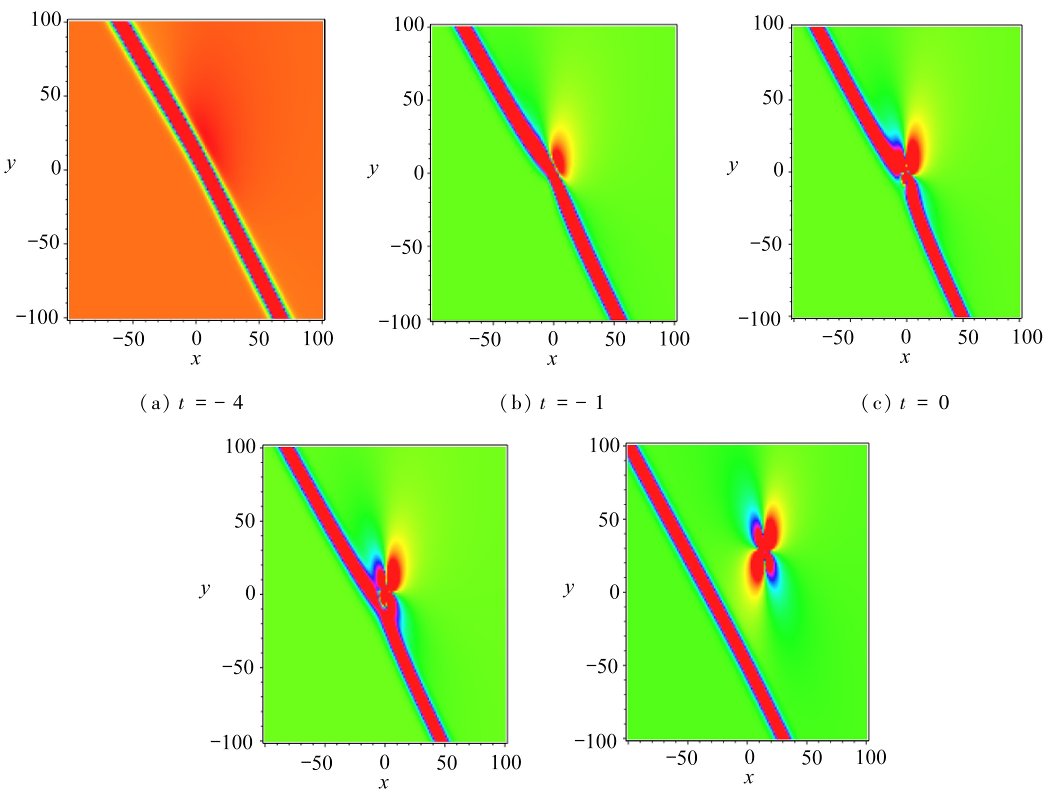
(d) $t = 2$ (e) $t = 6$

图 3 Lump 波与线孤子产生的“聚变”现象

Fig. 3 Fusion phenomena between the lump wave and the one-stripe soliton

如图 3(a)所示,当 $t = -5$ 时,解(24)包含一个 lump 波和一个线孤子.随着时间变化,lump 波和线孤子逐渐靠近并渐趋融合.当 $t = 6$ 时,lump 波完全融入线孤子中并最终消失,见图 3(e).

当 $k_2 < 0$, 图 4 展示了 lump 波与线孤子相互作用中产生的“裂变”现象[42-43].参数取值为 $\beta = -2, \gamma = -1, \delta = a_1 = a_5 = a_8 = a_{10} = k_3 = 1, a_3 = 5, a_7 = a_{11} = 0.5, k = 2, k_2 = -1/3, z = 0$.从图 4 中观察可见,当 $t = -4$ 时,解(24)只包含一个线孤子,随着时间变化,一个 lump 波逐渐从线孤子中分离;当 $t = 4$ 时,lump 波与线孤子完全分离,即产生所谓“裂变”,见图 4(e).



(a) $t = -4$ (b) $t = -1$ (c) $t = 0$

(d) $t = 0.5$ (e) $t = 4$

图 4 线孤子产生的“裂变”现象

Fig. 4 Fission phenomena produced by the one-stripe soliton

1.3 呼吸子解

为构造方程(1)的呼吸子解,假设[44-47]

$$f = k_1 e^{\xi_1} + k_2 \cos(\xi_2) + k_3 e^{-\xi_1}, \tag{26}$$

其中

$$\xi_1 = c_1x + c_2y + c_3z + c_4t,\ \xi_2 = c_5x + c_6y + c_7z + c_8t, \tag{27}$$

$c_i(i = 1,2,\cdots,8)$,$k_i(i = 1,2,3)$ 均为参数.类似于前面的计算,将表达式(26)和(27)代入双线性方程(8)中,求解 $e^{\pm\xi_1}$,$\cos(\xi_1)$ 和 $\sin(\xi_1)$ 的不同幂次系数组成的代数方程组,可得

$$c_1 = -\frac{\beta}{3c_2\alpha},\ c_4 = -\gamma c_2 - \delta c_3 + \frac{\beta^3}{27\alpha^2c_2^3},\ c_6 = \frac{\beta}{3c_5\alpha},\ c_8 = c_5^3\alpha - c_7\delta - \frac{\beta\gamma}{3c_5\alpha}, \tag{28}$$

且需满足如下约束:

$$\alpha c_2c_5 \neq 0. \tag{29}$$

由此,可得方程(1)的如下解:

$$v = \frac{6(k_1c_1c_2e^{\xi_1} - k_2c_5c_6\cos\xi_2 + k_3c_1c_2e^{-\xi_1})}{\alpha f} - \frac{6(k_1c_1e^{\xi_1} - k_2c_5\sin\xi_2 - k_3c_1e^{-\xi_1})(k_1c_2e^{\xi_1} - k_2c_6\sin\xi_2 - k_3c_2e^{-\xi_1})}{\alpha f^2}, \tag{30}$$

其中

$$\xi_1 = -\frac{\beta}{3c_2\alpha}x + c_2y + c_3z - \left(\gamma c_2 + \delta c_3 - \frac{\beta^3}{27\alpha^2c_2^3}\right)t,$$

$$\xi_2 = c_5x + \frac{\beta}{3c_5\alpha}y + c_7z + \left(c_5^3\alpha - c_7\delta - \frac{\beta\gamma}{3c_5\alpha}\right)t.$$

为便于理解呼吸子解(30)的特征,依次取 $z = 0$,$y = 0$,$x = 0$,图 5—7 分别展示了呼吸子解在时间 $t = 0$ 时的动力学特性.

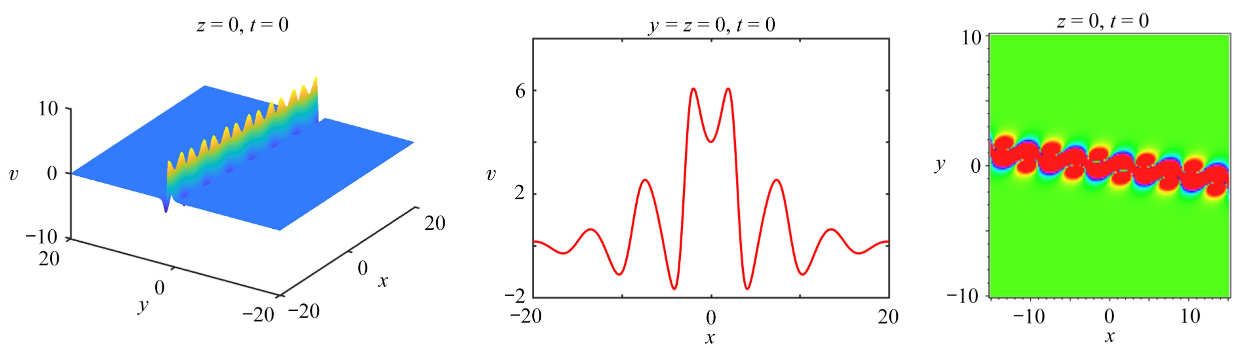


图 5　解(30)的三维图 (x-y-v),x 曲线图和密度图,参数取值为 $\alpha = \gamma = c_5 = c_7 = 1,\ \beta = -2,\ c_3 = 0,\ c_2 = k_1 = k_3 = 3,\ k_2 = 2$

Fig. 5　The 3D plot (x-y-v), the x- curve plot and the density plot of solution(30) with $\alpha = \gamma = c_5 = c_7 = 1,\ \beta = -2,\ c_3 = 0,\ c_2 = k_1 = k_3 = 3,\ k_2 = 2$

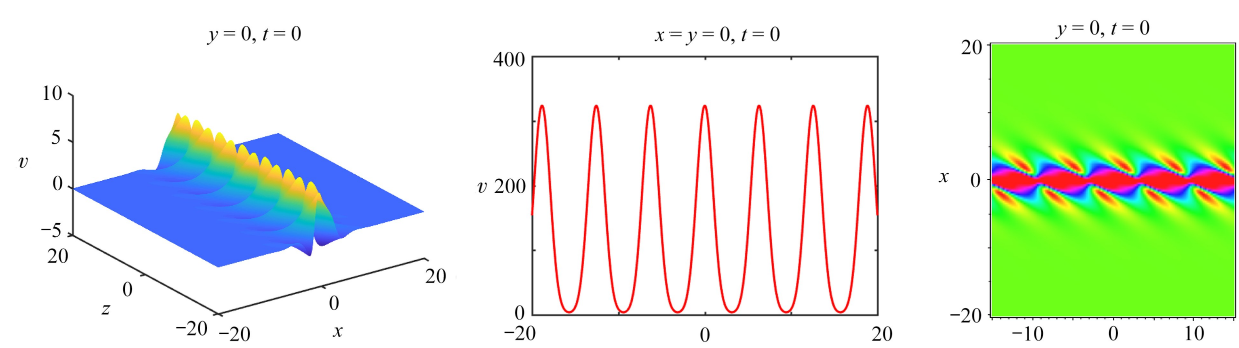


图 6　解(30)的三维图 (x-z-v), z 曲线图和密度图,参数取值为 $\alpha = \gamma = c_2 = c_5 = c_7 = 1,\ \beta = -2,\ c_3 = 0,\ k_1 = k_2 = k_3 = 2$

Fig. 6　The 3D plot (x-z-v), the z- curve plot and the density plot of solution(30) with $\alpha = \gamma = c_2 = c_5 = c_7 = 1,\ \beta = -2,\ c_3 = 0,\ k_1 = k_2 = k_3 = 2$

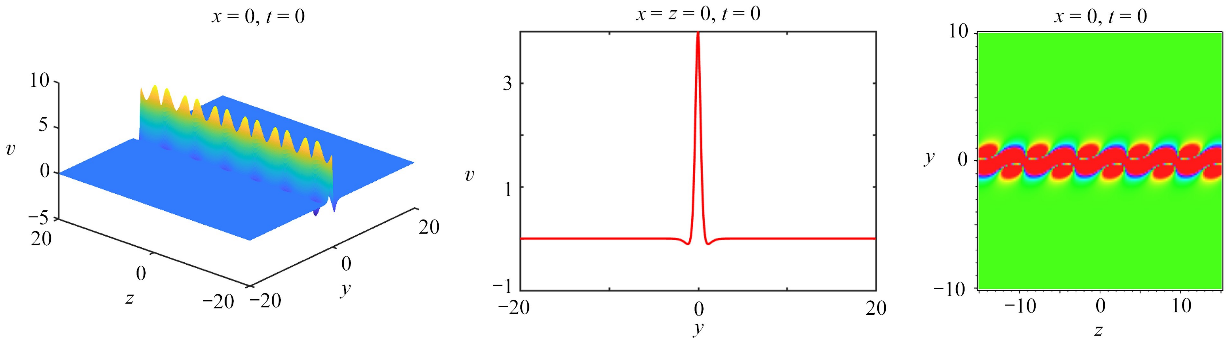


图 7 解(30)的三维图 (y-z-v), y 曲线图和密度图,参数取值为 $\alpha = \gamma = c_5 = c_7 = 1,\ \beta = -2,\ c_3 = 0,\ c_2 = k_1 = k_3 = 3,\ k_2 = 2$

Fig. 7 The 3D plot (y-z-v), the y- curve plot and the density plot of solution(30) with $\alpha = \gamma = c_5 = c_7 = 1,\ \beta = -2,\ c_3 = 0,\ c_2 = k_1 = k_3 = 3,\ k_2 = 2$

2 结 论

本文主要研究了广义(3+1)维 KdV 方程(1),得到了方程的 lump 解、相互作用解和呼吸子解.说明了 lump 解(13)的有理局域性,并通过图 1 和图 2 分别观察到 lump 解在 $y = 0$ 和 $x = 0$ 时均存在 3 个极值点.基于相互作用解(24),图 3 和图 4 分别展示了 lump 波与线孤子相互作用过程中的"聚变"和"裂变"现象.当 $k_2 > 0$ 时,图 3 展示了 lump 波与线孤子发生碰撞后的相互融合;当 $k_2 < 0$ 时,图 4 展示了线孤子分裂为线孤子和 lump 波的过程.最后,图 6 展示了解(30)的周期性传播特征.

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