http://www.applmathmech.cn

非 Lévy 型正交各向异性开口圆柱壳屈曲问题的辛叠加解析解*

刘明峰，　徐　典，　倪卓凡，　李逸豪，　李　锐

（大连理工大学 工程力学系 工业装备结构分析优化与 CAE 软件全国重点实验室，辽宁 大连 116024）

（我刊编委李锐来稿）

摘要： 该文基于笔者提出的辛叠加方法得到了经典解法难以直接获得的典型非 Lévy 型正交各向异性开口圆柱壳屈曲问题的解析解.首先,基于 Donnell 薄壳理论建立了正交各向异性开口圆柱壳屈曲问题的 Hamilton 体系控制方程,然后将非 Lévy 型边界下的原问题拆分为两个子问题,在 Hamilton 体系下利用分离变量和辛本征展开等数学手段对子问题进行求解,最后基于原问题边界条件,通过子问题解的叠加求得原问题的解析解.数值算例表明,辛叠加解析解与有限元数值解结果吻合良好.同时,定量研究了长度和厚度等参数对屈曲载荷的影响.相比于半逆解法等传统解析方法,辛叠加方法基于严格的数学推导,无需假定解的形式,可以获得更多类似问题的解析解.

关 键 词： 正交各向异性；开口圆柱壳；屈曲；辛叠加方法；解析解

中图分类号： O343.9　　**文献标志码：** A　　DOI：10.21656/1000-0887.440093

Symplectic Superposition-Based Analytical Solutions for Buckling of Non-Lévy-Type Orthotropic Cylindrical Shells

LIU Mingfeng,　XU Dian,　NI Zhuofan,　LI Yihao,　LI Rui

(*State Key Laboratory of Structural Analysis, Optimization and CAE Software for Industrial Equipment, Department of Engineering Mechanics, Dalian University of Technology, Dalian, Liaoning* 116024, *P.R.China*)

(Contributed by LI Rui, M.AMM Editorial Board)

Abstract: Based on the symplectic superposition method (SSM) pioneered by the authors, the buckling problem of typical non-Lévy-type orthotropic cylindrical shells was solved analytically, which is difficult to handle with conventional analytical methods. The Hamiltonian system-based governing equations for buckling of orthotropic cylindrical shells were firstly established based on Donnell's shell theory. The original problem under

* **收稿日期：** 2023-04-03；**修订日期：** 2023-04-25

基金项目： 国家自然科学基金项目(12022209;11972103)

作者简介： 刘明峰(1998—),男,硕士生(E-mail: lmf231@mail.dlut.edu.cn);
徐典(1999—),女,博士生(E-mail: dianxu@mail.dlut.edu.cn);
倪卓凡(1997—),男,博士生(E-mail: nizhuofan@mail.dlut.edu.cn);
李逸豪(1997—),男,硕士生(E-mail: lpy131419@mail.dlut.edu.cn);
李锐(1985—),男,教授,博士,博士生导师(通讯作者. E-mail: ruili@dlut.edu.cn).

引用格式： 刘明峰, 徐典, 倪卓凡, 李逸豪, 李锐. 非 Lévy 型正交各向异性开口圆柱壳屈曲问题的辛叠加解析解[J]. 应用数学和力学, 2023, **44**(12): 1428-1440.

non-Lévy-type boundary conditions was then divided into 2 subproblems, and each subproblem was solved with the mathematical techniques incorporating separation of variables and symplectic eigen expansion within the Hamiltonian framework. The analytical solution of the original problem was finally given through the superposition of the sub-solutions to satisfy the boundary conditions of the original problem. The numerical examples under consideration show that, the SSM-based analytical solutions are in good agreement with the finite element results. In addition, the effects of parameters including the length and the thickness on the critical buckling loads were quantitatively studied. Compared with the conventional analytical methods such as the semi-inverse method, the SSM works based on rigorous mathematical derivation without any assumption of the solution forms, and can obtain reliable analytical solutions to more similar issues.

Key words: orthotropic; cylindrical shell; buckling; symplectic superposition method; analytical solution

0 引　言

圆柱壳以其轻质属性和优异的力学性能,被广泛应用于航空航天[1-2]、船舶工程[3]、土建工程[4]等领域,其结构失稳可能导致严重的安全问题,这使得屈曲分析成为圆柱壳结构设计中的必要环节.轴向载荷作用下的屈曲是圆柱壳最重要的失效模式之一,许多学者采用各类数值方法对其屈曲问题开展了深入研究,例如有限单元法[5]、有限差分法[6]、Galerkin 方法[7-9]、Rayleigh-Ritz 法[10]、微分求积法[11]、无网格法[12]等.

在数值方法兴盛的同时,对解析解的研究仍然具有重要意义:解析解不仅可以作为检验各种数值方法的对比基准,还能够快速揭示不同参量对结构力学行为的影响,从而指导结构的高效设计.然而,对于圆柱壳屈曲问题的解析解研究并不多见,例如:Zou 和 Foster[13]基于 Flügge 的各向同性圆柱壳线性理论,推导了轴压和外压共同作用下圆柱壳屈曲的一般解,并应用于正交各向异性圆柱壳的求解;张俊霖等[14]使用辛方法开展了吸湿老化影响下天然纤维增强圆柱壳的屈曲分析;桂夷斐和马建敏[15]根据 Hamilton 变分原理求解了弹性介质中受轴向冲击载荷作用的圆柱壳的屈曲;Sun 等[16]发展了一种等效硬壳理论,用以预测准各向同性夹层圆柱壳的力学行为;Chen 等[17]利用状态空间法分析了受压功能梯度弹性空心圆柱的分叉屈曲问题.已有研究大多聚焦封闭圆柱壳的屈曲分析,对于工程中亦常见的开口圆柱壳的屈曲问题则少有讨论,而板壳力学中的经典解析解法——Navier 解法[18]和 Lévy 解法[19]又仅适用于四边简支和对边简支情况(即 Lévy 型开口圆柱壳),而对于非 Lévy 型情况则难以处理.

本文基于笔者近年来针对板壳力学问题提出的一种新解析方法:辛叠加方法[20-22],获得了经典解法难以直接求解的典型非 Lévy 型正交各向异性开口圆柱壳屈曲问题的解析解.该方法直接从正交各向异性圆柱壳的基本方程出发,将问题导入到 Hamilton 体系中,然后将非 Lévy 型边界下的原问题拆分为两个子问题,通过分离变量和辛本征展开求出子问题的解,进一步通过叠加求出原问题的解.在基于辛叠加方法的求解过程中,不需要预先选取试函数(如位移函数),而是通过逐步理性推导,严格得到问题的解,因而突破了半逆法的限制,可以求解传统框架下不易处理的壳体问题.通过数值算例验证了本文求解的正确性,并进一步开展了定量的参数分析,研究结果可以为开口圆柱壳的屈曲设计提供理论参考.

1 正交各向异性开口圆柱壳屈曲问题的控制方程

如图 1 所示,开口圆柱壳承受轴向力 N_1 的作用,柱壳的厚度为 δ,曲率半径为 R,轴向(α 方向)的弹性模量为 E_1、Poisson 比为 ν_{21},环向(β 方向)的弹性模量为 E_2, Poisson 比为 ν_{12},剪切模量为 G_{12}.柱壳沿轴向、环向和径向的位移函数分别记为 u,v 和 w.

1.1 基本方程

根据 Kirchhoff-Love 假设,壳体内部任意一点的应变分量可以表示为

$$\boldsymbol{e}=\{e_\alpha,e_\beta,e_{\alpha\beta}\}^{\mathrm{T}}=\boldsymbol{\varepsilon}+\gamma\boldsymbol{\chi}, \tag{1}$$

其中 $\boldsymbol{\varepsilon}=\{\varepsilon_\alpha,\varepsilon_\beta,\varepsilon_{\alpha\beta}\}^{\mathrm{T}}=\{\partial u/\partial\alpha,\partial v/\partial\beta+w/R,\partial u/\partial\beta+\partial v/\partial\alpha\}^{\mathrm{T}}$,$\varepsilon_\alpha$ 和 ε_β 为中面内各点沿 α 和 β 方向的线应变, $\varepsilon_{\alpha\beta}$ 为面内切应变; γ 为壳体中面法线方向的坐标;$\boldsymbol{\chi}=\{\chi_\alpha,\chi_\beta,\chi_{\alpha\beta}\}^{\mathrm{T}}=\{-\partial^2w/\partial\alpha^2,-\partial^2w/\partial\beta^2,$

$-\partial^2 w/\partial\alpha\partial\beta\}^{\mathrm{T}}$, χ_α 和 χ_β 为中面内各点主曲率的改变, $\chi_{\alpha\beta}$ 为扭率的改变.

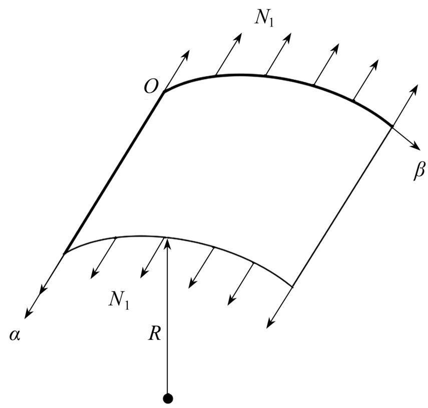


图 1 开口圆柱壳示意图

Fig. 1 Schematic diagram of a cylindrical shell

对于正交各向异性圆柱壳,物理方程为

$$\begin{Bmatrix}\sigma_\alpha\\ \sigma_\beta\\ \tau_{\alpha\beta}\end{Bmatrix}=\begin{Bmatrix}b_{11} & b_{12} & 0\\ b_{21} & b_{22} & 0\\ 0 & 0 & b_{66}\end{Bmatrix}\begin{Bmatrix}e_\alpha\\ e_\beta\\ e_{\alpha\beta}\end{Bmatrix}, \tag{2}$$

其中

$$b_{11}=E_1/(1-\nu_{21}\nu_{12}),\ b_{12}=E_1\nu_{21}/(1-\nu_{21}\nu_{12}),\ b_{22}=E_2/(1-\nu_{21}\nu_{12}),\ b_{66}=G_{12}.$$

进一步,基于 Donnell 薄壳理论[23],壳体所受内力和弯矩可以表示为

$$\begin{cases}F_{\mathrm{T1}}=\delta\left[b_{11}\dfrac{\partial u}{\partial\alpha}+b_{12}\left(\dfrac{\partial v}{\partial\beta}+\dfrac{w}{R}\right)\right]+N_1\dfrac{\partial u}{\partial\alpha},\ F_{\mathrm{T2}}=\delta\left[b_{21}\dfrac{\partial u}{\partial\alpha}+b_{22}\left(\dfrac{\partial v}{\partial\beta}+\dfrac{w}{R}\right)\right],\\ F_{\mathrm{T12}}=\delta b_{66}\left(\dfrac{\partial u}{\partial\beta}+\dfrac{\partial v}{\partial\alpha}\right)+N_1\dfrac{\partial v}{\partial\alpha},\ F_{\mathrm{T21}}=\delta b_{66}\left(\dfrac{\partial u}{\partial\beta}+\dfrac{\partial v}{\partial\alpha}\right),\\ M_1=\dfrac{(b_{11}\chi_\alpha+b_{12}\chi_\beta)\delta^3}{12},\ M_2=\dfrac{(b_{12}\chi_\alpha+b_{22}\chi_\beta)\delta^3}{12},\\ M_{12}=M_{21}=\dfrac{b_{66}\chi_{\alpha\beta}\delta^3}{6},\end{cases} \tag{3}$$

其中 F_{T1} 和 F_{T2} 分别为垂直于 α 和 β 轴的截面内单位宽度上的拉压力, F_{T12} 和 F_{T21} 分别为相应截面内的平错力, M_1 和 M_2 分别为弯矩, M_{12} 和 M_{21} 分别为扭矩.

等效剪力 V_1, V_2 可以表示为

$$\begin{cases}V_1=F_{\mathrm{S1}}+\dfrac{\partial M_{12}}{\partial\beta}+N_1\dfrac{\partial w}{\partial\alpha},\\ V_2=F_{\mathrm{S2}}+\dfrac{\partial M_{21}}{\partial\alpha},\end{cases} \tag{4}$$

其中 F_{S1} 和 F_{S2} 分别为垂直于 α 和 β 轴的截面内的横向剪力.

壳体的平衡方程可以表示为

$$\begin{cases}\dfrac{\partial F_{\mathrm{T1}}}{\partial\alpha}+\dfrac{\partial F_{\mathrm{T21}}}{\partial\beta}=0,\ \dfrac{\partial F_{\mathrm{T12}}}{\partial\alpha}+\dfrac{\partial F_{\mathrm{T2}}}{\partial\beta}=0,\\ \dfrac{\partial V_1}{\partial\alpha}+\dfrac{\partial V_2}{\partial\beta}-\dfrac{F_{\mathrm{T2}}}{R}-2\dfrac{\partial^2 M_{12}}{\partial\alpha\partial\beta}=0,\\ \dfrac{\partial M_1}{\partial\alpha}+\dfrac{\partial M_{12}}{\partial\beta}-F_{\mathrm{S1}}=0,\ \dfrac{\partial M_{21}}{\partial\alpha}+\dfrac{\partial M_2}{\partial\beta}-F_{\mathrm{S2}}=0.\end{cases} \tag{5}$$

1.2 导入 Hamilton 体系

利用 1.1 小节所示基本方程,可将开口圆柱壳屈曲问题导入 Hamilton 体系.定义

$$\frac{\partial w}{\partial \beta}=\theta_2, \tag{6}$$

由式(1)—(6)可以得到

$$\begin{cases}\dfrac{\partial u}{\partial \beta}=-\dfrac{T_{21}}{\delta b_{66}}+\dfrac{\partial \bar{v}}{\partial \alpha},\ \dfrac{\partial \bar{v}}{\partial \beta}=\dfrac{b_{12}}{b_{22}}\dfrac{\partial u}{\partial \alpha}+\dfrac{w}{R}-\dfrac{F_{\mathrm{T}2}}{b_{22}\delta},\\ \dfrac{\partial \theta}{\partial \beta}=-\dfrac{12M_2}{b_{22}\delta^3}-\dfrac{b_{21}}{b_{22}}\dfrac{\partial^2 w}{\partial \alpha^2},\\ \dfrac{\partial T_{21}}{\partial \beta}=-\dfrac{(b_{12}^2-b_{11}b_{22})\delta}{b_{22}}\dfrac{\partial^2 u}{\partial \alpha^2}+N_1\dfrac{\partial^2 u}{\partial \alpha^2}+\dfrac{b_{12}}{b_{22}}\dfrac{\partial F_{\mathrm{T}2}}{\partial \alpha},\\ \dfrac{\partial F_{\mathrm{T}2}}{\partial \beta}=\dfrac{\partial T_{21}}{\partial \alpha}+N_1\dfrac{\partial^2 \bar{v}}{\partial \alpha^2},\\ \dfrac{\partial T_2}{\partial \beta}=\dfrac{(b_{12}^2-b_{11}b_{22})\delta^3}{12b_{22}}\dfrac{\partial^4 w}{\partial \alpha^4}-\dfrac{F_{\mathrm{T}2}}{R}+\dfrac{b_{12}}{b_{22}}\dfrac{\partial^2 M_2}{\partial \alpha^2}+N_1\dfrac{\partial^2 w}{\partial \alpha^2},\\ \dfrac{\partial M_2}{\partial \beta}=\dfrac{b_{66}\delta^3}{3}\dfrac{\partial^2 \theta_2}{\partial \alpha^2}-T_2,\end{cases} \tag{7}$$

其中 $\bar{v}=-v, T_{21}=-F_{\mathrm{T}21}, T_2=-V_2$.将式(7)写成矩阵形式的 Hamilton 对偶方程,即

$$\frac{\partial \boldsymbol{Z}}{\partial \beta}=\boldsymbol{HZ}, \tag{8}$$

其中 $\boldsymbol{Z}=[u,\bar{v},w,\theta_2,T_{21},F_{\mathrm{T}2},T_2,M_2]^{\mathrm{T}}$,$\boldsymbol{H}$ 为 Hamilton 算子矩阵,满足 $(\boldsymbol{JH})^{\mathrm{T}}=\boldsymbol{JH}$[24],这里 $\boldsymbol{J}=\begin{bmatrix}\boldsymbol{0} & \boldsymbol{I}_4\\ -\boldsymbol{I}_4 & \boldsymbol{0}\end{bmatrix}$ 是单位辛矩阵, $\boldsymbol{I}_4$ 是单位矩阵, $\boldsymbol{H}$ 具体表示为

$$\boldsymbol{H}=\begin{bmatrix}\boldsymbol{F} & \boldsymbol{G}\\ \boldsymbol{Q} & -\boldsymbol{F}^{\mathrm{T}}\end{bmatrix}, \tag{9}$$

其中

$$\begin{cases}\boldsymbol{F}=\begin{bmatrix}0 & \dfrac{\partial}{\partial \alpha} & 0 & 0\\ \dfrac{b_{12}}{b_{22}}\dfrac{\partial}{\partial \alpha} & 0 & \dfrac{1}{R} & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -\dfrac{b_{12}}{b_{22}}\dfrac{\partial^2}{\partial \alpha^2} & 0\end{bmatrix},\ \boldsymbol{G}=\begin{bmatrix}-\dfrac{1}{b_{66}\delta} & 0 & 0 & 0\\ 0 & -\dfrac{1}{b_{22}\delta} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -\dfrac{12}{b_{22}\delta^3}\end{bmatrix},\\ \boldsymbol{Q}=\begin{bmatrix}\left[N_1-\dfrac{(b_{12}^2-b_{11}b_{22})\delta}{b_{22}}\right]\dfrac{\partial^2}{\partial \alpha^2} & 0 & 0 & 0\\ 0 & N_1\dfrac{\partial^2}{\partial \alpha^2} & 0 & 0\\ 0 & 0 & \dfrac{(b_{12}^2-b_{11}b_{22})\delta^3}{12b_{22}}\dfrac{\partial^4}{\partial \alpha^4}+N_1\dfrac{\partial^2}{\partial \alpha^2} & 0\\ 0 & 0 & 0 & \dfrac{b_{66}\delta^3}{3}\dfrac{\partial^2}{\partial \alpha^2}\end{bmatrix}.\end{cases} \tag{10}$$

采用类似的方法，定义 $\partial w/\partial\alpha = \theta_1$，对于子问题 2，Hamilton 对偶方程为

$$\frac{\partial \bar{\boldsymbol{Z}}}{\partial \alpha} = \bar{\boldsymbol{H}}\bar{\boldsymbol{Z}}, \tag{11}$$

其中 $\bar{\boldsymbol{Z}} = [u, v, T_1, M_1, F_{\mathrm{T1}}, F_{\mathrm{T12}}, w, \theta_1]^{\mathrm{T}}$，$T_1 = -V_1$，$\bar{\boldsymbol{H}}$ 为 Hamilton 算子矩阵[22]，具体表示为

$$\bar{\boldsymbol{H}} = \begin{bmatrix} \bar{\boldsymbol{F}} & \bar{\boldsymbol{G}} \\ \bar{\boldsymbol{Q}} & -\bar{\boldsymbol{F}}^{\mathrm{T}} \end{bmatrix}, \tag{12}$$

其中

$$\left\{\begin{aligned} \bar{\boldsymbol{F}} &= \begin{bmatrix} 0 & -\dfrac{1}{\tilde{N}_1}\dfrac{\partial}{\partial\beta} & 0 & 0 \\ -\dfrac{1}{\bar{N}_1}\dfrac{\partial}{\partial\beta} & 0 & 0 & 0 \\ 0 & \left(\dfrac{b_{12}\delta}{R\tilde{N}_1} - \dfrac{b_{22}\delta}{R}\right)\dfrac{\partial}{\partial\beta} & 0 & \dfrac{b_{12}}{b_{11}}\dfrac{\partial^2}{\partial\beta^2} \\ 0 & 0 & -1 & 0 \end{bmatrix}, \\ \bar{\boldsymbol{Q}} &= \begin{bmatrix} -\dfrac{N_1}{\bar{N}_1}\dfrac{\partial^2}{\partial\beta^2} & 0 & 0 & 0 \\ 0 & \left(\dfrac{b_{12}\delta}{\tilde{N}_1} - b_{22}\delta\right)\dfrac{\partial^2}{\partial\beta^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{12}{b_{11}\delta^3} \end{bmatrix}, \\ \bar{\boldsymbol{G}} &= \begin{bmatrix} \dfrac{1}{b_{12}\delta\tilde{N}_1} & 0 & -\dfrac{1}{R\tilde{N}_1} & 0 \\ 0 & \dfrac{1}{b_{66}\delta\bar{N}_1} & 0 & 0 \\ -\dfrac{1}{R\tilde{N}_1} & 0 & \dfrac{(b_{12}^2 - b_{11}b_{22})\delta^3}{12b_{11}}\dfrac{\partial^4}{\partial\beta^4} - \dfrac{(b_{22}\tilde{N}_1 - b_{12})\delta}{R^2\tilde{N}_1} & 0 \\ 0 & 0 & 0 & \dfrac{b_{66}\delta^3}{3}\dfrac{\partial^2}{\partial\beta^2} - N_1 \end{bmatrix}, \end{aligned}\right. \tag{13}$$

这里

$$\bar{N}_1 = (b_{66}\delta + N_1)/(b_{66}\delta),\ \tilde{N}_1 = (b_{11}\delta + N_1)/(b_{12}\delta).$$

在辛几何求解框架下，可以采用分离变量[25]使得

$$\boldsymbol{Z} = \boldsymbol{A}(\alpha)B(\beta), \tag{14}$$

其中 $\boldsymbol{A}(\alpha) = [u(\alpha), \bar{v}(\alpha), w(\alpha), \theta_2(\alpha), T_{21}(\alpha), F_{\mathrm{T2}}(\alpha), T_2(\alpha), M_2(\alpha)]^{\mathrm{T}}$. 将式(14)代入式(8)，可以得到

$$\left\{\begin{aligned} &\frac{\mathrm{d}B(\beta)}{\mathrm{d}\beta} = \mu B(\beta), \\ &\boldsymbol{H}\boldsymbol{A}(\alpha) = \mu\boldsymbol{A}(\alpha), \end{aligned}\right. \tag{15}$$

其中 μ 为本征值，$\boldsymbol{A}(\alpha)$ 为本征向量.

式(15)中第二式的通解可以写为

$$\begin{cases} u(\alpha)=\sum_{i=1}^{4}[C_{1i}\sinh(\lambda_i\alpha)+D_{1i}\cosh(\lambda_i\alpha)],\ \bar{v}(\alpha)=\sum_{i=1}^{4}[C_{2i}\sinh(\lambda_i\alpha)+D_{2i}\cosh(\lambda_i\alpha)], \\ w(\alpha)=\sum_{i=1}^{4}[C_{3i}\sinh(\lambda_i\alpha)+D_{3i}\cosh(\lambda_i\alpha)],\ \theta_2(\alpha)=\sum_{i=1}^{4}[C_{4i}\sinh(\lambda_i\alpha)+D_{4i}\cosh(\lambda_i\alpha)], \\ T_{21}(\alpha)=\sum_{i=1}^{4}[C_{5i}\sinh(\lambda_i\alpha)+D_{5i}\cosh(\lambda_i\alpha)],\ F_{\mathrm{T2}}(\alpha)=\sum_{i=1}^{4}[C_{6i}\sinh(\lambda_i\alpha)+D_{6i}\cosh(\lambda_i\alpha)], \\ T_2(\alpha)=\sum_{i=1}^{4}[C_{7i}\sinh(\lambda_i\alpha)+D_{7i}\cosh(\lambda_i\alpha)],\ M_2(\alpha)=\sum_{i=1}^{4}[C_{8i}\sinh(\lambda_i\alpha)+D_{8i}\cosh(\lambda_i\alpha)], \end{cases} \tag{16}$$

其中 λ_i 是微分方程的特征值,待定系数 C_{ji} 和 $D_{ji}(j=1,2,\cdots,8;i=1,2,3,4)$ 并不相互独立.将式(16)代回到式(15)的第二式,可以得到系数之间的关系:

$$\begin{cases} C(D)_{1i}=\dfrac{(b_{22}\lambda_i^2-b_{12}\bar{N}_1\mu^2)b_{66}\bar{N}_1\delta\mu C(D)_{7i}}{L_i}, \\ C(D)_{2i}=\dfrac{[b_{12}(b_{12}+b_{66})\bar{N}_1\delta\mu^2-b_{22}(\bar{N}_1\lambda_i^2+b_{66}\delta\lambda_i^2+b_{12}\bar{N}_1\tilde{N}_1\delta\lambda_i^2)]\lambda_i C(D)_{7i}}{L_i}, \\ C(D)_{3i}=\dfrac{[-12N_1+\delta^3(b_{12}\lambda_i^2+4b_{66}\lambda_i^2+b_{11}\mu^2)]\mu C(D)_{7i}}{12}, \\ C(D)_{4i}=-\dfrac{(b_{12}\lambda_i^2+b_{11}\mu^2)C(D)_{7i}\delta^3}{12}, \\ C(D)_{5i}=\dfrac{(N_1-b_{12}\delta+b_{22}\tilde{N}_1\delta)b_{12}b_{66}\bar{N}_1\delta\lambda_i^2\mu^2C(D)_{7i}}{L_i}, \\ C(D)_{6i}=-\dfrac{(b_{22}N_1\lambda_i^2-b_{12}^2\bar{N}_1\delta\mu^2+b_{12}b_{22}\bar{N}_1\tilde{N}_1\delta\mu^2)b_{66}\bar{N}_1\delta\lambda_i\mu C(D)_{7i}}{L_i}, \\ C(D)_{8i}=\mu C(D)_{7i}, \end{cases} \tag{17}$$

其中

$$L_i=(N_1\lambda_i^4+b_{66}\delta\lambda_i^4+b_{12}\bar{N}_1\tilde{N}_1\delta\lambda_i^2\mu^2)b_{22}R+[b_{66}N_1\lambda_i^2-b_{12}^2\delta\lambda_i^2+(\bar{N}_1\tilde{N}_1\mu^2-2\lambda_i^2)b_{12}b_{66}\delta]\bar{N}_1\mu^2.$$

2　非 Lévy 型正交各向异性开口圆柱壳屈曲问题的辛解析解

考虑如下边界条件的开口圆柱壳:

$$\begin{cases} F_{\mathrm{T1}}\big|_{\alpha=0,a}=0,\ F_{\mathrm{T12}}+\dfrac{M_{12}}{R}\bigg|_{\alpha=0,a}=0,\ w\big|_{\alpha=0,a}=0,\ \dfrac{\partial w}{\partial\alpha}\bigg|_{\alpha=0,a}=0, \\ F_{\mathrm{T2}}\big|_{\beta=0,b}=0,\ F_{\mathrm{T21}}\big|_{\beta=0,b}=0,\ w\big|_{\beta=0,b}=0,\ M_2\big|_{\beta=0,b}=0. \end{cases} \tag{18}$$

如图 2(a)所示的原问题边界条件为: $\alpha=0$ 和 $\alpha=a$ 边固支,用“C”表示, $\beta=0$ 和 $\beta=b$ 边简支,但并不满足 Lévy 型边界条件[23],用“S′”表示.将原问题拆分成两个子问题:子问题 1 是四边简支的开口圆柱壳施加了非零的轴向位移 $u|_{\beta=0,b}\neq 0$,子问题 2 是四边简支的开口圆柱壳施加了非零的环向位移 $v|_{\alpha=0,a}\neq 0$ 和弯矩 $M_1|_{\alpha=0,a}\neq 0$.两个子问题中的 Lévy 型简支边用“S”表示.

将式(16)代入 α 方向的对边简支边界条件,即

$$F_{\mathrm{T1}}\big|_{\alpha=0,a}=0,\ v\big|_{\alpha=0,a}=0,\ w\big|_{\alpha=0,a}=0,\ M_1\big|_{\alpha=0,a}=0, \tag{19}$$

可以得到

$$\sinh(a\lambda_1)\sinh(a\lambda_2)\sinh(a\lambda_3)\sinh(a\lambda_4)=0. \tag{20}$$

求解 $\mu\boldsymbol{I}-\boldsymbol{H}$ 的行列式展开可以得到重根零本征值 $\mu_{00}=0$ 和本征值 $\mu_{im}(i=1,2,\cdots,8;m=1,2,3,\cdots)$,对应

的本征向量为

$$\boldsymbol{A}_{00} = [1,0,0,0,0,0,0,0]^{\mathrm{T}}, \tag{21}$$

$$\boldsymbol{A}_{01} = [0,0,0,0,b_{66}\delta,0,0,0]^{\mathrm{T}}, \tag{22}$$

以及

$$\begin{aligned}\boldsymbol{A}_{im}(\alpha) = \Bigg[& \frac{(u_{im}+v_{im})a_m\cos(a_m\alpha)}{K_{im}}, \left(u_{im}-w_{im}+\frac{b_{12}+b_{66}}{N_1+b_{66}\delta}\delta v_{im}\right)\frac{\mu_{im}\sin(a_m\alpha)}{K_{im}}, \\ & \sin(a_m\alpha), \mu_{im}\sin(a_m\alpha), \left(w_{im}-\frac{b_{12}\delta-N_1}{N_1+b_{66}\delta}v_{im}\right)\frac{b_{66}\delta a_m\mu_{im}\cos(a_m\alpha)}{K_{im}}, \\ & -\left[N_1 w_{im}-b_{12}\delta v_{im}+\frac{b_{11}\delta a_m^2+(a_m^2-\mu_{im}^2)N_1}{\mu_{im}^2}u_{im}\right]\frac{\sin(a_m\alpha)a_m^2}{K_{im}}, \\ & \frac{[b_{22}\mu_{im}^2-(b_{12}+4b_{66})a_m^2]\mu_{im}\sin(a_m\alpha)\delta^3}{12}, \frac{(b_{12}a_m^2-b_{22}\mu_{im}^2)\sin(a_m\alpha)\delta^3}{12}\Bigg]^{\mathrm{T}},\end{aligned} \tag{23}$$

其中，式(21)和式(22)分别由 $\boldsymbol{A}\boldsymbol{A}_{00}=0$ 和 $\boldsymbol{H}\boldsymbol{A}_{01}=\boldsymbol{A}_{00}$ 得到，$a_m = m\pi/a\,(m=1,2,3,\cdots)$，$u_{im}$，$v_{im}$，$w_{im}$，$K_{im}$ 分别为

$$\begin{cases} u_{im}=b_{22}b_{66}\delta^2\mu_{im}^2,\ v_{im}=(N_1+b_{66}\delta)b_{12}a_m^2,\ w_{im}=(N_1+b_{11}\delta)b_{22}\delta a_m^2, \\ K_{im}=R\{[b_{66}\delta^2\mu_{im}^2-(N_1+b_{11}\delta)a_m^2]b_{22}\delta\mu_{im}^2+ \\ \qquad [b_{12}^2\delta^2\mu_{im}^2+(N_1+b_{11}\delta)N_1a_m^2+(N_1a_m^2-N_1\mu_{im}^2+2b_{12}\delta\mu_{im}^2+b_{11}\delta a_m^2)b_{66}\delta]a_m^2\}. \end{cases} \tag{24}$$

齐次方程(8)的解可以写为

$$\boldsymbol{Z} = [\boldsymbol{A}_{00},\boldsymbol{A}_{01},\boldsymbol{A}_{11}(\alpha),\boldsymbol{A}_{21}(\alpha),\cdots,\boldsymbol{A}_{81}(\alpha),\cdots,\boldsymbol{A}_{1m}(\alpha),\boldsymbol{A}_{2m}(\alpha),\cdots,\boldsymbol{A}_{8m}(\alpha),\cdots]\boldsymbol{B}(\beta), \tag{25}$$

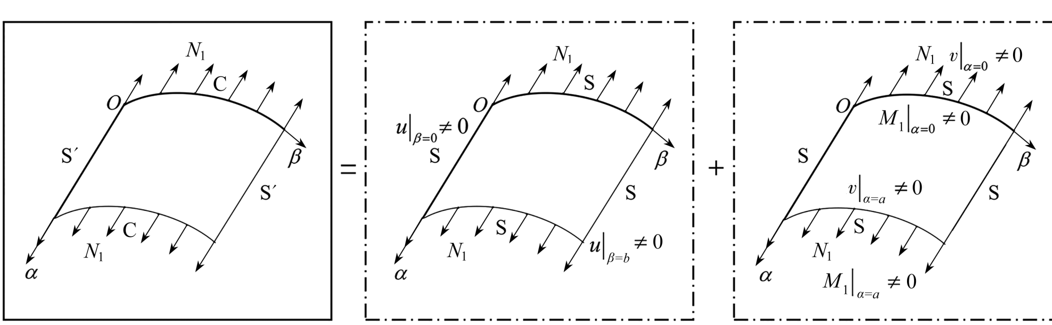
其中 $\boldsymbol{B}(\beta) = [B_{00}(\beta),B_{01}(\beta),B_{11}(\beta),B_{21}(\beta),\cdots,B_{81}(\beta),\cdots,B_{1m}(\beta),B_{2m}(\beta),\cdots,B_{8m}(\beta),\cdots]^{\mathrm{T}}$，由式(15)中的第一式可以得到

$$B_{01}(\beta)=c_{01},\ B_{00}(\beta)=c_{01}\beta+c_{00},\ B_{im}(\beta)=c_{im}\mathrm{e}^{\mu_{im}\beta}. \tag{26}$$

因此，子问题 1 的位移解可以写为

$$\begin{cases} u_1(\alpha,\beta)=c_{00}+c_{01}\beta+\sum\limits_{m=1,2,3,\cdots}^{\infty}\sum\limits_{i=1}^{8}c_{im}(u_{im}+v_{im})a_m\mathrm{e}^{\mu_{im}\beta}\cos(a_m\alpha), \\ v_1(\alpha,\beta)=\sum\limits_{m=1,2,3,\cdots}^{\infty}\sum\limits_{i=1}^{8}c_{im}\left(u_{im}-w_{im}+\dfrac{b_{12}+b_{66}}{N_1+b_{66}\delta}\delta v_{im}\right)\mu_{im}\mathrm{e}^{\mu_{im}\beta}\sin(a_m\alpha), \\ w_1(\alpha,\beta)=\sum\limits_{m=1,2,3,\cdots}^{\infty}\sum\limits_{i=1}^{8}c_{im}\mathrm{e}^{\mu_{im}\beta}\sin(a_m\alpha), \end{cases} \tag{27}$$

其中 c_{00}，c_{01} 和 c_{im} 为待定系数.



(a) 原问题　　　　(b) 子问题 1　　　　(c) 子问题 2

(a) The original problem　　　　(b) Subproblem 1　　　　(c) Subproblem 2

图 2　开口圆柱壳屈曲的辛叠加示意图

Fig. 2　Symplectic superposition for buckling of a cylindrical panel

在 $\beta=0$ 和 $\beta=b$ 处施加级数形式的位移,相关的边界条件可以表示为

$$\begin{cases} F_{\mathrm{T2}}|_{\beta=0}=0,\ F_{\mathrm{T2}}|_{\beta=b}=0, \\ u_1|_{\beta=0}=\sum\limits_{m=0,1,2,\cdots}^{\infty} u_{0m}\cos(a_m\alpha),\ u_1|_{\beta=b}=\sum\limits_{m=0,1,2,\cdots}^{\infty} u_{bm}\cos(a_m\alpha), \\ w_1|_{\beta=0}=0,\ w_1|_{\beta=b}=0,\ M_2|_{\beta=0}=0,\ M_2|_{\beta=b}=0. \end{cases} \tag{28}$$

将式(27)代入式(28),待定系数 c_{im} 得以由 u_{0m} 和 u_{bm} 表示.

对于子问题 2,代入相应的边界条件,同理可以得出位移解:

$$\begin{cases} u_2(\alpha,\beta)=-\sum\limits_{n=1,2,3,\cdots}^{\infty}\sum\limits_{i=1}^{8}\bar{c}_{in}\dfrac{(b_{12}\bar{N}_1\bar{\mu}_{in}^2+b_{22}b_n^2)b_{66}\bar{N}_1\delta\bar{\mu}_{in}\mathrm{e}^{\bar{\mu}_{in}\alpha}\sin(b_n\beta)}{\bar{u}_{in}}, \\ v_2(\alpha,\beta)=\bar{c}_{00}+\bar{c}_{01}\alpha+ \\ \quad\sum\limits_{n=1,2,3,\cdots}^{\infty}\sum\limits_{i=1}^{8}\bar{c}_{in}\dfrac{[b_{12}^2\bar{N}_1\delta\bar{\mu}_{in}^2+(b_{66}-b_{22}\tilde{N}_1)b_{12}\bar{N}_1\delta\bar{\mu}_{in}^2+(N_1+b_{66}\delta)b_{22}b_n^2]b_n\mathrm{e}^{\bar{\mu}_{in}\alpha}\cos(b_n\beta)}{\bar{u}_{in}}, \\ w_2(\alpha,\beta)=\sum\limits_{n=1,2,3,\cdots}^{\infty}\sum\limits_{i=1}^{8}\bar{c}_{in}\mathrm{e}^{\bar{\mu}_{in}\alpha}\sin(b_n\beta), \end{cases} \tag{29}$$

其中 $b_n=n\pi/b(n=1,2,3,\cdots)$,$\bar{c}_{00}$,$\bar{c}_{01}$ 和 $\bar{c}_{in}$ 是待定系数.

在 $\alpha=0$ 和 $\alpha=a$ 处施加级数形式的位移,相应的边界条件表示为

$$\begin{cases} F_{\mathrm{T1}}|_{\alpha=0}=0,\ F_{\mathrm{T1}}|_{\alpha=a}=0, \\ v_2|_{\alpha=0}=\sum\limits_{n=0,1,2,\cdots}^{\infty} v_{0n}\cos(b_n\beta),\ v_2|_{\alpha=a}=\sum\limits_{n=0,1,2,\cdots}^{\infty} v_{an}\cos(b_n\beta), \\ w_2|_{\alpha=0}=0,\ w_2|_{\alpha=a}=0, \\ M_1|_{\alpha=0}=\sum\limits_{n=1,2,3,\cdots}^{\infty} M_{0n}\sin(b_n\beta),\ M_1|_{\alpha=a}=\sum\limits_{n=1,2,3,\cdots}^{\infty} M_{an}\sin(b_n\beta). \end{cases} \tag{30}$$

将式(29)代入式(30),待定系数 $\bar{c}_{in}$ 得以由 v_{0n},v_{an},M_{0n} 和 M_{an} 表示.

3　非 Lévy 型正交各向异性开口圆柱壳屈曲问题的辛叠加解析解

将式(27)和式(29)中两个子问题的解进行叠加,为了与式(18)所对应的原问题的解等价,还需要满足的边界条件为

$$\begin{cases} F_{\mathrm{T12}}+\dfrac{M_{12}}{R}\Big|_{\alpha=0,a}=\sum\limits_{i=1}^{2}\left[N_1\dfrac{\partial v_i}{\partial\alpha}-\dfrac{\delta^3}{6R}b_{66}\dfrac{\partial^2 w_i}{\partial\alpha\partial\beta}+b_{66}\delta\left(\dfrac{\partial u_i}{\partial\beta}+\dfrac{\partial v_i}{\partial\alpha}\right)\right]\Big|_{\alpha=0,a}=0, \\ \dfrac{\partial w}{\partial\alpha}\Big|_{\alpha=0,a}=\sum\limits_{i=1}^{2}\dfrac{\partial w_i}{\partial\alpha}\Big|_{\alpha=0,a}=0, \\ F_{\mathrm{T21}}|_{\beta=0,b}=\sum\limits_{i=1}^{2}b_{66}\delta\left(\dfrac{\partial u_i}{\partial\beta}+\dfrac{\partial v_i}{\partial\alpha}\right)\Big|_{\beta=0,b}=0. \end{cases} \tag{31}$$

将式(27)和式(29)代入式(31),利用三角级数展开,具体地,由 $F_{\mathrm{T12}}+(M_{12}/R)|_{\alpha=0}=0$ 可以得到

$$\delta b_{66}c_{01}+(\delta b_{66}+N_1)\bar{c}_{01}=0, \tag{32}$$

$$\sum_{i=1}^{8}\frac{\bar{c}_{in}\bar{\mu}_{in}b_n}{\bar{u}_{in}}\Big\{-\frac{b_{66}\delta^3}{6R}+(b_{12}\bar{N}_1\bar{\mu}_{in}^2+b_{22}b_n^2)\bar{N}_1b_{66}^2\delta^2+$$

$$(\delta b_{66}+N_1)[b_{12}^2\bar{N}_1\delta\bar{\mu}_{in}^2+(b_{66}-b_{22}\tilde{N}_1)b_{12}\bar{N}_1\delta\bar{\mu}_{in}^2+(N_1+b_{66}\delta)b_{22}b_n^2]\Big\}+$$

$$\sum_{m=1,2,3,\cdots}^{\infty}\sum_{i=1}^{8}\frac{2c_{im}\mu_{im}^2a_m[\mathrm{e}^{b\mu_{im}}\cos(n\pi)-1]}{b(b_n^2+\mu_{im}^2)}\Big[-\frac{b_{66}\delta^3}{6R}+$$

$$(u_{im}+v_{im})b_{66}\delta+(b_{66}\delta+N_1)\left(u_{im}-w_{im}+\delta v_{im}\frac{b_{12}+b_{66}}{N_1+b_{66}\delta}\right)\Bigg]=0, \tag{33}$$

其中 $n=1,2,3,\cdots$.

由 $F_{\mathrm{T}12}+(M_{12}/R)\big|_{\alpha=a}=0$ 可以得到式(32)和

$$\sum_{m=1,2,3,\cdots}^{\infty}\sum_{i=1}^{8}\frac{2c_{im}\mu_{im}^2a_m\cos(m\pi)[\mathrm{e}^{b\mu_{im}}\cos(n\pi)-1]}{b(b_n^2+\mu_{im}^2)}\times$$
$$\left[(u_{im}+v_{im})b_{66}\delta+(b_{66}\delta+N_1)\left(u_{im}-w_{im}+\frac{b_{12}+b_{66}}{N_1+b_{66}\delta}\delta v_{im}\right)-\frac{b_{66}\delta^3}{6R}\right]+$$
$$\sum_{i=1}^{8}\frac{\bar{c}_{in}\bar{\mu}_{in}b_n\mathrm{e}^{a\bar{\mu}_{in}}}{\bar{u}_{in}}\Bigg\{-\frac{b_{66}\delta^3}{6R}-(b_{12}\bar{N}_1\bar{\mu}_{in}^2+b_{22}b_n^2)\bar{N}_1b_{66}^2\delta^2-$$
$$(\delta b_{66}+N_1)[b_{12}^2\bar{N}_1\delta\bar{\mu}_{in}^2+(b_{66}-b_{22}\tilde{N}_1)b_{12}\bar{N}_1\delta\bar{\mu}_{in}^2+(N_1+b_{66}\delta)b_{22}b_n^2]\Bigg\}=0, \tag{34}$$

其中 $n=1,2,3,\cdots$.

由 $\partial w/\partial\alpha\big|_{\alpha=0,a}=0$ 可以得到

$$\sum_{m=1,2,3\cdots}^{\infty}\sum_{i=1}^{8}\frac{2c_{im}a_mb_n[1-\mathrm{e}^{b\mu_{im}}\cos(n\pi)]}{b(b_n^2+\mu_{im}^2)}+\sum_{i=1}^{8}\bar{c}_{in}\bar{\mu}_{in}=0, \tag{35}$$

和

$$\sum_{m=1,2,3\cdots}^{\infty}\sum_{i=1}^{8}\frac{2c_{im}a_mb_n\cos(m\pi)[1-\mathrm{e}^{b\mu_{im}}\cos(n\pi)]}{b(b_n^2+\mu_{im}^2)}+\sum_{i=1}^{8}\bar{c}_{in}\bar{\mu}_{in}\mathrm{e}^{a\bar{\mu}_{in}}=0, \tag{36}$$

其中 $n=1,2,3,\cdots$.

由 $F_{\mathrm{T}21}\big|_{\beta=0,b}=0$ 可以得到

$$c_{01}+\bar{c}_{01}=0, \tag{37}$$

$$\sum_{i=1}^{8}\frac{2\delta b_{66}c_{im}\mu_{im}^2a_m[\mathrm{e}^{a\bar{\mu}_{in}}\cos(m\pi)-1]}{a(a_m^2+\bar{\mu}_{in}^2)}\left(2u_{im}+v_{im}-w_{im}+\frac{b_{12}+b_{66}}{N_1+b_{66}\delta}\delta v_{im}\right)+$$
$$\sum_{n=1,2,3\cdots}^{\infty}\sum_{i=1}^{8}\frac{\delta b_{66}\bar{c}_{in}\bar{\mu}_{in}b_n}{\bar{u}_{in}}\{(b_{12}\bar{N}_1\bar{\mu}_{in}^2+b_{22}b_n^2)b_{66}\bar{N}_1\delta-$$
$$[b_{12}^2\bar{N}_1\delta\bar{\mu}_{in}^2+(b_{66}-b_{22}\tilde{N}_1)b_{12}\bar{N}_1\delta\bar{\mu}_{in}^2+(N_1+b_{66}\delta)b_{22}b_n^2]\}=0, \tag{38}$$

以及

$$\sum_{i=1}^{8}\frac{2\delta b_{66}c_{im}\mu_{im}^2a_m\mathrm{e}^{b\mu_{im}}[\mathrm{e}^{a\bar{\mu}_{in}}\cos(m\pi)-1]}{a(a_m^2+\bar{\mu}_{in}^2)}\left(2u_{im}+v_{im}+\frac{b_{12}+b_{66}}{N_1+b_{66}\delta}\delta v_{im}-w_{im}\right)+$$
$$\sum_{n=1,2,3\cdots}^{\infty}\sum_{i=1}^{8}\frac{\delta b_{66}\bar{c}_{in}\bar{\mu}_{in}b_n\cos(n\pi)}{\bar{u}_{in}}\{(b_{12}\bar{N}_1\bar{\mu}_{in}^2+b_{22}b_n^2)b_{66}\bar{N}_1\delta-$$
$$[b_{12}^2\bar{N}_1\delta\bar{\mu}_{in}^2+(b_{66}-b_{22}\tilde{N}_1)b_{12}\bar{N}_1\delta\bar{\mu}_{in}^2+(N_1+b_{66}\delta)b_{22}b_n^2]\}=0, \tag{39}$$

其中 $m=1,2,3,\cdots$.

由式(32)和(37)可以得到, c_{01} 和 $\bar{c}_{01}$ 为 0,代表刚体位移的 c_{00} 和 $\bar{c}_{00}$ 也应该为 0.式(33)—(36)、(38)和(39)是 6 组关于常数 $u_{0m}, u_{bm}, v_{0n}, v_{an}, M_{0n}$ 和 M_{an}($m=1,2,3,\cdots;n=1,2,3,\cdots$)的无穷联立线性代数方程组,实际当中必须取有限组进行计算.以任意正整数 N 作为无穷级数的项数,生成一个 $6N$ 阶的齐次矩阵方程,通过求解系数矩阵行列式为 0,给出不同阶次的屈曲载荷解,进而得到对应的待定常数的非零解,然后将其代回式(29),求和后得到各阶屈曲模态.

4 数 值 算 例

下面给出开口圆柱壳屈曲问题的数值算例.选取由石墨和环氧树脂组成的正交各向异性复合材料[26],

其材料属性为 $E_1 = 128$ GPa，$E_2 = 11$ GPa，$\nu_{12} = 0.25$，$G_{12} = 4.48$ GPa.

表 1 为屈曲载荷解的收敛性研究，所研究的开口圆柱壳纵向尺寸与环向尺寸以及半径相同（$a = b = R = 1$ m），δ 分别为 0.01 m 和 0.005 m.显然，在两种厚度下，所有计算结果在级数取 20 项时均收敛到 5 位有效数字（表中粗体数据），表明本文方法得到的解析解收敛性非常好.由于文献中缺乏可供对比的解析解，因此将本文的计算结果与有限元软件 ABAQUS 得到的结果进行对比，其中有限元的参数按照如下设置：采用 S4R 单元，网格大小为壳最小面内尺寸的 1/100.表 2 和表 3 分别展示了不同尺寸下开口圆柱壳的屈曲载荷.当 $\delta = 0.01$ m 时，本文结果与有限元的差别均在 5%以内；当 $\delta = 0.005$ m 时，与有限元的差别均在 1%以内，验证了本文解析结果的正确性.除得到屈曲载荷之外，本文方法还得到了开口圆柱壳的屈曲模态，包括高阶模态.以 $a = b = R = 1$ m，$\delta = 0.005$ m 的圆柱壳为例，图 3 给出了其前十阶屈曲模态与有限元结果的对比，结果高度吻合.

表 1　$a = b = R = 1$ m 时，开口圆柱壳前十阶屈曲载荷的收敛性研究（单位：kN/m）

Table 1　Convergence of the 1st 10 buckling loads on the cylindrical shell with $a = b = R = 1$ m (unit: kN/m)

δ /m	number of series terms	mode									
		1	2	3	4	5	6	7	8	9	10
1	10	**1 220.2**	**1 392.5**	**1 501.8**	**1 531.1**	**1 830.1**	**1 887.1**	1 986.8	**2 049.8**	**2 201.7**	2 286.1
	20	1 220.2	1 392.5	1 501.8	1 531.1	1 830.1	1 887.1	**1 986.7**	2 049.8	2 201.7	**2 286.0**
	30	1 220.2	1 392.5	1 501.8	1 531.1	1 830.1	1 887.1	1 986.7	2 049.8	2 201.7	2 286.0
0.05	10	**285.61**	300.36	307.11	**322.16**	**367.86**	374.32	383.09	385.92	406.60	430.85
	20	285.61	**300.35**	**307.08**	322.16	367.86	**374.30**	**383.07**	**385.91**	**406.58**	**430.84**
	30	285.61	300.35	307.08	322.16	367.86	374.30	383.07	385.91	406.58	430.84

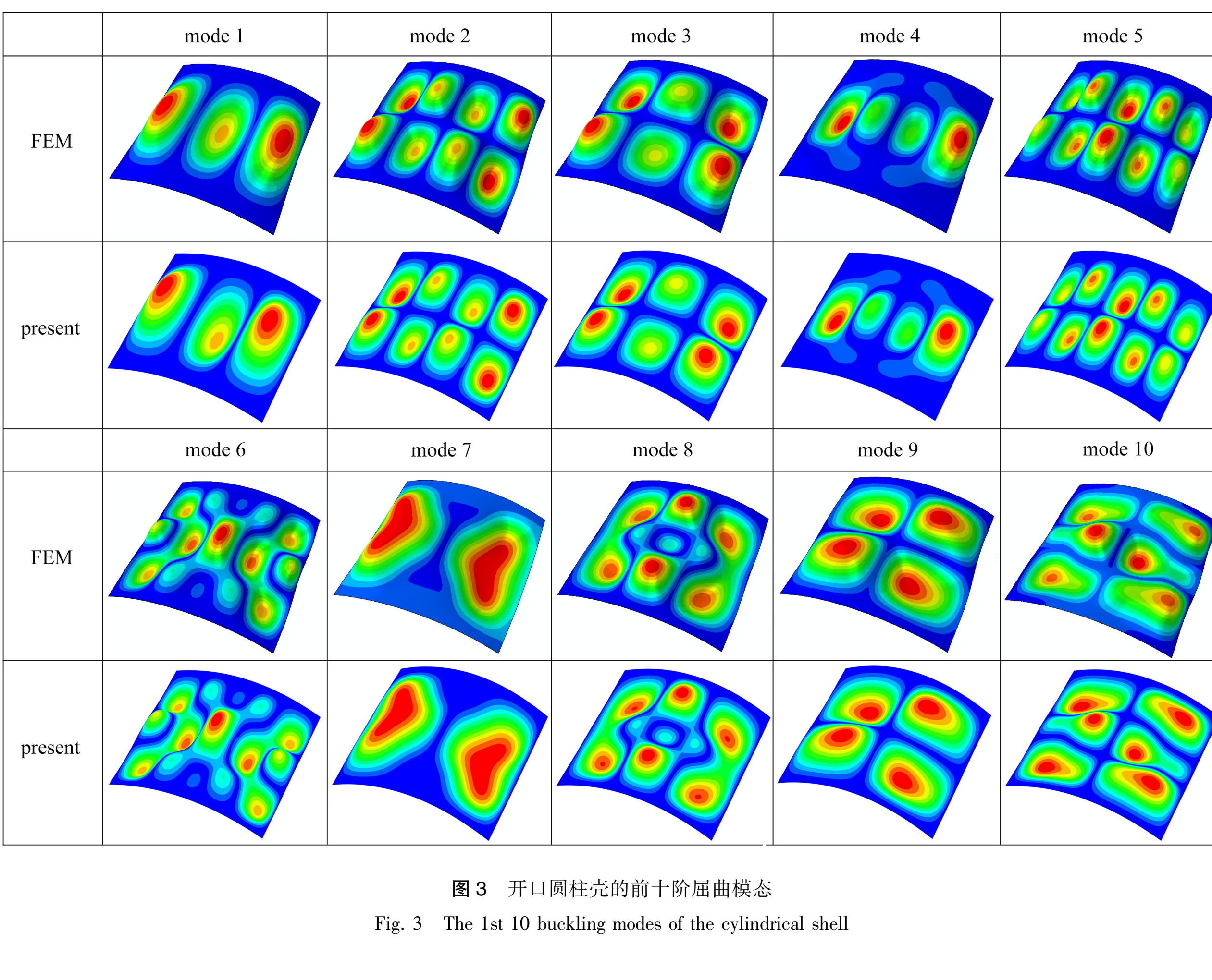


图 3　开口圆柱壳的前十阶屈曲模态

Fig. 3　The 1st 10 buckling modes of the cylindrical shell

表 2 $b=1$ m, $\delta=0.01$ m 时,开口圆柱壳的前十阶屈曲载荷(单位: kN/m)

Table 2 The 1st 10 buckling loads on the cylindrical shell with $b=1$ m and $\delta=0.01$ m (unit: kN/m)

a/m	R/m	method	mode 1	2	3	4	5	6	7	8	9	10
1	1	FEM	1 218.3	1 366.8	1 473.7	1 510.5	1 789.6	1 869.8	1 988.4	1 990.6	2 129.8	2 209.1
		present	1 220.2	1 392.5	1 501.8	1 531.1	1 830.1	1 887.1	1 986.7	2 049.8	2 201.7	2 286.0
	2	FEM	757.89	831.37	1 091.2	1 117.9	1 165.4	1 240.7	1 646.8	1 792.0	1 826.1	1 852.6
		present	765.18	835.57	1 114.0	1 138.6	1 182.1	1 268.5	1 685.3	1 862.5	1 899.7	1 911.4
	10	FEM	460.16	607.93	876.33	955.33	1 093.4	1 164.2	1 601.8	1 649.9	1 729.0	1 809.0
		present	465.05	614.49	897.36	978.51	1 108.2	1 191.8	1 639.5	1 720.4	1 802.8	1 867.2
2	1	FEM	1 074.7	1 089.4	1 140.3	1 169.6	1 249.8	1 363.0	1 377.4	1 472.8	1 569.4	1 580.9
		present	1 091.5	1 093.6	1 154.4	1 185.4	1 259.3	1 373.8	1 401.7	1 500.8	1 596.8	1 609.9
	2	FEM	580.60	630.27	667.79	766.72	838.42	842.57	902.04	903.58	935.23	965.62
		present	584.52	635.65	668.25	767.03	843.13	850.34	913.12	914.70	947.30	975.79
	10	FEM	172.86	262.98	413.54	460.57	468.95	663.31	721.07	752.43	815.28	852.77
		present	173.45	264.58	416.49	465.46	473.21	674.36	727.81	765.29	824.88	863.90

表 3 $b=1$ m, $\delta=0.005$ m 时,开口圆柱壳的前十阶屈曲载荷(单位: kN/m)

Table 3 The 1st 10 buckling loads on the cylindrical shell with $b=1$ m and $\delta=0.005$ m (unit: kN/m)

a/m	R/m	method	mode 1	2	3	4	5	6	7	8	9	10
1	1	FEM	284.75	299.42	306.74	320.64	366.45	372.05	382.91	384.16	407.24	429.56
		present	285.61	300.35	307.08	322.16	367.86	374.30	383.07	385.91	406.58	430.84
	2	FEM	154.95	174.48	188.02	192.21	228.72	237.54	252.14	255.64	274.23	284.70
		present	154.94	175.10	188.71	192.72	229.59	238.04	251.97	257.08	276.15	286.74
	10	FEM	63.880	79.045	115.44	123.84	139.42	149.47	204.84	215.42	224.94	232.71
		present	64.008	79.224	116.00	124.47	139.74	150.21	205.71	217.29	226.92	234.12
2	1	FEM	267.91	270.50	272.42	274.90	291.75	300.12	306.17	309.58	313.35	334.04
		present	268.73	270.68	273.02	275.64	292.06	301.30	306.66	311.87	314.53	335.13
	2	FEM	138.49	138.74	144.98	148.91	159.28	172.95	175.71	187.93	199.89	201.64
		present	138.79	138.92	145.27	149.24	159.52	173.18	176.28	188.62	200.38	202.34
	10	FEM	29.704	41.073	54.651	61.639	63.996	88.736	92.837	97.687	104.21	109.03
		present	29.727	41.116	54.736	61.757	64.127	89.029	93.008	98.038	104.43	109.30

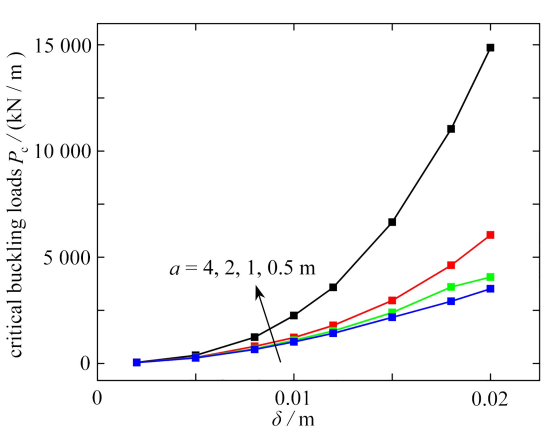


图 4 不同长度下,开口圆柱壳临界屈曲载荷随厚度的变化曲线

Fig. 4 Thickness-dependent critical buckling loads on cylindrical shells with different lengths

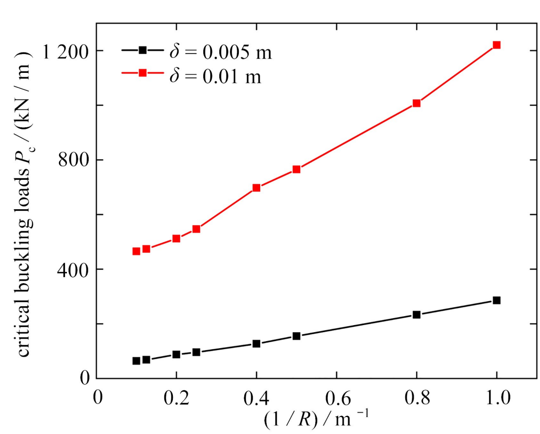


图 5 不同厚度下,开口圆柱壳临界屈曲载荷随曲率的变化曲线

Fig. 5 Curvature-dependent critical buckling loads on cylindrical shells with different thicknesses

利用所得解析解进一步考查主要参数对开口圆柱壳屈曲结果的影响.图 4 给出了在不同长度下厚度对临界屈曲载荷的影响(取 $b = R = 1$ m).可以看到,随着厚度增加临界载荷均显著增大,而且在壳的长度较短时,这种效应更加明显.图 5 给出了两种厚度下曲率 $1/R$ 对临界载荷的影响(取 $a = b = 1$ m),在曲率较小时,壳的形态接近于平板,随着曲率的增大,对轴向刚度的贡献加强,因而提高了壳的承载能力.上述结果可以为开口圆柱壳的抗屈曲设计提供参考.

5 结　论

由于高阶偏微分控制方程的求解困难,开口圆柱壳的解析求解本就是一项具有挑战性的课题,而对于非 Lévy 边界的正交各向异性开口圆柱壳的屈曲问题,求解难度无疑会进一步增加.本文从 Donnell 壳体理论的基本方程出发,导出了正交各向异性开口圆柱壳屈曲的 Hamilton 对偶方程,利用辛叠加方法,将原问题拆分,采用分离变量、辛本征展开求解得到子问题的解,最后根据叠加后的解与原问题解的等价性要求得到关于待定常数的齐次线性方程组,根据方程组有非零解的条件得到屈曲载荷解,进而得到相应的屈曲模态.由于文献中缺乏对应的解析解,因此采用有限元数值解对本文结果进行了对比验证,同时还利用本文的解研究了壳的尺寸对屈曲结果的影响.辛叠加方法在求解过程中无须假定试函数,兼备了辛方法的理性和叠加法的规律性,有望推广到更多壳体问题的求解当中.

参考文献(References):

[1] 熊健, 李志彬, 刘惠彬, 等. 航空航天轻质复合材料壳体结构研究进展[J]. 复合材料, 2021, **38**(6): 1629-1650. (XIONG Jian, LI Zhibin, LIU Huibin, et al. Advances in aerospace lightweight composite shell structure[J]. *Acta Materiae Compositae Sinica*, 2021, **38**(6): 1629-1650. (in Chinese))

[2] WEI K, PENG Y, QU Z, et al. Lightweight composite lattice cylindrical shells with novel character of tailorable thermal expansion[J]. *International Journal of Mechanical Sciences*, 2018, **137**: 77-85.

[3] 王晓旭, 张典堂, 钱坤, 等. 深海纤维增强树脂复合材料圆柱耐压壳力学性能的研究进展[J]. 复合材料学报, 2020, **37**(1): 16-26. (WANG Xiaoxu, ZHANG Diantang, QIAN Kun, et al. Research progress on mechanical properties of deep-sea fiber reinforced resin composite cylindrical pressure shells[J]. *Acta Materiae Compositae Sinica*, 2020, **37**(1): 16-26. (in Chinese))

[4] TENG J G, YU T, FERNANDO D. Strengthening of steel structures with fiber-reinforced polymer composites[J]. *Journal of Constructional Steel Research*, 2012, **78**: 131-143.

[5] MOROZOV E V, LOPATIN A V, NESTEROV V A. Finite-element modelling and buckling analysis of anisogrid composite lattice cylindrical shells[J]. *Composite Structures*, 2011, **93**(2): 308-323.

[6] RAVARI M R K, TALEBI S, SHAHIDI A R. Analysis of the buckling of rectangular nanoplates by use of finite-difference method[J]. *Meccanica*, 2014, **49**(6): 1443-1455.

[7] LOPATIN A V, MOROZOV E V. Buckling of the composite sandwich cylindrical shell with clamped ends under uniform external pressure[J]. *Composite Structures*, 2015, **122**: 209-216.

[8] LOPATIN A V, MOROZOV E V. Buckling of composite cylindrical shells with rigid end disks under hydrostatic pressure[J]. *Composite Structures*, 2017, **173**: 136-143.

[9] 张磊, 张文明, 王林, 等. 基于小波 Galerkin 法的矩形薄板二次屈曲分析[J]. 应用数学和力学, 2023, **44**(1): 25-35. (ZHANG Lei, ZHANG Wenming, WANG Lin, et al. Secondary buckling analysis of thin rectangular plates based on the wavelet Galerkin method[J]. *Applied Mathematics and Mechanics*, 2023, **44**(1): 25-35. (in Chinese))

[10] GHAHFAROKHI D S, RAHIMI G. An analytical approach for global buckling of composite sandwich cylindrical shells with lattice cores[J]. *International Journal of Solids and Structures*, 2018, **146**: 69-79.

[11] SAFARPOUR H, GHANBARI B, GHADIRI M. Buckling and free vibration analysis of high speed rotating carbon nanotube reinforced cylindrical piezoelectric shell[J]. *Applied Mathematical Modelling*, 2019, **65**: 428-442.

[12] YOSHIDA K, SADAMOTO S, SETOYAMA Y, et al. Meshfree flat-shell formulation for evaluating linear buck-

ling loads and mode shapes of structural plates[J]. *Journal of Marine Science and Technology*, 2017, **22**(3): 501-512.

[13] ZOU R D, FOSTER C G. Simple solution for buckling of orthotropic circular cylindrical shells[J]. *Thin-Walled Structures*, 1995, **22**(3): 143-158.

[14] 张俊霖, 倪一文, 李庆东, 等. 吸湿老化影响下天然纤维增强复合圆柱壳屈曲分析的辛方法[J]. 应用数学和力学, 2021, **42**(12): 1238-1247.(ZHANG Junlin, NI Yiwen, LI Qingdong, et al. A symplectic approach for buckling analysis of natural fiber reinforced composite shells under hygrothermal aging[J]. *Applied Mathematics and Mechanics*, 2021, **42**(12): 1238-1247.(in Chinese))

[15] 桂夷斐, 马建敏. 弹性介质中受轴向冲击载荷作用的圆柱壳的屈曲临界载荷计算分析[J]. 应用数学和力学, 2020, **41**(4): 353-366.(GUI Yifei, MA Jianmin. Buckling critical load calculation and analysis of axially impacted cylindrical shells embedded in elastic media[J]. *Applied Mathematics and Mechanics*, 2020, **41**(4): 353-366.(in Chinese))

[16] SUN F, FAN H, ZHOU C, et al. Equivalent analysis and failure prediction of quasi-isotropic composite sandwich cylinder with lattice core under uniaxial compression[J]. *Composite Structures*, 2013, **101**: 180-190.

[17] CHEN W, LIU D, KITIPORNCHAI S, et al. Bifurcation of pressurized functionally graded elastomeric hollow cylinders[J]. *Composites (Part B): Engineering*, 2017, **109**: 259-276.

[18] REDDY J, STARNES J. General buckling of stiffened circular cylindrical-shells according to a layerwise theory [J]. *Computers & Structures*, 1993, **49**(4): 605-616.

[19] SHADMEHRI F, HOA S, HOJJATI M. The effect of displacement field on bending, buckling, and vibration of cross-ply circular cylindrical shells[J]. *Mechanics of Advanced Materials and Structures*, 2014, **21**(1): 14-22.

[20] LI R, ZHENG X, WANG H, et al. New analytic buckling solutions of rectangular thin plates with all edges free [J]. *International Journal of Mechanical Sciences*, 2018, **144**: 67-73.

[21] LI R, ZHENG X, WANG P, et al. New analytic free vibration solutions of orthotropic rectangular plates by a novel symplectic approach[J]. *Acta Mechanica*, 2019, **230**(9): 3087-3101.

[22] 王海阳. 矩形板屈曲问题的若干新解析解研究[D]. 硕士学位论文. 大连: 大连理工大学, 2019.(WANG Haiyang. New analytical buckling solutions of rectangular plates[D]. Master Thesis. Dalian: Dalian University of Technology, 2019.(in Chinese))

[23] DONNELL L. A new theory for the buckling of thin cylinders under axial compression and bending[J]. *Trans ASME*, 1934, **13**: 795-806.

[24] 姚伟岸, 钟万勰. 辛弹性力学[M]. 北京: 高等教育出版社, 2002.(YAO Weian, ZHONG Wanxie. *Symplectic Elasticity*[M]. Beijing: Higher Education Press, 2002.(in Chinese))

[25] ZHENG X, NI Z, XU D, et al. New analytic buckling solutions of non-Lévy-type cylindrical panels within the symplectic framework[J]. *Applied Mathematical Modelling*, 2021, **98**: 398-415.

[26] CRAWLEY E. Natural-modes of graphite-epoxy cantilever plates and shells[J]. *Journal of Composite Materials*, 1979, **13**: 195-205.