ⓒ 应用数学和力学编委会,ISSN 1000-0887　　http://www.applmathmech.cn

混合边界约束下矩形薄板自由振动问题的有限积分变换解*

李逸豪，　徐　典，　陈一鸣，　安东琦，　李　锐

(大连理工大学 工程力学系 工业装备结构分析优化与 CAE 软件全国重点实验室, 辽宁 大连 116024)

(我刊编委李锐来稿)

摘要： 解析解可以作为经验公式以及数值方法对比的基准、快速参数分析和优化的工具以及实验设计的理论依据,具有独特的研究价值,而传统解析方法(如 Lévy 解法)只能求解对边简支板壳的力学问题,对于复杂边界约束下的板壳力学问题难以获得解析解.笔者等近年来发展了板壳力学问题的有限积分变换法,实现了非 Lévy 型板壳力学问题的求解,但仍无法直接求解由混合边界约束引起的板壳高阶偏微分方程复杂边值问题.该文首次结合有限积分变换与子域分解方法,实现了混合边界约束下矩形薄板自由振动问题的解析求解.首先根据混合边界约束将矩形板拆分为两部分,然后通过有限积分变换法对两部分分别进行求解,最后引入连续性条件,获得了原问题的解析解.以工程中常见的边缘点焊悬臂板为背景,具体分析了一边固支-简支混合约束、其余三边自由的矩形薄板自由振动问题,获得的固有频率和振型结果均与有限元数值解及文献结果高度吻合,验证了该文推导和结果的准确性.有限积分变换法的求解从基本控制方程出发,无需预先假设解的形式,因此是一种严格的分析方法,可以广泛求解以板壳力学问题为代表的高阶偏微分方程复杂边值问题.

关　键　词： 有限积分变换法；　混合边界约束；　矩形薄板；　自由振动；　解析解

中图分类号： O302　　**文献标志码：** A　　DOI：10.21656/1000-0887.440051

Finite Integral Transform Solutions for Free Vibrations of Rectangular Thin Plates With Mixed Boundary Constraints

LI Yihao,　XU Dian,　CHEN Yiming,　AN Dongqi,　LI Rui

(*State Key Laboratory of Structural Analysis, Optimization and CAE Software for Industrial Equipment, Department of Engineering Mechanics, Dalian University of Technology, Dalian, Liaoning* 116024, *P.R.China*)

(Contributed by LI Rui, M. AMM Editorial Board)

Abstract: Analytical solutions, with unique research value, can serve as benchmarks for empirical formulas

* **收稿日期：** 2023-02-28；**修订日期：** 2023-04-03

基金项目： 国家自然科学基金项目(12022209;11972103)

作者简介： 李逸豪(1997—),男,硕士生(E-mail：lpy131419@mail.dlut.edu.cn)；
徐典(1999—),女,博士生(E-mail：dianxu@mail.dlut.edu.cn)；
陈一鸣(1999—),男,硕士生(E-mail：yimingchen@mail.dlut.edu.cn)；
安东琦(1996—),男,博士生(E-mail：adq96@mail.dlut.edu.cn)；
李锐(1985—),男,教授,博士,博士生导师(通讯作者. E-mail：ruili@dlut.edu.cn).

引用格式： 李逸豪, 徐典, 陈一鸣, 安东琦, 李锐. 混合边界约束下矩形薄板自由振动问题的有限积分变换解[J]. 应用数学和力学, 2023, **44**(9): 1112-1121.

and numerical methods, a tool for rapid parameter analysis and optimization, and a theoretical basis for experimental designs. Conventional analytical methods, e.g., the Lévy solution method, are only applicable to mechanical problems of plates and shells with opposite simply-supported edges, which, however, may fail to obtain analytical solutions for the issues with complex boundary constraints. In recent years, the finite integral transform method for plate and shell problems was developed to deal with non-Lévy-type plates and shells, but it is still infeasible to solve the mixed boundary constrains-induced complex boundary value problems of higher-order partial differential equations. Herein, for the first time, the finite integral transform method was combined with the sub-domain decomposition technique to solve the free vibrations of rectangular thin plates with mixed boundary constraints. The rectangular plate was first divided into 2 sub-domains according to the mixed boundary constraints, and the 2 sub-domains were solved analytically with the finite integral transform method. Finally, the continuity conditions were introduced to obtain the analytical solution of the original problem. Based on the side spot-welded cantilever plates commonly used in engineering, the free vibration problem of a rectangular thin plate with 1 edge subjected to clamped-simply supported constraints and the other 3 edges free, was analyzed. The obtained natural frequencies and mode shapes are in good agreement with those from the finite element method as well as the solutions in literature, thus verifying the accuracy of the proposed method. The solution procedure of the finite integral transform method can be implemented based on the governing equations without any assumption of the solution form. Therefore, this strict analytical method is widely applicable to complex boundary value problems of higher-order partial differential equations for such mechanical problems of plates and shells.

Key words: finite integral transform method; mixed boundary constraint; rectangular thin plate; free vibration; analytical solution

0　引　　言

弹性矩形薄板广泛应用于土木工程、海洋工程及机械工程等领域,其力学行为一直是学者们的研究重点之一[1-4].其中,弹性薄板的自由振动问题因其与结构安全高度相关而备受关注,此类问题的解析求解对于结构的快速分析和初步设计具有重要意义.针对板的自由振动问题,其核心是在满足给定的边界条件下,通过求解该问题的高阶偏微分控制方程来获得板的固有频率和振型.然而,由于高阶偏微分控制方程求解过程的复杂性,相关边值问题的解析求解成为一类难题.传统解析方法,例如 Navier 解法[5]和 Lévy 解法[6]只能求解对边简支约束下的板问题,其余边界条件下的矩形薄板自由振动问题都不易获得解析解.

实际工程应用对于板的边界约束提出了更高的要求,一边受到混合约束(即同时存在固支、简支或自由中的两种或以上约束)的板进入学者们的视野.例如,在汽车制造领域广泛应用的边缘点焊钢板[7]就是一种典型的混合边界约束板,其点焊位置在分析时可等效为固支边界条件,而同一边上其余位置仍然可视为简支等边界条件.然而,对于这类复杂约束下板的振动问题,传统方法更难以解析求解.针对上述情况,学者们通常采用数值方法来解决相关问题,例如有限元法[8-9]、有限条法[10]、有限差分法[11-12]、边界元法[13-14]、微分求积法[15]、微分容积法[16]等.

上述各类数值方法可以得到满足工程需要的结果,但是解析解仍然具有重要的地位.首先,解析解具有独特的研究价值,例如:其可以作为经验公式以及数值方法对比的基准、快速参数分析和优化的工具,同时也可以为实验提供理论依据等.此外,数值方法所求结果通常为近似解,未必总能达到高精度要求,且求解输入量和输出量较多,过程比较复杂.因此,发展解析方法不仅对理论方法的发展和完善具有重要价值,也对工程结构分析与设计具有重要的指导意义.基于上述背景,本文拟寻求一种新的解析方法来处理混合边界约束下矩形薄板的自由振动问题.

有限积分变换法作为求解数学物理方程的一类重要方法,近年来被笔者等进一步发展,用于板壳力学一般问题的解析求解[17-23].该方法的思路是:将待求问题的控制方程转换到积分变换域内,得到含有待定系数的位移函数变换式,再由边界条件求解待定系数,最后通过积分逆变换可得到原问题的解析解.因有限积分

变换法简洁有效,更容易被工程师所理解和接受,因此有望作为一种通用的理论分析工具.然而,以往关于有限积分变换法的研究均聚焦一边仅有单一约束的问题,对于一边受到混合约束的问题则未能直接应用该方法.为此,有必要将有限积分变换法进一步发展,推广至混合边界板问题的求解中.

本文将有限积分变换与子域分解法结合,首次实现了混合边界矩形薄板自由振动问题的解析求解.首先根据混合边界条件将矩形板拆分为两个子域,然后应用有限积分变换法对两部分分别解析求解,最后通过满足子域间的连续性条件得到该问题最终的解析解.限于篇幅,本文聚焦工程中典型的边缘点焊悬臂板的自由振动问题,将其归结为一边固支-简支混合约束、其余三边自由的矩形薄板自由振动问题进行求解.数值算例表明,无论是固有频率还是振型,本文的结果均与精细有限元及文献结果高度吻合,求解结果的精度不亚于其他复杂的解析方法.本文方法及相关结果有望作为检验各类数值方法精度的对比基准,求解思路也可推广至其他复杂边界约束下的板壳力学问题.

1 薄板自由振动问题的控制方程

基于 Kirchhoff 薄板理论,矩形薄板自由振动问题的控制方程如下:

$$D\left[\frac{\partial^4 W(x,y,t)}{\partial x^4}+2\frac{\partial^4 W(x,y,t)}{\partial x^2\partial y^2}+\frac{\partial^4 W(x,y,t)}{\partial y^4}\right]+\rho h\frac{\partial^2 W(x,y,t)}{\partial t^2}=0,\tag{1}$$

式中 $W(x,y,t)$ 表示 t 时刻的挠度;$D=Eh^3/[12(1-\mu^2)]$ 表示抗弯刚度,其中 E 为弹性模量,h 为板的厚度,μ 为 Poisson 比;ρ 表示薄板密度.根据振动理论,板在自由振动时的动挠度方程为 $W(x,y,t)=w(x,y)\sin(\omega t)$,其中 $w(x,y)$ 为振型函数,ω 为固有频率.将 W 代入式(1),得到矩形薄板振型微分方程如下:

$$D\left[\frac{\partial^4 w(x,y)}{\partial x^4}+2\frac{\partial^4 w(x,y)}{\partial x^2\partial y^2}+\frac{\partial^4 w(x,y)}{\partial y^4}\right]-\rho h\omega^2 w(x,y)=0.\tag{2}$$

板的内力可由 w 表示为

$$\begin{cases}M_x=-D\left(\dfrac{\partial^2 w}{\partial x^2}+\mu\dfrac{\partial^2 w}{\partial y^2}\right),\\ V_x=-D\dfrac{\partial}{\partial x}\left[\dfrac{\partial^2 w}{\partial x^2}+(2-\mu)\dfrac{\partial^2 w}{\partial y^2}\right],\\ M_y=-D\left(\dfrac{\partial^2 w}{\partial y^2}+\mu\dfrac{\partial^2 w}{\partial x^2}\right),\\ V_y=-D\dfrac{\partial}{\partial y}\left[\dfrac{\partial^2 w}{\partial y^2}+(2-\mu)\dfrac{\partial^2 w}{\partial x^2}\right],\end{cases}\tag{3}$$

其中 M_x 和 M_y 表示 Oy 轴向和 Ox 轴向的弯矩,V_x 和 V_y 表示垂直于 Ox 轴和 Oy 轴的横截面中的等效剪力.

2 混合边界约束下矩形薄板自由振动问题的有限积分变换解

针对混合边界矩形薄板,本文以“C”“S”“F”分别表示固支、简支、自由这三种边界条件,并从左下方边界开始以顺时针方向通过字母对板命名.限于篇幅,本文以工程中典型的边缘点焊悬臂板、即 CS-F-F-F 型混合边界矩形薄板为对象进行求解.如图 1 所示,板的尺寸表示为 a,b,b_1 及 b_2,其中 $b=b_1+b_2$;坐标轴 Ox 与 Oy 分别与板的下边界与左边界重合.

将原板按照边界条件拆分为①、②两个子域,即将原问题拆分为两个子问题,随后对两个子域分别采用有限积分变换法进行求解,在求解过程中每个子域需要满足相应的边界条件及内部连续性条件.各子域的几何模型如图 2 所示.

对于每个子域,在矩形域 $0\leqslant x_i\leqslant a,0\leqslant y_i\leqslant b_i$ 内定义二维有限余弦积分变换如下:

$$\bar{w}_i(m,n)=\int_0^{b_i}\int_0^a w_i(x_i,y_i)\cos[\alpha(m)x_i]\cos[\beta_i(n)y_i]\,\mathrm{d}x_i\mathrm{d}y_i,\tag{4}$$

式中 i 表示子域编号 1、2,w_i 和(x_i,y_i) 分别表示子域的位移函数和局部坐标,$\alpha(m)=m\pi/a$,$\beta_i(n)=n\pi/b_i$,

其中 $m=0,1,2,\cdots,n=0,1,2,\cdots$.

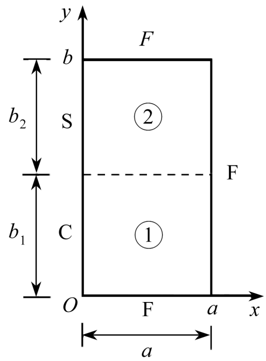


图 1　CS-F-F-F 型混合边界矩形薄板

Fig. 1　The rectangular thin plate under CS-F-F-F mixed boundary constraints

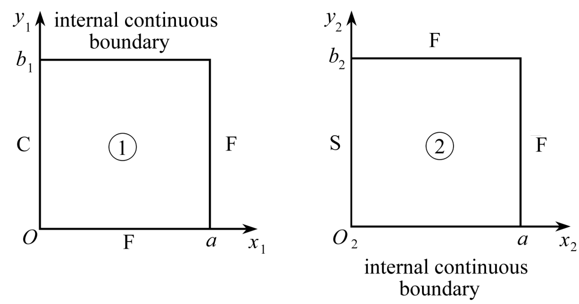


图 2　CS-F-F-F 型混合边界矩形薄板各子域示意图

Fig. 2　Schematic diagram of the sub-domains of CS-F-F-F-F rectangular thin plates

逆变换的表达式如下:

$$w_i(x_i,y_i)=\frac{1}{ab_i}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\varepsilon(m)\varepsilon(n)\bar{w}_i(m,n)\cos[\alpha(m)x_i]\cos[\beta_i(n)y_i]\,. \tag{5}$$

上式中当 $m=0$ 时,$\varepsilon(m)=1$;当 $m=1,2,\cdots$ 时, $\varepsilon(m)=2$;当 $n=0$ 时, $\varepsilon(n)=1$;当 $n=1,2,\cdots$ 时, $\varepsilon(n)=2$.依据式(4)对式(2)所示高阶偏微分方程各相关项进行二维有限余弦积分变换,可得

$$\begin{aligned}&\int_0^{b_i}\int_0^a\frac{\partial^4 w_i}{\partial x_i^4}\cos[\alpha(m)x_i]\cos[\beta_i(n)y_i]\mathrm{d}x_i\mathrm{d}y_i=\\&\quad\alpha^4(m)\bar{w}_i(m,n)+\int_0^{b_i}\cos[\beta_i(n)y_i]\left[-\left.\frac{\partial^3 w_i}{\partial x_i^3}\right|_{x_i=0}+(-1)^m\left.\frac{\partial^3 w_i}{\partial x_i^3}\right|_{x_i=a}\right]\mathrm{d}y_i+\\&\quad\alpha^2(m)\int_0^{b_i}\cos[\beta_i(n)y_i]\left[\left.\frac{\partial w_i}{\partial x_i}\right|_{x_i=0}-(-1)^m\left.\frac{\partial w_i}{\partial x_i}\right|_{x_i=a}\right]\mathrm{d}y_i,\end{aligned} \tag{6}$$

$$\begin{aligned}&\int_0^{b_i}\int_0^a\frac{\partial^4 w_i}{\partial y_i^4}\cos[\alpha(m)x_i]\cos[\beta_i(n)y_i]\mathrm{d}x_i\mathrm{d}y_i=\\&\quad\beta_i^4(n)\bar{w}_i(m,n)+\int_0^{a}\cos[\alpha(m)x_i]\left[-\left.\frac{\partial^3 w_i}{\partial y_i^3}\right|_{y_i=0}+(-1)^n\left.\frac{\partial^3 w_i}{\partial y_i^3}\right|_{y_i=b_i}\right]\mathrm{d}x_i+\\&\quad\beta_i^2(n)\int_0^{a}\cos[\alpha(m)x_i]\left[\left.\frac{\partial w_i}{\partial y_i}\right|_{y_i=0}-(-1)^n\left.\frac{\partial w_i}{\partial y_i}\right|_{y_i=b_i}\right]\mathrm{d}x_i,\end{aligned} \tag{7}$$

$$\begin{aligned}&\int_0^{b_i}\int_0^a\frac{\partial^4 w_i}{\partial x_i^2\partial y_i^2}\cos[\alpha(m)x_i]\cos[\beta_i(n)y_i]\mathrm{d}x_i\mathrm{d}y_i=\\&\quad\alpha^2(m)\beta_i^2(n)\bar{w}_i(m,n)+\underline{\frac{\partial^2 w_i}{\partial x_i\partial y_i}\cos[\alpha(m)x_i]\cos[\beta_i(n)y_i]\Big|_{x_i=0}^{x_i=a}\Big|_{y_i=0}^{y_i=b_i}}+\\&\quad\alpha^2(m)\int_0^{a}\cos[\alpha(m)x_i]\left[\left.\frac{\partial w_i}{\partial y_i}\right|_{y_i=0}-(-1)^n\left.\frac{\partial w_i}{\partial y_i}\right|_{y_i=b_i}\right]\mathrm{d}x_i+\\&\quad\beta_i^2(n)\int_0^{b_i}\cos[\beta_i(n)y_i]\left[\left.\frac{\partial w_i}{\partial x_i}\right|_{x_i=0}-(-1)^m\left.\frac{\partial w_i}{\partial x_i}\right|_{x_i=a}\right]\mathrm{d}y_i,\end{aligned} \tag{8}$$

$$\int_0^{b_i}\int_0^a\rho h\omega^2 w_i\cos[\alpha(m)x_i]\cos[\beta_i(n)y_i]\mathrm{d}x_i\mathrm{d}y_i=\rho h\omega^2\bar{w}_i(m,n)\,. \tag{9}$$

由余弦函数的性质可以消去式(8)中的下划线部分,随后将式(6)—(9)代入式(2)的变换式,可得

$$
\begin{aligned}
&\left\{[\alpha^2(m)+\beta_i^2(n)]^2-\frac{\rho h\omega^2}{D}\right\}\bar{w}_i(m,n)= \\
&\quad \int_0^a \cos[\alpha(m)x_i]\left[\left.\frac{\partial^3 w_i}{\partial y_i^3}\right|_{y_i=0}-(-1)^n\left.\frac{\partial^3 w_i}{\partial y_i^3}\right|_{y_i=b_i}\right]\mathrm{d}x_i+ \\
&\quad \int_0^{b_i} \cos[\beta_i(n)y_i]\left[\left.\frac{\partial^3 w_i}{\partial x_i^3}\right|_{x_i=0}-(-1)^m\left.\frac{\partial^3 w_i}{\partial x_i^3}\right|_{x_i=a}\right]\mathrm{d}y_i+ \\
&\quad [2\alpha^2(m)+\beta_i^2(n)]\int_0^a \cos[\alpha(m)x_i]\left[(-1)^n\left.\frac{\partial w_i}{\partial y_i}\right|_{y_i=b_i}-\left.\frac{\partial w_i}{\partial y_i}\right|_{y_i=0}\right]\mathrm{d}x_i+ \\
&\quad [\alpha^2(m)+2\beta_i^2(n)]\int_0^{b_i} \cos[\beta_i(n)y_i]\left[(-1)^m\left.\frac{\partial w_i}{\partial x_i}\right|_{x_i=a}-\left.\frac{\partial w_i}{\partial x_i}\right|_{x_i=0}\right]\mathrm{d}y_i.
\end{aligned}
\tag{10}
$$

为方便书写,记

$$
\begin{cases}
I_i^b(m)=\displaystyle\int_0^a \left.\frac{\partial^3 w_i}{\partial y_i^3}\right|_{y_i=b_i}\cos[\alpha(m)x_i]\mathrm{d}x_i,\ I_i^0(m)=\int_0^a \left.\frac{\partial^3 w_i}{\partial y_i^3}\right|_{y_i=0}\cos[\alpha(m)x_i]\mathrm{d}x_i, \\
J_i^b(m)=\displaystyle\int_0^a \left.\frac{\partial w_i}{\partial y_i}\right|_{y_i=b_i}\cos[\alpha(m)x_i]\mathrm{d}x_i,\ J_i^0(m)=\int_0^a \left.\frac{\partial w_i}{\partial y_i}\right|_{y_i=0}\cos[\alpha(m)x_i]\mathrm{d}x_i, \\
K_i^a(n)=\displaystyle\int_0^{b_i} \left.\frac{\partial^3 w_i}{\partial x_i^3}\right|_{x_i=a}\cos[\beta_i(n)y_i]\mathrm{d}y_i,\ K_i^0(n)=\int_0^{b_i} \left.\frac{\partial^3 w_i}{\partial x_i^3}\right|_{x_i=0}\cos[\beta_i(n)y_i]\mathrm{d}y_i, \\
L_i^a(n)=\displaystyle\int_0^{b_i} \left.\frac{\partial w_i}{\partial x_i}\right|_{x_i=a}\cos[\beta_i(n)y_i]\mathrm{d}y_i,\ L_i^0(n)=\int_0^{b_i} \left.\frac{\partial w_i}{\partial x_i}\right|_{x_i=0}\cos[\beta_i(n)y_i]\mathrm{d}y_i, \\
C_i(m,n)=\{[\alpha^2(m)+\beta_i^2(n)]^2-\gamma^2\}^{-1},
\end{cases}
\tag{11}
$$

其中 $\gamma=\omega\sqrt{\rho h/D}$.则式(10)可以进一步简化为

$$
\begin{aligned}
\bar{w}_i(m,n)=&C_i(m,n)\{[I_i^0(m)-(-1)^n I_i^b(m)]+[K_i^0(n)-(-1)^m K_i^a(n)]+ \\
&[2\alpha^2(m)+\beta_i^2(n)][(-1)^n J_i^b(m)-J_i^0(m)]+ \\
&[\alpha^2(m)+2\beta_i^2(n)][(-1)^m L_i^a(n)-L_i^0(n)]\}.
\end{aligned}
\tag{12}
$$

根据式(3)—(5),并结合 Stokes 变换[24],可得边界处的弯矩表达式如下:

$$
\begin{aligned}
M_{x_i}\big|_{x_i=0}=&-D\left(\left.\frac{\partial^2 w_i}{\partial x_i^2}\right|_{x_i=0}+\mu\left.\frac{\partial^2 w_i}{\partial y_i^2}\right|_{x_i=0}\right)= \\
&-\frac{D}{ab_i}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\varepsilon(m)\varepsilon(n)\{(-1)^m L_i^a(n)-L_i^0(n)+ \\
&\mu[(-1)^n J_i^b(m)-J_i^0(m)]-[\alpha^2(m)+\mu\beta_i^2(n)]\bar{w}_i(m,n)\}\cos[\beta_i(n)y_i],
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
M_{x_i}\big|_{x_i=a}=&-D\left(\left.\frac{\partial^2 w_i}{\partial x_i^2}\right|_{x_i=a}+\mu\left.\frac{\partial^2 w_i}{\partial y_i^2}\right|_{x_i=a}\right)= \\
&-\frac{D}{ab_i}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\varepsilon(m)\varepsilon(n)(-1)^m\{(-1)^m L_i^a(n)-L_i^0(n)+ \\
&\mu[(-1)^n J_i^b(m)-J_i^0(m)]-[\alpha^2(m)+\mu\beta_i^2(n)]\bar{w}_i(m,n)\}\cos[\beta_i(n)y_i],
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
M_{y_i}\big|_{y_i=0}=&-D\left(\left.\frac{\partial^2 w_i}{\partial y_i^2}\right|_{y_i=0}+\mu\left.\frac{\partial^2 w_i}{\partial x_i^2}\right|_{y_i=0}\right)= \\
&-\frac{D}{ab_i}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\varepsilon(m)\varepsilon(n)\{\mu[(-1)^m L_i^a(n)-L_i^0(n)]+ \\
&(-1)^n J_i^b(m)-J_i^0(m)-[\mu\alpha^2(m)+\beta_i^2(n)]\bar{w}_i(m,n)\}\cos[\alpha(m)x_i],
\end{aligned}
\tag{15}
$$

$$M_{y_i}\big|_{y_i=b_i}=-D\left(\frac{\partial^2 w_i}{\partial y_i^2}\bigg|_{y_i=b_i}+\mu\frac{\partial^2 w_i}{\partial x_i^2}\bigg|_{y_i=b_i}\right)=$$
$$-\frac{D}{ab_i}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\varepsilon(m)\varepsilon(n)(-1)^n\{\mu[(-1)^m L_i^a(n)-L_i^0(n)]+$$
$$(-1)^n J_i^b(m)-J_i^0(m)-[\mu\alpha^2(m)+\beta_i^2(n)]\bar{w}_i(m,n)\}\cos[\alpha(m)x_i].\tag{16}$$

同理,可得边界处等效剪力的表达式如下:

$$V_{x_i}\big|_{x_i=a}=-D\left[\frac{\partial^3 w_i}{\partial x_i^3}\bigg|_{x_i=a}+(2-\mu)\frac{\partial^3 w_i}{\partial x_i\partial y_i^2}\bigg|_{x_i=a}\right]=$$
$$-\frac{D}{b_i}\sum_{n=0}^{\infty}\varepsilon_n[K_i^a(n)-(2-\mu)\beta_i^2(n)L_i^a(n)]\cos[\beta_i(n)y_i],\tag{17}$$

$$V_{y_i}\big|_{y_i=0}=-D\left[\frac{\partial^3 w_i}{\partial y_i^3}\bigg|_{y_i=0}+(2-\mu)\frac{\partial^3 w_i}{\partial y_i\partial x_i^2}\bigg|_{y_i=0}\right]=$$
$$-\frac{D}{a}\sum_{m=0}^{\infty}\varepsilon_m[I_i^0(m)-(2-\mu)\alpha^2(m)J_i^0(m)]\cos[\alpha(m)x_i],\tag{18}$$

$$V_{y_i}\big|_{y_i=b_i}=-D\left[\frac{\partial^3 w_i}{\partial y_i^3}\bigg|_{y_i=b_i}+(2-\mu)\frac{\partial^3 w_i}{\partial y_i\partial x_i^2}\bigg|_{y_i=b_i}\right]=$$
$$-\frac{D}{a}\sum_{m=0}^{\infty}\varepsilon_m[I_i^b(m)-(2-\mu)\alpha^2(m)J_i^b(m)]\cos[\alpha(m)x_i].\tag{19}$$

根据图 2 可知:子域①需要满足的边界条件为

$$\begin{cases}w_1\big|_{x_1=0}=0,\ \dfrac{\partial w_1}{\partial x_1}\bigg|_{x_1=0}=0,\\ M_{x_1}\big|_{x_1=a}=0,\ V_{x_1}\big|_{x_1=a}=0,\\ M_{y_1}\big|_{y_1=0}=0,\ V_{y_1}\big|_{y_1=0}=0;\end{cases}\tag{20}$$

需要满足的连续性条件为

$$\begin{cases}w_1\big|_{y_1=b_1}=w_2\big|_{y_2=0},\ \dfrac{\partial^2 w_1}{\partial y_1^2}\bigg|_{y_1=b_1}=\dfrac{\partial^2 w_2}{\partial y_2^2}\bigg|_{y_2=0},\\ \dfrac{\partial w_1}{\partial y_1}\bigg|_{y_1=b_1}=\dfrac{\partial w_2}{\partial y_2}\bigg|_{y_2=0},\ \dfrac{\partial^3 w_1}{\partial y_1^3}\bigg|_{y_1=b_1}=\dfrac{\partial^3 w_2}{\partial y_2^3}\bigg|_{y_2=0}.\end{cases}\tag{21}$$

将式(20)和(21)中的 $(\partial w_1/\partial x_1)|_{x_1=0}=0$, $(\partial w_1/\partial y_1)|_{y_1=b_1}=(\partial w_2/\partial y_2)|_{y_2=0}$, $(\partial^3 w_1/\partial y_1^3)|_{y_1=b_1}=(\partial^3 w_2/\partial y_2^3)|_{y_2=0}$ 代入式(11)中可得

$$\begin{cases}L_1^0(n)=0,\\ I_1^b(m)=I_2^0(m),\\ J_1^b(m)=J_2^0(m),\end{cases}\tag{22}$$

则子域①的位移函数积分变换式为

$$\bar{w}_1(m,n)=C_1(m,n)\{[I_1^0(m)-(-1)^n I_1^b(m)]+[K_1^0(n)-(-1)^m K_1^a(n)]+$$
$$[2\alpha^2(m)+\beta_1^2(n)][(-1)^n J_1^b(m)-J_1^0(m)]+$$
$$[\alpha^2(m)+2\beta_1^2(n)](-1)^m L_1^a(n)\}.\tag{23}$$

根据图 2 可知子域②需要满足的边界条件为

$$\begin{cases}w_2\big|_{x_2=0}=0,\ M_{x_2}\big|_{x_2=0}=0,\\ M_{x_2}\big|_{x_2=a}=0,\ V_{x_2}\big|_{x_2=a}=0,\\ M_{y_2}\big|_{y_2=b_2}=0,\ V_{y_2}\big|_{y_2=b_2}=0;\end{cases}\tag{24}$$

需要满足的连续性条件为

$$\begin{cases} w_1\big|_{y_1=b_1} = w_2\big|_{y_2=0},\ \dfrac{\partial^2 w_1}{\partial y_1^2}\bigg|_{y_1=b_1} = \dfrac{\partial^2 w_2}{\partial y_2^2}\bigg|_{y_2=0}, \\ \dfrac{\partial w_1}{\partial y_1}\bigg|_{y_1=b_1} = \dfrac{\partial w_2}{\partial y_2}\bigg|_{y_2=0},\ \dfrac{\partial^3 w_1}{\partial y_1^3}\bigg|_{y_1=b_1} = \dfrac{\partial^3 w_2}{\partial y_2^3}\bigg|_{y_2=0}. \end{cases} \tag{25}$$

将式(25)中的 $(\partial w_1/\partial y_1)\big|_{y_1=b_1} = (\partial w_2/\partial y_2)\big|_{y_2=0}$,$(\partial^3 w_1/\partial y_1^3)\big|_{y_1=b_1} = (\partial^3 w_2/\partial y_2^3)\big|_{y_2=0}$ 代入式(11)中可得

$$\begin{cases} I_1^b(m) = I_2^0(m), \\ J_1^b(m) = J_2^0(m). \end{cases} \tag{26}$$

则子域②的位移函数积分变换式为

$$\begin{aligned} \bar{w}_2(m,n) = C_2(m,n)\{&[I_2^0(m) - (-1)^n I_2^b(m)] + [K_2^0(n) - (-1)^m K_2^a(n)] + \\ &[2\alpha^2(m) + \beta_2^2(n)][(-1)^n J_2^b(m) - J_2^0(m)] + \\ &[\alpha^2(m) + 2\beta_2^2(n)][(-1)^m L_2^a(n) - L_2^0(n)]\}. \end{aligned} \tag{27}$$

综上,还未满足的边界条件和连续性条件有

$$\begin{cases} w_1\big|_{x_1=0} = 0,\ M_{x_1}\big|_{x_1=a} = 0,\ V_{x_1}\big|_{x_1=a} = 0,\ M_{y_1}\big|_{y_1=0} = 0, \\ V_{y_1}\big|_{y_1=0} = 0,\ w_2\big|_{x_2=0} = 0,\ M_{x_2}\big|_{x_2=0} = 0,\ M_{x_2}\big|_{x_2=a} = 0, \\ V_{x_2}\big|_{x_2=a} = 0,\ M_{y_2}\big|_{y_2=b_2} = 0,\ V_{y_2}\big|_{y_2=b_2} = 0,\ w_1\big|_{y_1=b_1} = w_2\big|_{y_2=0}, \\ \dfrac{\partial^2 w_1}{\partial y_1^2}\bigg|_{y_1=b_1} = \dfrac{\partial^2 w_2}{\partial y_2^2}\bigg|_{y_2=0}. \end{cases} \tag{28}$$

将式(5)和式(13)—(19)分别结合两个子域的位移函数积分变换式(23)、(27)代入式(28)中,可得 13 组齐次线性方程 ($m = 0,1,2,\cdots,M$;$n = 0,1,2,\cdots,N$),当 m,n 分别取上限 M,N 时,可以得到关于 13 组未知数 $I_1^0(m)$,$I_1^b(m)$,$J_1^0(m)$,$J_1^b(m)$,$K_1^0(n)$,$K_1^a(n)$,$L_1^a(n)$,$I_2^b(m)$,$J_2^b(m)$,$K_2^0(n)$,$K_2^a(n)$,$L_2^0(n)$,$L_2^a(n)$ 的 $6(M+1) + 7(N+1)$ 个齐次线性方程.令该线性方程组的系数行列式为 0,即可得到各阶固有频率 ω,之后将频率解代回式(23)和(27)中,再将求得的完整表达式代入式(5),最后通过式(29)获得整个矩形板的振型解:

$$w(x,y) = \begin{cases} w_1(x,y), & 0 \leqslant y < b_1, \\ w_2(x, y - b_1), & b_1 \leqslant y \leqslant b. \end{cases} \tag{29}$$

3 典型算例分析

为了验证本文求解的正确性,表 1 给出了 CS-F-F-F 方板前十阶无量纲固有频率 $\omega b^2\sqrt{\rho h/D}$ 的收敛性分析.可以看到,当 $M(N) = 150$ 时,所有结果可以收敛至四位有效数字,收敛后的结果在表中加粗标注.因此,本文算例分析中均取 $M = N = 150$,以保证所有计算结果的收敛精度.

表 1 CS-F-F-F 方板前十阶无量纲固有频率 $\omega b^2\sqrt{\rho h/D}$ 收敛性研究

Table 1 Convergence study of the first 10 non-dimensional natural frequencies and $\omega b^2\sqrt{\rho h/D}$ of CS-F-F-F square plates

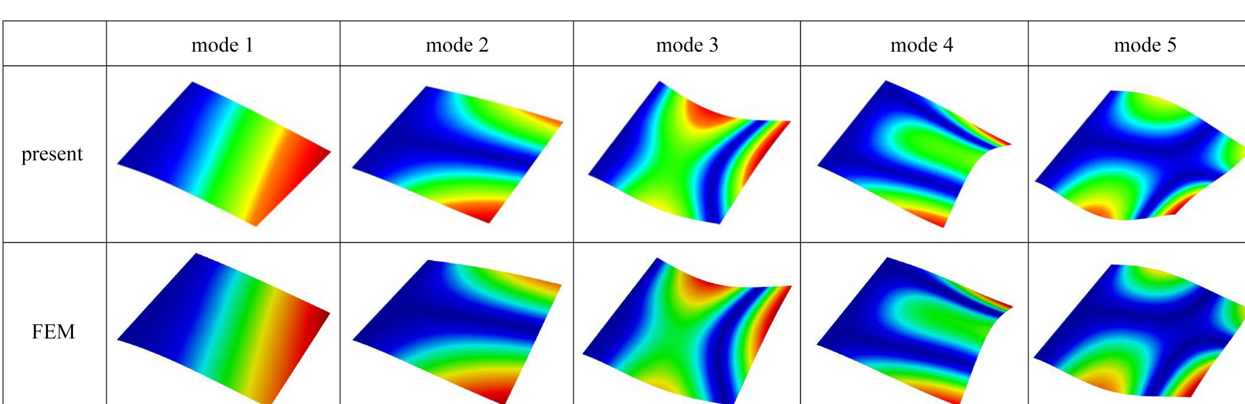
$M(N)$	mode									
	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
10	2.759	7.541	18.21	26.79	28.53	51.14	55.05	63.68	67.75	89.96
20	2.769	7.547	18.23	26.80	28.57	51.15	55.12	63.74	67.77	90.03
30	2.773	7.548	18.24	26.80	28.57	51.15	55.14	63.75	**67.78**	90.05
50	2.775	7.549	**18.25**	**26.81**	28.57	**51.16**	55.16	63.75	67.78	90.07
100	2.777	**7.550**	18.25	26.81	**28.58**	51.16	55.17	**63.76**	67.78	**90.08**
150	**2.778**	7.550	18.25	26.81	28.58	51.16	**55.18**	63.76	67.78	90.08
200	2.778	7.550	18.25	26.81	28.58	51.16	55.18	63.76	67.78	90.08

表 2 给出了不同长宽比的 CS-F-F-F 板在 $b_1 = b_2$ 条件下的无量纲固有频率.通过与精细有限元分析(采用 ABAQUS 软件中的 S4R 单元,网格尺寸为 0.002 5a) 的收敛结果以及文献的结果对比可知,对于不同尺寸的板,本文的固有频率解均与参考结果吻合良好,证明了本文求解方法的有效性和求解结果的准确性.图 3 给出了 CS-F-F-F 方板前十阶振型,结果也与有限元解高度吻合.

表 2　不同长宽比 CS-F-F-F 板在 $b_1 = b_2$ 条件下的无量纲固有频率

Table 2　Non-dimensional natural frequencies of CS-F-F-F plates with different aspect ratios, with $b_1 = b_2$

a/b	method	mode									
		1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
0.5	present	8.906	18.63	38.19	68.19	74.32	93.39	116.9	134.0	162.9	204.4
	FEM	8.915	18.63	38.19	68.20	74.32	93.40	116.9	134.0	162.9	204.4
	XU et al. [4]	8.910	18.62	38.18	68.17	74.33	93.37	116.9	134.0	162.9	204.4
1	present	2.778	7.550	18.25	26.81	28.58	51.16	55.18	63.76	67.78	90.08
	FEM	2.780	7.550	18.26	26.81	28.58	51.16	55.20	63.76	67.79	90.09
	XU et al. [4]	2.778	7.552	18.25	26.81	28.58	51.16	55.18	63.77	67.80	90.10
1.5	present	1.320	4.710	8.531	16.20	23.39	25.29	33.63	37.35	49.03	57.74
	FEM	1.321	4.710	8.534	16.20	23.39	25.29	33.63	37.35	49.04	57.75
	XU et al. [4]	1.320	4.710	8.533	16.20	23.39	25.29	33.63	37.35	49.03	57.74
2	present	0.767 2	3.413	4.923	11.19	14.10	21.71	23.26	28.12	31.15	36.46
	FEM	0.767 9	3.413	4.924	11.19	14.10	21.71	23.26	28.12	31.15	36.47
	XU et al. [4]	0.767 2	3.412	4.923	11.19	14.10	21.71	23.25	28.11	31.15	36.46
2.5	present	0.500 4	2.659	3.224	8.344	9.362	15.89	18.13	22.94	25.61	28.03
	FEM	0.501 3	2.660	3.224	8.345	9.363	15.89	18.13	22.95	25.61	28.03
	XU et al. [4]	0.500 4	2.658	3.223	8.344	9.362	15.88	18.13	22.95	25.61	28.04
3	present	0.351 9	2.106	2.352	6.151	7.193	12.04	13.22	19.45	21.12	22.86
	FEM	0.352 4	2.107	2.352	6.153	7.193	12.04	13.23	19.45	21.13	22.86
	XU et al. [4]	0.351 9	2.106	2.352	6.150	7.194	12.04	13.23	19.45	21.12	22.86
3.5	present	0.260 8	1.613	1.933	4.590	5.984	9.073	10.69	15.01	16.45	21.95
	FEM	0.261 3	1.614	1.934	4.591	5.984	9.075	10.69	15.01	16.45	21.96
	XU et al. [4]	0.260 8	1.613	1.933	4.589	5.983	9.073	10.69	15.00	16.45	21.95
4	present	0.201 0	1.252	1.670	3.541	5.143	7.008	9.058	11.65	13.66	17.42
	FEM	0.201 6	1.253	1.670	3.542	5.143	7.009	9.058	11.66	13.66	17.43
	XU et al. [4]	0.200 9	1.252	1.669	3.540	5.143	7.006	9.060	11.66	13.66	17.43



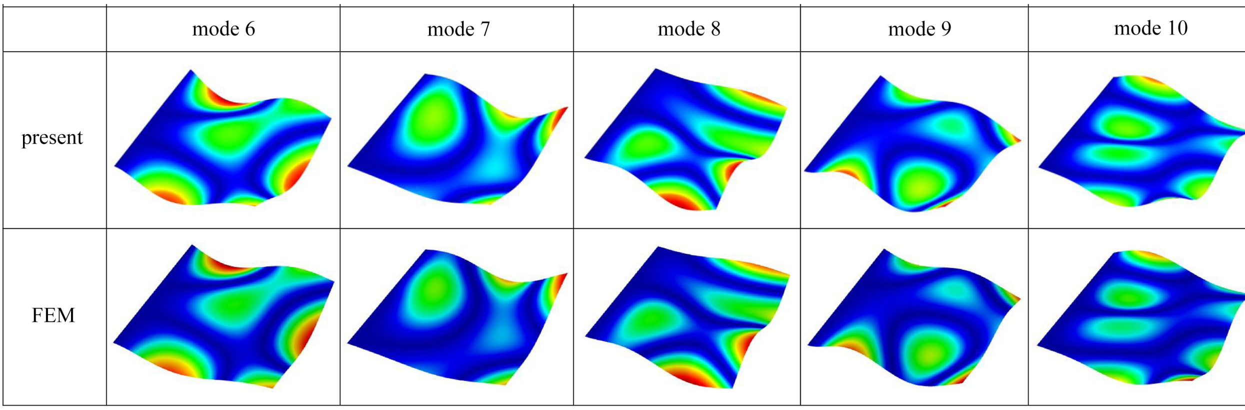


图 3　CS-F-F-F 方板的前十阶振型

Fig. 3　The first 10 mode shapes of CS-F-F-F square plates

4　结　　论

本文首次将有限积分变换法扩展至混合边界约束薄板自由振动问题的解析求解,以工程中常见的边缘点焊悬臂板为背景,具体求解了 CS-F-F-F 型板的自由振动问题.分析过程中采用双余弦形式的积分核对两个子域分别解析求解,最后通过连续性条件获得该问题完整的解析解.本文给出的数值算例表明获得的解析解与精细有限元分析及文献结果吻合良好,证明了求解方法的有效性及所求结果的准确性,同时为检验各类数值方法提供了对比基准.本文发展的有限积分变换结合子域分解的方法在求解过程中无需预先假设解的形式,而是从基本控制方程出发并逐步推导以获得结果,因此是一种严格的求解方法,可为复杂边界约束下板壳力学问题的解析求解提供一种新思路.

参考文献(References):

[1]　田宇. 求解矩形薄板问题的辛-叠加方法[D]. 硕士学位论文. 大连: 大连理工大学, 2018. (TIAN Yu. On the symplectic superposition method for rectangular thin plates[D]. Master Thesis. Dalian: Dalian University of Technology, 2018. (in Chinese))

[2]　李若愚, 王天宏. 薄板热力耦合的屈曲分析[J]. 应用数学和力学, 2020, **41**(8): 877-886. (LI Ruoyu, WANG Tianhong. Thermo-mechanical buckling analysis of thin plates[J]. *Applied Mathematics and Mechanics*, 2020, **41**(8): 877-886. (in Chinese))

[3]　张磊, 张文明, 王林, 等. 基于小波 Galerkin 法的矩形薄板二次屈曲分析[J]. 应用数学和力学, 2023, **44**(1): 25-35. (ZHANG Lei, ZHANG Wenming, WANG Lin, et al. Secondary buckling analysis of thin rectangular plates based on the wavelet Galerkin method[J]. *Applied Mathematics and Mechanics*, 2023, **44**(1): 25-35. (in Chinese))

[4]　XU D, NI Z F, LI Y H, et al. On the symplectic superposition method for free vibration of rectangular thin plates with mixed boundary constraints on an edge[J]. *Theoretical and Applied Mechanics Letters*, 2021, **11**(5): 100293.

[5]　王磊, 蔡松柏. 简支任意四边形板的弯曲、稳定和振动的 Navier 解法[J]. 湖南大学学报, 1989, **16**(1): 104-114. (WANG Lei, CAI Songbo. Navier's method for solving the problem of bending, buckling and vibration of all edges simply supported quadrilateral plates[J]. *Journal of Hunan University*, 1989, **16**(1): 104-114.(in Chinese))

[6]　THAI H T, CHOI D H. Levy solution for free vibration analysis of functionally graded plates based on a refined plate theory[J]. *KSCE Journal of Civil Engineering*, 2014, **18**(6): 1813-1824.

[7]　丁鹏飞. 超高强度汽车钢板电阻点焊结构强度的优化分析[D]. 硕士学位论文. 上海: 华东理工大学, 2016: 1-26. (DING Pengfei. Optimization on structure strength of resistance spot welding of ultra high strength steel

[D]. Master Thesis. Shanghai: East China University of Science and Technology, 2016: 1-26. (in Chinese))

[8] ZIENKIEWICZ O C, CHEUNG Y K. The finite element method for analysis of elastic isotropic and orthotropic slabs[J]. *ICE Proceedings*, 1964, **28**(4):471-488.

[9] BUCZKOWSKI R, TORBACKI W. Finite element modelling of thick plates on two-parameter elastic foundation [J]. *International Journal for Numerical and Analytical Methods in Geomechanics*, 2001, **25**(14): 1409-1427.

[10] CHEUNG B Y K. *Finite Strip Method in Structural Analysis*[M]. London: Pergamon Press, 1976.

[11] HENWOOD D J, WHITEMAN J R, YETTRAM A L. Finite difference solution of a system of first-order partial differential equations[J]. *International Journal for Numerical Methods in Engineering*, 2010, **17**(9): 1385-1395.

[12] CHAKRAVORTY A K, GHOSH A. Finite difference solution for circular plates on elastic foundations[J]. *International Journal for Numerical Methods in Engineering*, 2010, **9**(1): 73-84.

[13] BREBBIA C A. *The Boundary Element Method for Engineers*[M]. New York: Halsted Press, 1978.

[14] PEREIRA WL A, KARAM V J, CARRER J A M, et al. A dynamic formulation for the analysis of thick elastic plates by the boundary element method[J]. *Engineering Analysis With Boundary Elements*, 2012, **36**(7): 1138-1150.

[15] LIEW K M, HAN J B. Bending solution for thick plates with quadrangular boundary[J]. *Journal of Engineering Mechanics*, 1998, **124**(1): 9-17.

[16] LIU F L, LIEW K M. Differential cubature method for static solutions of arbitrarily shaped thick plates[J]. *International Journal of Solids Structures*,1998, **35**(28/29): 3655-3674.

[17] ZHONG Y, SUN A M, ZHOU F L, et al. Analytical solution for rectangular thin plate on elastic foundation with four edges free by finite cosine integral transform method[J]. *Chinese Journal of Geotechnical Engineering*, 2006, **28**(11): 2019-2022.

[18] ZHONG Y, WANG G X, SUN A M. Vibration of a thin plate on Winkler foundation with completely free boundary by finite cosine integral transform method[J]. *Journal of Dalian University of Technology*, 2007, **47**(1): 73-77.

[19] ZHONG Y, YIN J H. Free vibration analysis of a plate on foundation with completely free boundary by finite integral transform method[J]. *Mechanics Research Communications*, 2008, **35**(4): 268-275.

[20] LI R, ZHONG Y,TIAN B, et al. On the finite integral transform method for exact bending solutions of fully clamped orthotropic rectangular thin plates[J]. *Applied Mathematics Letters*, 2009, **22**(12): 1821-1827.

[21] ZHONG Y, ZHAO X F, LI R. Freevibration analysis of rectangular cantilever plates by finite integral transform method[J]. *International Journal for Computational Methods in Engineering Science and Mechanics*, 2013, **14**(3): 221-226.

[22] AN D Q, XU D, NI Z F, et al. Finite integral transform method for analytical solutions of static problems of cylindrical shell panels[J]. *European Journal of Mechanics A: Solids*, 2020, **83**: 104033.

[23] CHEN Y M, AN D Q, ZHOU C, et al. Analytical free vibration solutions of rectangular edge-cracked plates by the finite integral transform method[J]. *International Journal of Mechanical Sciences*, 2023, **243**: 108032.

[24] KHALILI M R, MALEKZADEH K, MITTAL R K. A new approach to static and dynamic analysis of composite plates with different boundary conditions[J]. *Composite Structures*, 2005, **69**(2): 149-155.