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三维稳态磁流体动力学方程的 Liouville 定理*

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摘要： 研究了三维稳态磁流体动力学方程的 Liouville 定理.首先由能量估计建立了一个 Caccioppoli 型不等式,再结合 Sobolev 嵌入得到了 Liouville 定理成立的 3 个充分条件,其中一个充分条件表明:若三维稳态磁流体动力学方程的光滑解 $(\boldsymbol{u},\boldsymbol{b}) \in L^p, 3/2 < p < 3$,则 $\boldsymbol{u} = \boldsymbol{b} \equiv \mathbf{0}$.该结果在不需要有限 Dirichlet 积分的条件下,将 Lebesgue 空间中可积指标的下界从 2 扩展至3/2,改进和推广了已有关于磁流体动力学方程 Liouville 定理的一些结论.

关　键　词： 磁流体动力学方程；　Liouville 定理；　Caccioppoli 型不等式
中图分类号： O175.2　　　**文献标志码：** A　　　DOI：10.21656/1000-0887.430375

On the Liouville Theorems for 3D Stationary Magnetohydrodynamic Equations

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Abstract: The Liouville theorems for 3D stationary magnetohydrodynamic equations were studied. First, a Caccioppoli type inequality was obtained with the energy method, then 3 sufficient conditions for the Liouville theorems were obtained based on the Sobolev embedding theorems, of which 1 sufficient condition indicates that, given a smooth solution to the 3D stationary magnetohydrodynamic equation satisfying $(\boldsymbol{u},\boldsymbol{b}) \in L^p, 3/2 < p < 3$, equality $\boldsymbol{u} = \boldsymbol{b} \equiv \mathbf{0}$ will be tenable. This work extends the lower bound of the integrable index in the Lebesgue space from 2 to 3/2 without the finite Dirichlet integral condition, which improves and generalizes some conclusions about the Liouville theorems for stationary magnetohydrodynamic equations.

Key words: magnetohydrodynamic equations; Liouville theorems; Caccioppoli type inequality

0　引　　言

考虑如下三维稳态磁流体动力学方程：

* **收稿日期：** 2022-11-22；**修订日期：** 2023-03-04
基金项目： 国家自然科学基金项目(11871305;12271296)
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引用格式： 田琴，向长林，别群益. 三维稳态磁流体动力学方程的 Liouville 定理[J]. 应用数学和力学，2023，**44**(10)：1250-1259.

$$\begin{cases} -\Delta \boldsymbol{u} + \boldsymbol{u}\cdot\nabla \boldsymbol{u} - \boldsymbol{b}\cdot\nabla \boldsymbol{b} + \nabla P = \boldsymbol{0}, \\ -\Delta \boldsymbol{b} + \boldsymbol{u}\cdot\nabla \boldsymbol{b} - \boldsymbol{b}\cdot\nabla \boldsymbol{u} = \boldsymbol{0}, \\ \nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{b} = 0, \end{cases} \tag{1}$$

这里 $\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{x})=(u_1,u_2,u_3)$ 是磁流体的速度场,$\boldsymbol{b}=\boldsymbol{b}(\boldsymbol{x})=(b_1,b_2,b_3)$ 表示磁流体的磁场,$P=P(\boldsymbol{x})$ 表示压力,方程(1)描述了导电流体的磁性能.

当 $\boldsymbol{b}=\boldsymbol{0}$ 时,方程(1)变为稳态 Navier-Stokes 方程.近几十年来对 Navier-Stokes 方程的研究已经取得了很多成果,其中包括解的稳定性、正则性、一致渐近性以及 Liouville 定理等[1-12].同样对磁流体动力学方程的 Liouville 定理也有一些研究结果.2016 年,Chae 和 Weng[4] 证明了若方程(1)光滑解 $(\boldsymbol{u},\boldsymbol{b})\in L^3(\mathbb{R}^3)$, 且满足有限 Dirichlet 积分的条件 $(\nabla\boldsymbol{u},\nabla\boldsymbol{b})\in L^2(\mathbb{R}^3)$,有 $\boldsymbol{u}=\boldsymbol{b}\equiv\boldsymbol{0}$.2019 年,Schulz[13] 证明了若 $(\boldsymbol{u},\boldsymbol{b})\in L^p(\mathbb{R}^3)$,其中 $2<p<6$,且 $(\boldsymbol{u},\boldsymbol{b})\in \mathrm{BMO}^{-1}(\mathbb{R}^3)$, 则 Liouville 定理成立.2020 年,Yuan 和 Xiao[14] 在不需要有限 Dirichlet 积分的条件下,证明了以下定理:若

$$(\boldsymbol{u},\boldsymbol{b})\in L^p(\mathbb{R}^3),\qquad 2<p<9/2, \tag{2}$$

则 $\boldsymbol{u}=\boldsymbol{b}\equiv\boldsymbol{0}$.2023 年,周艳平等[15] 在局部 Morrey 空间框架下,得到了若干充分条件以保证方程(1)存在唯一平凡解,并利用 Lorentz 空间和局部 Morrey 空间的嵌入关系,给出了该系统在 Lorentz 空间中的 Liouville 定理,所得结果不需要有限 Dirichlet 积分条件.

本文深入研究文献[8]的证明思想,并结合周艳平和别群益等[15] 工作中的方法,在不需要有限 Dirichlet 积分的条件下,进一步得到如下结论:

定理 1　假设 $(\boldsymbol{u},\boldsymbol{b},P)$ 是方程(1)的光滑解,若 $(\boldsymbol{u},\boldsymbol{b})\in L^p(\mathbb{R}^3)$,$3/2<p<3$,则 $\boldsymbol{u}=\boldsymbol{b}\equiv\boldsymbol{0}$,$P=0$.

注 1　定理 1 中 Lebesgue 空间的可积指标的取值范围是 $3/2<p<3$, 相比于文献[14]的结果(见式(2)),定理 1 将可积指标 p 的下界从 2 扩展至3/2.

为了进一步介绍本文的主要结果,记 $B(R)=\{\boldsymbol{x}\in\mathbb{R}^3: |\boldsymbol{x}|<R\}$,$\mathcal{C}(\rho,R)=\{\boldsymbol{x}\in\mathbb{R}: \rho<|\boldsymbol{x}|<R\}$.对于 $E\subset\mathbb{R}^3$,记 E 在 $\mathbb{R}^3$ 中的测度为 $|E|$.若 $f\in L^1(E)$, 定义

$$f_E=\frac{1}{|E|}\int_E f\,\mathrm{d}\boldsymbol{x}.$$

由 Hölder 不等式,对任意 $f\in L^p(E)$, 有

$$\|f-f_E\|_{L^p(E)}\leqslant C\|f\|_{L^p(E)}.$$

下面介绍 $\mathrm{BMO}(\mathbb{R}^3)$ 空间与 $\mathrm{BMO}^{-1}(\mathbb{R}^3)$ 空间(可参考文献[16-17]).对任意 $r>0$, 若

$$\sup_{r>0}\frac{1}{|B(r)|}\int_{B(r)}|\boldsymbol{\Phi}-\boldsymbol{\Phi}_{B(r)}|\,\mathrm{d}\boldsymbol{x}<+\infty,$$

则称 $\boldsymbol{\Phi}\in\mathrm{BMO}(\mathbb{R}^3)$.在 $\mathrm{BMO}(\mathbb{R}^3)$ 空间中还有如下等价定义[17]:

$$\sup_{r>0}\frac{1}{|B(r)|}\int_{B(r)}|\boldsymbol{\Phi}-\boldsymbol{\Phi}_{B(r)}|^p\,\mathrm{d}\boldsymbol{x}<+\infty,\qquad \forall\, 1\leqslant p<+\infty.$$

若存在 $\boldsymbol{\Phi}\in\mathrm{BMO}(\mathbb{R}^3)$,使得 $\boldsymbol{u}=\nabla\cdot\boldsymbol{\Phi}$,则称 $\boldsymbol{u}\in\mathrm{BMO}^{-1}(\mathbb{R}^3)$.下面给出方程(1)满足 Liouville 定理的另外两个充分条件.

定理 2　假设 $(\boldsymbol{u},\boldsymbol{b})$ 是方程(1)的光滑解,且 $\boldsymbol{u}=\nabla\times\boldsymbol{H}$,$\boldsymbol{b}=\nabla\times\boldsymbol{V}$,其中 $\boldsymbol{H}$ 和 $\boldsymbol{V}$ 是光滑向量场.若

$$K_{\alpha_1}(t)=\sup_{R>0}R^{\alpha_1}\left(\frac{1}{|B(R)|}\int_{B(R)}|\boldsymbol{H}-\boldsymbol{H}_{B(R)}|^t\,\mathrm{d}\boldsymbol{x}\right)^{1/t}<\infty,$$

$$K_{\alpha_2}(t)=\sup_{R>0}R^{\alpha_2}\left(\frac{1}{|B(R)|}\int_{B(R)}|\boldsymbol{V}-\boldsymbol{V}_{B(R)}|^t\,\mathrm{d}\boldsymbol{x}\right)^{1/t}<\infty,$$

其中

$$\alpha_1,\ \alpha_2>-\frac{t-3}{6(t-1)},\ t>3,$$

则在 $\mathbb{R}^3$ 中 $\boldsymbol{u}=\boldsymbol{b}\equiv\boldsymbol{0}$.

注 2 特别地,若 $\alpha_1 = \alpha_2 = 0$, 即当 $(\boldsymbol{u},\boldsymbol{b}) \in \mathrm{BMO}^{-1}(\mathbb{R}^3)$ 时,定理 2 成立.

定理 3 假设 $(\boldsymbol{u},\boldsymbol{b})$ 是方程(1)的光滑解.若满足

$$M_{\beta_1}(q) = \sup_{R>0} R^{\beta_1}\left(\frac{1}{|B(R)|}\int_{B(R)} |\boldsymbol{u}|^q \mathrm{d}\boldsymbol{x}\right)^{1/q} < \infty,$$

$$M_{\beta_2}(q) = \sup_{R>0} R^{\beta_2}\left(\frac{1}{|B(R)|}\int_{B(R)} |\boldsymbol{b}|^q \mathrm{d}\boldsymbol{x}\right)^{1/q} < \infty,$$

其中

$$\beta_1,\beta_2 > \frac{6q-3}{8q-6},\ \frac{3}{2} < q < 3,$$

则在 $\mathbb{R}^3$ 中 $\boldsymbol{u} = \boldsymbol{b} \equiv \mathbf{0}$.

1 预备知识

设 $R/2 \leqslant \rho < r \leqslant R$,定义截断函数 $\varphi(\boldsymbol{x}) \in C_c^\infty(B(r))$,使得 $0 \leqslant \varphi(\boldsymbol{x}) \leqslant 1$.当 $|\boldsymbol{x}| \leqslant \rho$ 时,$\varphi(\boldsymbol{x}) = 1$;当 $|\boldsymbol{x}| \geqslant r$ 时,$\varphi(\boldsymbol{x}) = 0$, 且满足

$$|\nabla\varphi(\boldsymbol{x})| < \frac{C}{r-\rho},\ |\nabla^2\varphi(\boldsymbol{x})| < \frac{C}{(r-\rho)^2}.$$

引理 1[6] 对任意 $2 < s < +\infty$, 存在一个常数 $c_0 = c_0(s) > 0$ 和一个向量函数 $\boldsymbol{\omega} \in W^{1,s}(B(r))$, 使得 $\nabla\cdot\boldsymbol{\omega} = \nabla\varphi\cdot\boldsymbol{u}$,且在 $\partial B(r)$ 上 $\boldsymbol{\omega} = \mathbf{0}$, 有

$$\int_{B(r)} |\nabla\boldsymbol{\omega}|^s \mathrm{d}\boldsymbol{x} \leqslant c_0(s)\int_{B(r)} |\nabla\varphi\cdot\boldsymbol{u}|^s \mathrm{d}\boldsymbol{x}.$$

引理 2[7] 令 $f(\boldsymbol{x})$ 是非负有界函数, 其定义域是 $[r_1,r_2] \subset \mathbb{R}_+$.若存在正数 $k > l > m$,非负常数 K,L,M,N 及参数 $\gamma \in (0,1)$, 使得对 $r_1 \leqslant s < t \leqslant r_2$, 有

$$f(s) \leqslant \gamma f(t) + \frac{K}{(t-s)^k} + \frac{L}{(t-s)^l} + \frac{M}{(t-s)^m} + N,$$

则

$$f(s) \leqslant C(k,l,m,\gamma)\left[\frac{K}{(t-s)^k} + \frac{L}{(t-s)^l} + \frac{M}{(t-s)^m} + N\right].$$

引理 3[8] 令 ϕ 是 $B(R)$ 上具有紧支集的非负有界二次可微函数,$\tilde{\boldsymbol{H}}$ 是光滑向量, 使得 $\boldsymbol{u} = \nabla\times\tilde{\boldsymbol{H}}$, 则

$$\int_{B(R)} \phi|\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x} \leqslant C\left(\int_{B(R)} \phi|\nabla\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}\left(\int_{B(R)} \phi|\tilde{\boldsymbol{H}}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2} + C\left(\int_{B(R)} |\nabla^2\phi||\tilde{\boldsymbol{H}}|^2\mathrm{d}\boldsymbol{x}\right).$$

下面给出 Caccioppoli 型不等式.

引理 4 设 $(\boldsymbol{u},\boldsymbol{b})$ 是方程(1)的光滑解.且 $\boldsymbol{u} = \nabla\times\bar{\boldsymbol{H}} = \nabla\times\tilde{\boldsymbol{H}}$,$\boldsymbol{b} = \nabla\times\bar{\boldsymbol{V}} = \nabla\times\tilde{\boldsymbol{V}}$, 其中 $\bar{\boldsymbol{H}}$,$\tilde{\boldsymbol{H}}$,$\bar{\boldsymbol{V}}$ 和 $\tilde{\boldsymbol{V}}$ 是光滑向量.则存在常数 $C > 0$, 使得对任意 $R > 1$, 有

$$\int_{B(R/2)}(|\nabla\boldsymbol{u}|^2 + |\nabla\boldsymbol{b}|^2)\mathrm{d}\boldsymbol{x} \leqslant$$

$$\frac{C}{R^4}\int_{\mathcal{C}(R/2,R)}(|\tilde{\boldsymbol{H}}|^2 + |\tilde{\boldsymbol{V}}|^2)\mathrm{d}\boldsymbol{x}\left[1 + \left(\frac{1}{R^3}\int_{B(R)} |\bar{\boldsymbol{H}}|^t\mathrm{d}\boldsymbol{x}\right)^{4/(t-3)} + \left(\frac{1}{R^3}\int_{B(R)} |\bar{\boldsymbol{V}}|^t\mathrm{d}\boldsymbol{x}\right)^{4/(t-3)}\right], \quad (3)$$

其中 $t > 3$.

证明 方程(1)第 1 式的两边乘以 $\varphi\boldsymbol{u} - \boldsymbol{\omega}$, 方程(1)第 2 式的两边乘以 $\varphi\boldsymbol{b}$,然后在 $B(r)$ 上积分,得

$$\int_{B(r)}(|\nabla\boldsymbol{u}|^2 + |\nabla\boldsymbol{b}|^2)\varphi\mathrm{d}\boldsymbol{x} =$$

$$-\int_{B(r)} \nabla\boldsymbol{u} : (\nabla\varphi\otimes\boldsymbol{u})\mathrm{d}\boldsymbol{x} + \int_{B(r)} \nabla\boldsymbol{u} : \nabla\boldsymbol{\omega}\mathrm{d}\boldsymbol{x} + \int_{B(r)} (\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{\omega}\mathrm{d}\boldsymbol{x} -$$

$$\int_{B(r)} (\boldsymbol{b}\cdot\nabla\boldsymbol{b})\cdot\boldsymbol{\omega}\mathrm{d}\boldsymbol{x} - \int_{B(r)} \nabla\boldsymbol{b} : (\nabla\varphi\otimes\boldsymbol{b})\mathrm{d}\boldsymbol{x} - \int_{B(r)} (\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\varphi\boldsymbol{u}\mathrm{d}\boldsymbol{x} -$$

$$\int_{B(r)}(\boldsymbol{u}\cdot\nabla\boldsymbol{b})\cdot\varphi\boldsymbol{b}\mathrm{d}\boldsymbol{x}+\int_{B(r)}(\boldsymbol{b}\cdot\nabla\boldsymbol{b})\cdot\varphi\boldsymbol{u}\mathrm{d}\boldsymbol{x}+\int_{B(r)}(\boldsymbol{b}\cdot\nabla\boldsymbol{u})\cdot\varphi\boldsymbol{b}\mathrm{d}\boldsymbol{x}=$$
$$I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8+I_9.$$

由引理 1 和 Hölder 不等式,有

$$|I_1+I_2|\leqslant\left|\int_{B(r)}\nabla\boldsymbol{u}:(\nabla\varphi\otimes\boldsymbol{u})\mathrm{d}\boldsymbol{x}\right|+\left|\int_{B(r)}\nabla\boldsymbol{u}:\nabla\boldsymbol{\omega}\mathrm{d}\boldsymbol{x}\right|\leqslant$$
$$C\left(\int_{B(r)}|\nabla\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}\left(\int_{B(r)}|\nabla\varphi|^2|\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}.$$

由 $\boldsymbol{u}=\nabla\times\bar{\boldsymbol{H}}$、引理 1 和 Hölder 不等式,有

$$|I_3+I_4|\leqslant\left|\int_{B(r)}(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{\omega}\mathrm{d}\boldsymbol{x}\right|+\left|\int_{B(r)}(\boldsymbol{b}\cdot\nabla\boldsymbol{b})\cdot\boldsymbol{\omega}\mathrm{d}\boldsymbol{x}\right|=$$
$$\left|\int_{B(r)}(\nabla\times\bar{H})_j\partial_j u_k\omega_k\mathrm{d}\boldsymbol{x}\right|+\left|\int_{B(r)}(\nabla\times\bar{V})_j\partial_j b_k\omega_k\mathrm{d}\boldsymbol{x}\right|=$$
$$\left|\int_{B(r)}\bar{\boldsymbol{H}}\cdot(\nabla u_k\times\nabla\omega_k)\mathrm{d}\boldsymbol{x}\right|+\left|\int_{B(r)}\bar{\boldsymbol{V}}\cdot(\nabla b_k\times\nabla\omega_k)\mathrm{d}\boldsymbol{x}\right|\leqslant$$
$$C\left(\int_{B(r)}|\nabla\varphi|^s|\boldsymbol{u}|^s\mathrm{d}\boldsymbol{x}\right)^{1/s}\left[\left(\int_{B(r)}|\nabla\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}\left(\int_{B(r)}|\bar{\boldsymbol{H}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{(s-2)/(2s)}+\right.$$
$$\left.\left(\int_{B(r)}|\nabla\boldsymbol{b}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}\left(\int_{B(r)}|\bar{\boldsymbol{V}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{(s-2)/(2s)}\right].$$

对于 I_5, 利用 Hölder 不等式,可得

$$|I_5|\leqslant\left|\int_{B(r)}\nabla\boldsymbol{b}:(\nabla\varphi\otimes\boldsymbol{b})\mathrm{d}\boldsymbol{x}\right|\leqslant C\left(\int_{B(r)}|\nabla\boldsymbol{b}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}\left(\int_{B(r)}|\nabla\varphi|^2|\boldsymbol{b}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}.$$

同样, I_6 可化为

$$|I_6|\leqslant\left|\int_{B(r)}(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\varphi\boldsymbol{u}\mathrm{d}\boldsymbol{x}\right|=\frac{1}{2}\left|\int_{B(r)}\nabla\times\bar{\boldsymbol{H}}\cdot(\nabla|\boldsymbol{u}|^2\varphi)\mathrm{d}\boldsymbol{x}\right|=$$
$$\frac{1}{2}\left|\int_{B(r)}\bar{\boldsymbol{H}}\cdot(\nabla\times(\nabla|\boldsymbol{u}|^2\varphi))\mathrm{d}\boldsymbol{x}\right|=\frac{1}{2}\left|\int_{B(r)}\bar{\boldsymbol{H}}\cdot(\nabla\varphi\times(\nabla|\boldsymbol{u}|^2))\mathrm{d}\boldsymbol{x}\right|\leqslant$$
$$\left(\int_{B(r)}|\nabla\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}\left(\int_{B(r)}|\nabla\varphi|^s|\boldsymbol{u}|^s\mathrm{d}\boldsymbol{x}\right)^{1/s}\left(\int_{B(r)}|\bar{\boldsymbol{H}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{(s-2)/(2s)}.$$

对于 I_7,利用 $\boldsymbol{u}=\nabla\times\bar{\boldsymbol{H}}$ 和 Hölder 不等式,可得

$$|I_7|=\left|\int_{B(r)}(\boldsymbol{u}\cdot\nabla\boldsymbol{b})\cdot\varphi\boldsymbol{b}\mathrm{d}\boldsymbol{x}\right|=\left|\int_{B(r)}\bar{\boldsymbol{H}}\cdot\nabla\varphi\times(\boldsymbol{b}\cdot\nabla\boldsymbol{b})\mathrm{d}\boldsymbol{x}\right|\leqslant$$
$$\left(\int_{B(r)}|\nabla\boldsymbol{b}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}\left(\int_{B(r)}|\nabla\varphi|^s|\boldsymbol{b}|^s\mathrm{d}\boldsymbol{x}\right)^{1/s}\left(\int_{B(r)}|\bar{\boldsymbol{H}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{(s-2)/(2s)}.$$

对于 I_8 和 I_9,由 $\boldsymbol{b}=\nabla\times\bar{\boldsymbol{V}}$ 和 Hölder 不等式,得

$$|I_8+I_9|=\left|\int_{B(r)}(\boldsymbol{b}\cdot\nabla\boldsymbol{b})\cdot\varphi\boldsymbol{u}\mathrm{d}\boldsymbol{x}+\int_{B(r)}(\boldsymbol{b}\cdot\nabla\boldsymbol{u})\cdot\varphi\boldsymbol{b}\mathrm{d}\boldsymbol{x}\right|=$$
$$\left|\int_{B(r)}b_i\partial_i(b_ju_j)\varphi\mathrm{d}\boldsymbol{x}\right|=\left|\int_{B(r)}\nabla\times\bar{\boldsymbol{V}}\cdot(\nabla(\boldsymbol{bu})\varphi)\mathrm{d}\boldsymbol{x}\right|=$$
$$\left|\int_{B(r)}\bar{\boldsymbol{V}}\cdot(\nabla\times(\nabla(\boldsymbol{bu})\varphi))\mathrm{d}\boldsymbol{x}\right|=\left|\int_{B(r)}\bar{\boldsymbol{V}}\cdot(\nabla\varphi\times(\nabla(\boldsymbol{bu})))\mathrm{d}\boldsymbol{x}\right|\leqslant$$
$$C\left(\int_{B(r)}|\bar{\boldsymbol{V}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{(s-2)/(2s)}\left[\left(\int_{B(r)}|\nabla\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}\left(\int_{B(r)}|\nabla\varphi|^s|\boldsymbol{b}|^s\mathrm{d}\boldsymbol{x}\right)^{1/s}+\right.$$
$$\left.\left(\int_{B(r)}|\nabla\boldsymbol{b}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}\left(\int_{B(r)}|\nabla\varphi|^s|\boldsymbol{u}|^s\mathrm{d}\boldsymbol{x}\right)^{1/s}\right].$$

联立 $I_1\sim I_9$, 再由 Hölder 不等式,得

$$\int_{B(\rho)}(|\nabla\boldsymbol{u}|^2+|\nabla\boldsymbol{b}|^2)\mathrm{d}\boldsymbol{x}\leqslant\int_{B(r)}(|\nabla\boldsymbol{u}|^2+|\nabla\boldsymbol{b}|^2)\varphi\mathrm{d}\boldsymbol{x}\leqslant$$

$$C\left(\int_{B(r)}|\nabla\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}\left(\int_{B(r)}|\nabla\varphi|^2|\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}+$$
$$C\left(\int_{B(r)}|\nabla\boldsymbol{b}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}\left(\int_{B(r)}|\nabla\varphi|^2|\boldsymbol{b}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}+$$
$$C\left(\int_{B(r)}|\nabla\varphi|^s|\boldsymbol{u}|^s\mathrm{d}\boldsymbol{x}\right)^{1/s}J_1+C\left(\int_{B(r)}|\nabla\varphi|^s|\boldsymbol{b}|^s\mathrm{d}\boldsymbol{x}\right)^{1/s}J_2,\tag{4}$$

其中

$$J_1=\left(\int_{B(r)}|\nabla\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}\left(\int_{B(r)}|\bar{\boldsymbol{H}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{(s-2)/(2s)}+$$
$$\left(\int_{B(r)}|\nabla\boldsymbol{b}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}\left(\int_{B(r)}|\bar{\boldsymbol{V}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{(s-2)/(2s)},$$
$$J_2=\left(\int_{B(r)}|\nabla\boldsymbol{b}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}\left(\int_{B(r)}|\bar{\boldsymbol{H}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{(s-2)/(2s)}+$$
$$\left(\int_{B(r)}|\nabla\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}\left(\int_{B(r)}|\bar{\boldsymbol{V}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{(s-2)/(2s)}.$$

对于式(4),由 Young 不等式,有

$$\int_{B(\rho)}(|\nabla\boldsymbol{u}|^2+|\nabla\boldsymbol{b}|^2)\mathrm{d}\boldsymbol{x}\leqslant$$
$$\frac{1}{8}\int_{B(r)}(|\nabla\boldsymbol{u}|^2+|\nabla\boldsymbol{b}|^2)\mathrm{d}\boldsymbol{x}+C\left(\int_{B(r)}|\nabla\varphi|^2|\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}+\int_{B(r)}|\nabla\varphi|^2|\boldsymbol{b}|^2\mathrm{d}\boldsymbol{x}\right)+$$
$$C\left[\left(\int_{B(r)}|\nabla\varphi|^s|\boldsymbol{u}|^s\mathrm{d}\boldsymbol{x}\right)^{2/s}+\left(\int_{B(r)}|\nabla\varphi|^s|\boldsymbol{b}|^s\mathrm{d}\boldsymbol{x}\right)^{2/s}\right]\times$$
$$\left[\left(\int_{B(r)}|\bar{\boldsymbol{H}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{(s-2)/s}+\left(\int_{B(r)}|\bar{\boldsymbol{V}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{(s-2)/s}\right].\tag{5}$$

当 $2<s<6$ 时, 由 Gagliardo-Nirenberg-Sobolev 不等式,有

$$\left(\int_{B(r)}|\nabla\varphi|^s|\boldsymbol{u}|^s\mathrm{d}\boldsymbol{x}\right)^{2/s}\leqslant$$
$$C\left(\int_{B(r)}|\nabla\varphi|^2|\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}\right)^{1-\theta}\left[\left(\int_{B(r)}|\nabla\varphi|^2|\nabla\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}\right)^{\theta}+\left(\int_{B(r)}|\nabla^2\varphi|^2|\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}\right)^{\theta}\right],$$
$$\left(\int_{B(r)}|\nabla\varphi|^s|\boldsymbol{b}|^s\mathrm{d}\boldsymbol{x}\right)^{2/s}\leqslant$$
$$C\left(\int_{B(r)}|\nabla\varphi|^2|\boldsymbol{b}|^2\mathrm{d}\boldsymbol{x}\right)^{1-\theta}\left[\left(\int_{B(r)}|\nabla\varphi|^2|\nabla\boldsymbol{b}|^2\mathrm{d}\boldsymbol{x}\right)^{\theta}+\left(\int_{B(r)}|\nabla^2\varphi|^2|\boldsymbol{b}|^2\mathrm{d}\boldsymbol{x}\right)^{\theta}\right],$$

其中 $\theta=3(s-2)/(2s)$.通过 Young 不等式,式(5)可化为

$$\int_{B(\rho)}(|\nabla\boldsymbol{u}|^2+|\nabla\boldsymbol{b}|^2)\mathrm{d}\boldsymbol{x}\leqslant$$
$$\frac{1}{4}\int_{B(r)}(|\nabla\boldsymbol{u}|^2+|\nabla\boldsymbol{b}|^2)\mathrm{d}\boldsymbol{x}+C\left(\int_{B(r)}|\nabla\varphi|^2|\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}+\int_{B(r)}|\nabla\varphi|^2|\boldsymbol{b}|^2\mathrm{d}\boldsymbol{x}\right)+$$
$$C\left[\left(\int_{B(r)}|\bar{\boldsymbol{H}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{(s-2)/s}+\left(\int_{B(r)}|\bar{\boldsymbol{V}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{(s-2)/s}\right]\times$$
$$\left[\left(\int_{B(r)}|\nabla\varphi|^2|\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}\right)^{1-\theta}\left(\int_{B(r)}|\nabla^2\varphi|^2|\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}\right)^{\theta}+\right.$$
$$\left.\left(\int_{B(r)}|\nabla\varphi|^2|\boldsymbol{b}|^2\mathrm{d}\boldsymbol{x}\right)^{1-\theta}\left(\int_{B(r)}|\nabla^2\varphi|^2|\boldsymbol{b}|^2\mathrm{d}\boldsymbol{x}\right)^{\theta}\right]+$$
$$C\left(\frac{1}{(r-\rho)^2}\right)^{\theta/(1-\theta)}\left(\int_{B(r)}|\nabla\varphi|^2|\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}+\int_{B(r)}|\nabla\varphi|^2|\boldsymbol{b}|^2\mathrm{d}\boldsymbol{x}\right)\times$$
$$\left[\left(\int_{B(r)}|\bar{\boldsymbol{H}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{(s-2)/(s(1-\theta))}+\left(\int_{B(r)}|\bar{\boldsymbol{V}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{(s-2)/(s(1-\theta))}\right]\leqslant$$
$$\frac{1}{4}\int_{B(r)}(|\nabla\boldsymbol{u}|^2+|\nabla\boldsymbol{b}|^2)\mathrm{d}\boldsymbol{x}+C\left(\int_{B(r)}|\nabla\varphi|^2|\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}+\int_{B(r)}|\nabla\varphi|^2|\boldsymbol{b}|^2\mathrm{d}\boldsymbol{x}\right)\times$$

$$\left[1+\left(\int_{B(r)}|\nabla\varphi|^2|\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}\right)^{-\theta}\left(\int_{B(r)}|\nabla^2\varphi|^2|\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}\right)^{\theta}+\right.$$
$$\left.\left(\int_{B(r)}|\nabla\varphi|^2|\boldsymbol{b}|^2\mathrm{d}\boldsymbol{x}\right)^{-\theta}\left(\int_{B(r)}|\nabla^2\varphi|^2|\boldsymbol{b}|^2\mathrm{d}\boldsymbol{x}\right)^{\theta}\right]\times$$
$$\left[\left(\int_{B(r)}|\bar{\boldsymbol{H}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{(s-2)/s}+\left(\int_{B(r)}|\bar{\boldsymbol{V}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{(s-2)/s}\right]+$$
$$C\left(\frac{1}{(r-\rho)^2}\right)^{\theta/(1-\theta)}\left(\int_{B(r)}|\nabla\varphi|^2|\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}+\int_{B(r)}|\nabla\varphi|^2|\boldsymbol{b}|^2\mathrm{d}\boldsymbol{x}\right)\times$$
$$\left[\left(\int_{B(r)}|\bar{\boldsymbol{H}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{(s-2)/(s(1-\theta))}+\left(\int_{B(r)}|\bar{\boldsymbol{V}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{(s-2)/(s(1-\theta))}\right].\tag{6}$$

下面估计式(6)中含有 $|\nabla\varphi|^2$ 和 $|\nabla^2\varphi|^2$ 的项.根据引理 3,可得

$$\int_{B(r)}|\nabla\varphi|^2|\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}\leqslant$$
$$\frac{C}{(r-\rho)^2}\left[\left(\int_{\mathcal{C}(\rho,r)}|\tilde{\boldsymbol{H}}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}\left(\int_{\mathcal{C}(\rho,r)}|\nabla\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}+\frac{1}{(r-\rho)^2}\int_{\mathcal{C}(\rho,r)}|\tilde{\boldsymbol{H}}|^2\mathrm{d}\boldsymbol{x}\right],$$
$$\int_{B(r)}|\nabla\varphi|^2|\boldsymbol{b}|^2\mathrm{d}\boldsymbol{x}\leqslant$$
$$\frac{C}{(r-\rho)^2}\left[\left(\int_{\mathcal{C}(\rho,r)}|\tilde{\boldsymbol{V}}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}\left(\int_{\mathcal{C}(\rho,r)}|\nabla\boldsymbol{b}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}+\frac{1}{(r-\rho)^2}\int_{\mathcal{C}(\rho,r)}|\tilde{\boldsymbol{V}}|^2\mathrm{d}\boldsymbol{x}\right],$$
$$\int_{B(r)}|\nabla^2\varphi|^2|\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}\leqslant$$
$$\frac{C}{(r-\rho)^4}\left[\left(\int_{\mathcal{C}(\rho,r)}|\tilde{\boldsymbol{H}}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}\left(\int_{\mathcal{C}(\rho,r)}|\nabla\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}+\frac{1}{(r-\rho)^2}\int_{\mathcal{C}(\rho,r)}|\tilde{\boldsymbol{H}}|^2\mathrm{d}\boldsymbol{x}\right],$$
$$\int_{B(r)}|\nabla^2\varphi|^2|\boldsymbol{b}|^2\mathrm{d}\boldsymbol{x}\leqslant$$
$$\frac{C}{(r-\rho)^4}\left[\left(\int_{\mathcal{C}(\rho,r)}|\tilde{\boldsymbol{V}}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}\left(\int_{\mathcal{C}(\rho,r)}|\nabla\boldsymbol{b}|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}+\frac{1}{(r-\rho)^2}\int_{\mathcal{C}(\rho,r)}|\tilde{\boldsymbol{V}}|^2\mathrm{d}\boldsymbol{x}\right].$$

由上面 4 个式子以及 $\theta=3(s-2)/(2s)$,可得

$$\left(\int_{B(r)}|\nabla\varphi|^2|\boldsymbol{u}|^2\mathrm{d}x\right)^{-\theta}\left(\int_{B(r)}|\nabla^2\varphi|^2|\boldsymbol{u}|^2\mathrm{d}x\right)^{\theta}=\frac{C}{(r-\rho)^{2\theta}}=C\left(\frac{1}{(r-\rho)^3}\right)^{(s-2)/s},$$
$$\left(\int_{B(r)}|\nabla\varphi|^2|\boldsymbol{b}|^2\mathrm{d}x\right)^{-\theta}\left(\int_{B(r)}|\nabla^2\varphi|^2|\boldsymbol{b}|^2\mathrm{d}x\right)^{\theta}=\frac{C}{(r-\rho)^{2\theta}}=C\left(\frac{1}{(r-\rho)^3}\right)^{(s-2)/s}.$$

因为 $\theta=\dfrac{3(s-2)}{2s}$, 可得 $\dfrac{s-2}{s(1-\theta)}=\dfrac{2(s-2)}{6-s}$, $\dfrac{\theta}{1-\theta}=\dfrac{3(s-2)}{6-s}$, 有

$$C\left(\frac{1}{(r-\rho)^2}\right)^{\theta/(1-\theta)}\left(\int_{B(r)}|\bar{\boldsymbol{H}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{(s-2)/(s(1-\theta))}=C\left(\frac{1}{(r-\rho)^3}\int_{B(r)}|\bar{\boldsymbol{H}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{2(s-2)/(6-s)},$$
$$C\left(\frac{1}{(r-\rho)^2}\right)^{\theta/(1-\theta)}\left(\int_{B(r)}|\bar{\boldsymbol{V}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{(s-2)/(s(1-\theta))}=C\left(\frac{1}{(r-\rho)^3}\int_{B(r)}|\bar{\boldsymbol{V}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{2(s-2)/(6-s)}.$$

将上述式子代入式(6),再由 Young 不等式,有

$$\int_{B(\rho)}(|\nabla\boldsymbol{u}|^2+|\nabla\boldsymbol{b}|^2)\mathrm{d}\boldsymbol{x}\leqslant$$
$$\frac{1}{2}\int_{B(r)}(|\nabla\boldsymbol{u}|^2+|\nabla\boldsymbol{b}|^2)\mathrm{d}\boldsymbol{x}+\frac{C}{(r-\rho)^4}\left(\int_{\mathcal{C}(\rho,r)}|\tilde{\boldsymbol{H}}|^2\mathrm{d}\boldsymbol{x}+\int_{\mathcal{C}(\rho,r)}|\tilde{\boldsymbol{V}}|^2\mathrm{d}\boldsymbol{x}\right)\times$$
$$\left[1+\left(\frac{1}{(r-\rho)^3}\int_{B(r)}|\bar{\boldsymbol{H}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{4(s-2)/(6-s)}+\left(\frac{1}{(r-\rho)^3}\int_{B(r)}|\bar{\boldsymbol{V}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{4(s-2)/(6-s)}\right].$$

由引理 2,可得

$$\int_{B(\rho)}(|\nabla \boldsymbol{u}|^2+|\nabla \boldsymbol{b}|^2)\mathrm{d}\boldsymbol{x} \leqslant$$
$$\frac{C}{(r-\rho)^4}\left(\int_{\mathcal{C}(\rho,r)}|\tilde{\boldsymbol{H}}|^2\mathrm{d}\boldsymbol{x}+\int_{\mathcal{C}(\rho,r)}|\tilde{\boldsymbol{V}}|^2\mathrm{d}\boldsymbol{x}\right)\times$$
$$\left[1+\left(\frac{1}{(r-\rho)^3}\int_{B(r)}|\bar{\boldsymbol{H}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{4(s-2)/(6-s)}+\left(\frac{1}{(r-\rho)^3}\int_{B(r)}|\bar{\boldsymbol{V}}|^{2s/(s-2)}\mathrm{d}\boldsymbol{x}\right)^{4(s-2)/(6-s)}\right], \tag{7}$$

其中 $2<s<6$,令 $t=2s/(s-2)$,因此 $t>3$.令 $\rho=R/2$,$r=R$, 则式(7)可化为

$$\int_{B(R/2)}(|\nabla \boldsymbol{u}|^2+|\nabla \boldsymbol{b}|^2)\mathrm{d}\boldsymbol{x} \leqslant$$
$$\frac{C}{R^4}\int_{\mathcal{C}(R/2,R)}(|\tilde{\boldsymbol{H}}|^2+|\tilde{\boldsymbol{V}}|^2)\mathrm{d}\boldsymbol{x}\times$$
$$\left[1+\left(\frac{1}{R^3}\int_{B(R)}|\bar{\boldsymbol{H}}|^t\mathrm{d}\boldsymbol{x}\right)^{4/(t-3)}+\left(\frac{1}{R^3}\int_{B(R)}|\bar{\boldsymbol{V}}|^t\mathrm{d}\boldsymbol{x}\right)^{4/(t-3)}\right].$$

此即式(3),故引理 4 得证.

2 主要结果的证明

定理 1 的证明 若 $(\boldsymbol{u},\boldsymbol{b})$ 是方程(1)的光滑解, 假设 $(\boldsymbol{u},\boldsymbol{b})\in L^p(\mathbb{R}^3)$, 其中 $3/2<p<3$. 定义如下向量:

$$\boldsymbol{H}=\frac{1}{-\Delta}(\nabla\times\boldsymbol{u}),\ \boldsymbol{V}=\frac{1}{-\Delta}(\nabla\times\boldsymbol{b}).$$

利用 $\nabla\cdot\boldsymbol{u}=\nabla\cdot\boldsymbol{b}=0$, 有

$$\nabla\times\boldsymbol{H}=\nabla\times\left(\frac{1}{-\Delta}(\nabla\times\boldsymbol{u})\right)=\nabla\times\left(\nabla\times\left(\frac{1}{-\Delta}\right)\boldsymbol{u}\right)=\boldsymbol{u}+\nabla\left(\nabla\cdot\left(\frac{1}{-\Delta}\right)\boldsymbol{u}\right)=\boldsymbol{u},$$
$$\nabla\times\boldsymbol{V}=\nabla\times\left(\frac{1}{-\Delta}(\nabla\times\boldsymbol{b})\right)=\nabla\times\left(\nabla\times\left(\frac{1}{-\Delta}\right)\boldsymbol{b}\right)=\boldsymbol{b}+\nabla\left(\nabla\cdot\left(\frac{1}{-\Delta}\right)\boldsymbol{b}\right)=\boldsymbol{b}.$$

在引理 4 中, 令 $t=q=3p/(3-p)$,因为 $3/2<p<3$,所以 $q>3$.令 $\boldsymbol{H}=\bar{\boldsymbol{H}}=\tilde{\boldsymbol{H}}$,$\boldsymbol{V}=\bar{\boldsymbol{V}}=\tilde{\boldsymbol{V}}$.将 $\boldsymbol{H}$ 和 $\boldsymbol{V}$ 代入引理 4,对任意 $R>1$, 有

$$\int_{B(R/2)}(|\nabla \boldsymbol{u}|^2+|\nabla \boldsymbol{b}|^2)\mathrm{d}\boldsymbol{x} \leqslant$$
$$\frac{C}{R^4}\int_{\mathcal{C}(R/2,R)}(|\boldsymbol{H}|^2+|\boldsymbol{V}|^2)\mathrm{d}\boldsymbol{x}\times$$
$$\left[1+\left(\frac{1}{R^3}\int_{B(R)}|\boldsymbol{H}|^q\mathrm{d}\boldsymbol{x}\right)^{4/(q-3)}+\left(\frac{1}{R^3}\int_{B(R)}|\boldsymbol{V}|^q\mathrm{d}\boldsymbol{x}\right)^{4/(q-3)}\right]. \tag{8}$$

通过 Riesz 变换 $\mathscr{R}_i=\partial_i/\sqrt{-\Delta}$, 可得

$$\|\boldsymbol{H}\|_{L^q(\mathbb{R}^3)}\leqslant C\left\|\frac{1}{\sqrt{-\Delta}\sqrt{-\Delta}}(\nabla\times\boldsymbol{u})\right\|_{L^q(\mathbb{R}^3)}\leqslant C\left\|\frac{1}{\sqrt{-\Delta}}(\nabla\times\boldsymbol{u})\right\|_{L^p(\mathbb{R}^3)}\leqslant C\|\boldsymbol{u}\|_{L^p(\mathbb{R}^3)},$$
$$\|\boldsymbol{V}\|_{L^q(\mathbb{R}^3)}\leqslant C\left\|\frac{1}{\sqrt{-\Delta}\sqrt{-\Delta}}(\nabla\times\boldsymbol{b})\right\|_{L^q(\mathbb{R}^3)}\leqslant C\left\|\frac{1}{\sqrt{-\Delta}}(\nabla\times\boldsymbol{b})\right\|_{L^p(\mathbb{R}^3)}\leqslant C\|\boldsymbol{b}\|_{L^p(\mathbb{R}^3)}.$$

利用 Hölder 不等式,式(8)可化为

$$\int_{B(R/2)}(|\nabla \boldsymbol{u}|^2+|\nabla \boldsymbol{b}|^2)\mathrm{d}\boldsymbol{x} \leqslant$$
$$CR^{-1-6/q}\left[\left(\int_{\mathcal{C}(R/2,R)}|\boldsymbol{H}|^q\mathrm{d}\boldsymbol{x}\right)^{2/q}+\left(\int_{\mathcal{C}(R/2,R)}|\boldsymbol{V}|^q\mathrm{d}\boldsymbol{x}\right)^{2/q}\right]\times$$
$$\left[1+\left(\frac{1}{R^3}\int_{B(R)}|\boldsymbol{H}|^q\mathrm{d}\boldsymbol{x}\right)^{4/(q-3)}+\left(\frac{1}{R^3}\int_{B(R)}|\boldsymbol{V}|^q\mathrm{d}\boldsymbol{x}\right)^{4/(q-3)}\right].$$

当 $R\to+\infty$ 时,可得

$$\int_{\mathbb{R}^3}(|\nabla\boldsymbol{u}|^2+|\nabla\boldsymbol{b}|^2)\mathrm{d}\boldsymbol{x}\to 0.$$

由 Sobolev 嵌入,有

$$\|\boldsymbol{u}\|_{L^6(\mathbb{R}^3)}\leqslant C\|\nabla\boldsymbol{u}\|_{L^2(\mathbb{R}^3)},\quad \|\boldsymbol{b}\|_{L^6(\mathbb{R}^3)}\leqslant C\|\nabla\boldsymbol{b}\|_{L^2(\mathbb{R}^3)},$$

则 $\boldsymbol{u}=\boldsymbol{b}\equiv\boldsymbol{0}$.最后,对方程(1)第 1 式的两边取散度,利用 $\nabla\cdot\boldsymbol{u}=\nabla\cdot\boldsymbol{b}=0$, 则有

$$\Delta P=-\nabla\cdot\nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u}+\boldsymbol{b}\otimes\boldsymbol{b}).$$

因此 $P=\mathscr{R}_i\mathscr{R}_j(\boldsymbol{u}\otimes\boldsymbol{u}+\boldsymbol{b}\otimes\boldsymbol{b})=0$.

定理 2 的证明 令 $\bar{\boldsymbol{H}}=\tilde{\boldsymbol{H}}=\boldsymbol{H}-\boldsymbol{H}_{B(R)}$, $\bar{\boldsymbol{V}}=\tilde{\boldsymbol{V}}=\boldsymbol{V}-\boldsymbol{V}_{B(R)}$, 则式(3)可化为

$$\begin{aligned}&\int_{B(R/2)}(|\nabla\boldsymbol{u}|^2+|\nabla\boldsymbol{b}|^2)\mathrm{d}\boldsymbol{x}\leqslant\\&\quad\frac{C}{R^4}\int_{\mathcal{C}(R/2,R)}(|\boldsymbol{H}-\boldsymbol{H}_{B(R)}|^2+|\boldsymbol{V}-\boldsymbol{V}_{B(R)}|^2)\mathrm{d}\boldsymbol{x}\times\\&\quad\left[1+\left(\frac{1}{R^3}\int_{B(R)}|\boldsymbol{H}-\boldsymbol{H}_{B(R)}|^t\mathrm{d}\boldsymbol{x}\right)^{4/(t-3)}+\left(\frac{1}{R^3}\int_{B(R)}|\boldsymbol{V}-\boldsymbol{V}_{B(R)}|^t\mathrm{d}\boldsymbol{x}\right)^{4/(t-3)}\right]\leqslant\\&\quad C[(K_{\alpha_1}(t))^2R^{-1-2\alpha_1}+(K_{\alpha_2}(t))^2R^{-1-2\alpha_2}]\times\\&\quad[1+R^{(-4t/(t-3))\alpha_1}(K_{\alpha_1}(t))^{4t/(t-3)}+R^{(-4t/(t-3))\alpha_2}(K_{\alpha_2}(t))^{4t/(t-3)}],\end{aligned}\tag{9}$$

其中

$$\alpha_1,\alpha_2>-\frac{t-3}{6(t-1)},\ t>3.$$

令 $R\to\infty$, 由式(9),有

$$\int_{\mathbb{R}^3}(|\nabla\boldsymbol{u}|^2+|\nabla\boldsymbol{b}|^2)\mathrm{d}\boldsymbol{x}\to 0,$$

再由 Sobolev 嵌入 $\dot{W}^{1,2}(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$, 得

$$\|\boldsymbol{u}\|_{L^6(\mathbb{R}^3)}\leqslant\|\nabla\boldsymbol{u}\|_{L^2(\mathbb{R}^3)}\to 0,\quad \|\boldsymbol{b}\|_{L^6(\mathbb{R}^3)}\leqslant\|\nabla\boldsymbol{b}\|_{L^2(\mathbb{R}^3)}\to 0,$$

因此,在 $\mathbb{R}^3$中有 $\boldsymbol{u}=\boldsymbol{b}\equiv\boldsymbol{0}$.

定理 3 的证明 令 $\bar{\boldsymbol{H}}=\tilde{\boldsymbol{H}}=\boldsymbol{H},\bar{\boldsymbol{V}}=\tilde{\boldsymbol{V}}=\boldsymbol{V}$.这里 $\boldsymbol{H}$ 和 $\boldsymbol{V}$ 是以下边值问题的唯一解:

$$\begin{cases}\nabla\times\boldsymbol{H}=\boldsymbol{u},\ \nabla\cdot\boldsymbol{H}=0,\\\nabla\times\boldsymbol{V}=\boldsymbol{b},\ \nabla\cdot\boldsymbol{V}=0,\end{cases}$$

且在 $\partial B(R)$ 上 $\boldsymbol{H}\cdot\boldsymbol{\nu}_1=\boldsymbol{0},\boldsymbol{V}\cdot\boldsymbol{\nu}_2=\boldsymbol{0}$,其中 $\boldsymbol{\nu}_1$ 和 $\boldsymbol{\nu}_2$ 是垂直于 $\partial B(R)$ 的法向量.将 $\boldsymbol{H}$ 和 $\boldsymbol{V}$ 代入式(3)可得

$$\begin{aligned}&\int_{B(R/2)}(|\nabla\boldsymbol{u}|^2+|\nabla\boldsymbol{b}|^2)\mathrm{d}\boldsymbol{x}\leqslant\\&\quad\frac{C}{R^4}\int_{\mathcal{C}(R/2,R)}(|\boldsymbol{H}|^2+|\boldsymbol{V}|^2)\mathrm{d}\boldsymbol{x}\times\\&\quad\left[1+\left(\frac{1}{R^3}\int_{B(R)}|\boldsymbol{H}|^t\mathrm{d}\boldsymbol{x}\right)^{4/(t-3)}+\left(\frac{1}{R^3}\int_{B(R)}|\boldsymbol{V}|^t\mathrm{d}\boldsymbol{x}\right)^{4/(t-3)}\right].\end{aligned}\tag{10}$$

由嵌入关系

$$\begin{cases}\left(\int_{B(R)}|\boldsymbol{H}|^{3q/(3-q)}\mathrm{d}\boldsymbol{x}\right)^{(3-q)/(3q)}\leqslant C\left(\int_{B(R)}|\nabla\times\boldsymbol{H}|^q\mathrm{d}\boldsymbol{x}\right)^{1/q},\\\left(\int_{B(R)}|\boldsymbol{V}|^{3q/(3-q)}\mathrm{d}\boldsymbol{x}\right)^{(3-q)/(3q)}\leqslant C\left(\int_{B(R)}|\nabla\times\boldsymbol{V}|^q\mathrm{d}\boldsymbol{x}\right)^{1/q}\end{cases}\tag{11}$$

以及 Hölder 不等式,有

$$\begin{aligned}&\int_{B(R)}|\boldsymbol{H}|^2\mathrm{d}\boldsymbol{x}\leqslant\left(\int_{B(R)}|\boldsymbol{H}|^{3q/(3-q)}\mathrm{d}\boldsymbol{x}\right)^{2(3-q)/(3q)}\left(\int_{B(R)}1\mathrm{d}\boldsymbol{x}\right)^{(5q-6)/(3q)}\leqslant\\&\quad C\left(\int_{B(R)}|\nabla\times\boldsymbol{H}|^q\mathrm{d}\boldsymbol{x}\right)^{2/q}R^{5-6/q},\end{aligned}\tag{12}$$

$$\int_{B(R)}|\boldsymbol{V}|^2\mathrm{d}\boldsymbol{x}\leqslant\left(\int_{B(R)}|\boldsymbol{V}|^{3q/(3-q)}\mathrm{d}\boldsymbol{x}\right)^{2(3-q)/(3q)}\left(\int_{B(R)}1\mathrm{d}\boldsymbol{x}\right)^{(5q-6)/(3q)}\leqslant$$
$$C\left(\int_{B(R)}|\nabla\times\boldsymbol{V}|^q\mathrm{d}\boldsymbol{x}\right)^{2/q}R^{5-6/q}.\tag{13}$$

令 $t=3q/(3-q)$，将式(11)—(13)代入式(10)可得

$$\int_{B(R/2)}(|\nabla\boldsymbol{u}|^2+|\nabla\boldsymbol{b}|^2)\mathrm{d}\boldsymbol{x}\leqslant$$
$$\frac{C}{R^4}\left[\int_{\mathcal{C}(R/2,R)}|\boldsymbol{H}|^2\mathrm{d}\boldsymbol{x}+\int_{\mathcal{C}(R/2,R)}|\boldsymbol{V}|^2\mathrm{d}\boldsymbol{x}\right]\times$$
$$\left[1+R^{(-12+4q)/(2q-3)}\left(\int_{B(R)}|\boldsymbol{H}|^{3q/(3-q)}\mathrm{d}\boldsymbol{x}\right)^{(12-4q)/(6q-9)}+\right.$$
$$\left.R^{(-12+4q)/(2q-3)}\left(\int_{B(R)}|\boldsymbol{V}|^{3q/(3-q)}\mathrm{d}\boldsymbol{x}\right)^{(12-4q)/(6q-9)}\right]\leqslant$$
$$CR\left[\left(\frac{1}{|B(R)|}\int_{\mathcal{C}(R/2,R)}|\boldsymbol{u}|^q\mathrm{d}\boldsymbol{x}\right)^{2/q}+\left(\frac{1}{|B(R)|}\int_{\mathcal{C}(R/2,R)}|\boldsymbol{b}|^q\mathrm{d}\boldsymbol{x}\right)^{2/q}\right]\times$$
$$\left[1+R^{4q/(2q-3)}\left(\frac{1}{|B(R)|}\int_{B(R)}|\boldsymbol{u}|^q\mathrm{d}\boldsymbol{x}\right)^{4/(2q-3)}+R^{4q/(2q-3)}\left(\frac{1}{|B(R)|}\int_{B(R)}|\boldsymbol{b}|^q\mathrm{d}\boldsymbol{x}\right)^{4/(2q-3)}\right]\leqslant$$
$$CR\left[\left(R^{\beta_1}\left(\frac{1}{|B(R)|}\int_{\mathcal{C}(R/2,R)}|\boldsymbol{u}|^q\mathrm{d}\boldsymbol{x}\right)^{1/q}\right)^2R^{-2\beta_1}+\left(R^{\beta_2}\left(\frac{1}{|B(R)|}\int_{\mathcal{C}(R/2,R)}|\boldsymbol{b}|^q\mathrm{d}\boldsymbol{x}\right)^{1/q}\right)^2R^{-2\beta_2}\right]\times$$
$$\left[1+R^{4q/(2q-3)}\left(R^{\beta_1}\left(\frac{1}{|B(R)|}\int_{B(R)}|\boldsymbol{u}|^q\mathrm{d}\boldsymbol{x}\right)^{1/q}\right)^{4q/(2q-3)}R^{4q\beta_1/(2q-3)}+\right.$$
$$\left.R^{4q/(2q-3)}\left(R^{\beta_2}\left(\frac{1}{|B(R)|}\int_{B(R)}|\boldsymbol{b}|^q\mathrm{d}\boldsymbol{x}\right)^{1/q}\right)^{4q/(2q-3)}R^{-4q\beta_2/(2q-3)}\right]\leqslant$$
$$C[(M_{\beta_1}(q))^2R^{1-2\beta_1}+(M_{\beta_2}(q))^2R^{1-2\beta_2}]\times$$
$$[1+R^{(4q/(2q-3))(1-\beta_1)}(M_{\beta_1}(q))^{4q/(2q-3)}+R^{(4q/(2q-3))(1-\beta_2)}(M_{\beta_2}(q))^{4q/(2q-3)}],\tag{14}$$

其中

$$\beta_1,\beta_2>\frac{6q-3}{8q-6},\ \frac{3}{2}<q<3.$$

令 $R\to\infty$，则由式(14)可得

$$\int_{\mathbb{R}^3}(|\nabla\boldsymbol{u}|^2+|\nabla\boldsymbol{b}|^2)\to 0.$$

再由 Sobolev 嵌入 $\dot{W}^{1,2}(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$，有 $\boldsymbol{u}=\boldsymbol{b}\equiv\boldsymbol{0}$.

3 结　　论

磁流体动力学方程的研究对象是与磁场相互作用的流体运动.为了使三维稳态磁流体动力学方程只有平凡的零解，通过 Caccioppoli 型不等式，结合 Sobolev 嵌入定理，得到使其成立的充分条件，即为 Liouville 定理.本文证明了三维稳态磁流体动力学方程满足 Liouville 定理的 3 个充分条件，其中之一是：若其光滑解 $(\boldsymbol{u},\boldsymbol{b})\in L^p(\mathbb{R}^3)$，$p\in(3/2,3)$，有 $\boldsymbol{u}=\boldsymbol{b}\equiv\boldsymbol{0}$.这个结论把三维稳态磁流体动力学方程的解在 L^p 空间中的可积指标 p 的下限从 2 扩展到3/2，推广了已有的结果.

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