

# 基于比例边界有限元的复合梁自由振动频率计算\*

李文武<sup>1</sup>, 王 为<sup>2</sup>

- (1. 湖南省交通规划勘察设计院有限公司, 长沙 410219;
- 2. 湖南省建设投资集团有限责任公司, 长沙 410009)

**摘要:** 将比例边界有限元方法(SBFEM)拓展用于计算复合梁的自由振动频率,该方法将梁简化为一维模型,并且仅选用  $x$  和  $z$  方向的弹性线位移作为基本未知量,从弹性力学基本方程出发,通过比例边界坐标、虚功原理和对偶变量技术推导得到了复合梁的一阶常微分比例边界有限元动力控制方程,其通解为解析的矩阵指数函数.利用 Padé 级数求解矩阵指数函数可得各个梁层的动力刚度矩阵,根据自由度匹配原则组装得到复合梁的整体刚度和质量矩阵.求解特征值方程,最终可得复合梁的自由振动频率.该方法对复合梁的层数和边界条件均无限制,具有广泛的适用性.将该文的解与三层、四层和十层复合梁振动频率的数值参考解以及阶梯型悬臂梁固有频率的实验实测值进行对比,验证了比例边界有限元算法的准确性、高效性和快速收敛性.

**关键词:** 复合梁; 自由振动频率; 比例边界有限元; Padé 级数

**中图分类号:** TB332      **文献标志码:** A      **DOI:** 10.21656/1000-0887.440208

## Natural Vibration Frequencies of Laminated Composite Beams Based on the Scaled Boundary Finite Element Method

LI Wenwu<sup>1</sup>, WANG Wei<sup>2</sup>

- (1. Hunan Provincial Communications Planning, Survey & Design Institute Co., Ltd.,  
Changsha 410219, P.R.China;
- 2. Hunan Construction Investment Group, Changsha 410009, P.R.China)

**Abstract:** The scaled boundary finite element method (SBFEM) was extended to calculate the natural frequencies of laminated composite beams. With this method, the beam was simplified as a 1D model. Only the displacement components along the  $x$  and  $z$  directions were selected as the fundamental unknowns. Based on the fundamental equations of elasticity and the scaled boundary coordinates, under the principle of virtual work and with the dual vector technique, the 1st-order ordinary differential scaled boundary finite element dynamic equation for composite beams was obtained, with its general solution in the form of the analytical matrix exponential function. The Padé expansion was utilized to solve the matrix exponential function and the dynamic stiffness matrix for each beam layer was acquired. According to the principle of matching degrees of freedom, the global stiffness and mass matrices of the laminated beam were gained. The eigenvalue equation was solved to give the natural vibration frequencies of the laminated composite beam. The results show that, the proposed method is

\* 收稿日期: 2023-07-10; 修订日期: 2024-03-04

基金项目: 国家自然科学基金(51875159)

作者简介: 李文武(1980—),男,博士,教授级高级工程师(通讯作者. E-mail: m19818937339@163.com);  
王为(1973—),男,高级工程师(E-mail: tian3316625@163.com).

引用格式: 李文武, 王为. 基于比例边界有限元的复合梁自由振动频率计算[J]. 应用数学和力学, 2024, 45(7): 936-948.

widely applicable without limitation on the layer number and boundary conditions. Comparisons between the numerical natural frequencies and the experiment results of 3-, 4- and 10-layered step-shaped cantilever beams, validate the accuracy, high efficiency and fast convergence of the SBFEM.

**Key words:** laminate composite beam; natural frequency; scaled boundary finite element method (SBFEM); Padé expansion

## 0 引 言

复合梁,即复合材料层合梁,由构成材料不同的梁层按一定顺序组合而成,具有强度高、刚度大、重量轻、耐腐蚀以及耐疲劳等优异性能,并且可以通过改变梁层纤维走向和排列顺序来满足特殊的需求,具有良好的设计灵活性,因而在航空航天、土木工程、风力发电、机械制造、交通运输、生物医学和军事科技等领域拥有广阔的应用前景.复合梁用于制造机翼、汽车传动轴、直升机叶片、风力发电机扇叶等结构时,经常承受各种动力荷载,产生动态响应.模态分析作为动力响应研究的重要组成部分,同时为了保证上述装置的安全稳定运行,需研究复合梁的自由振动响应.

为了分析复合梁的力学特性,学者们提出了等效单层理论<sup>[1-2]</sup>.该理论将复合梁简化为一等效单层梁,计算方便,求解效率高.等效单层理论包括经典复合梁理论、一阶剪切变形梁理论和高阶剪切变形梁理论.经典复合梁理论中假设垂直于中面的平面变形后仍保持为平面且垂直于中面,因而忽略了剪切变形的影响,导致其求解得到的自振频率偏大.一阶剪切变形梁理论可考虑剪切刚度的影响.宋丽红等<sup>[3]</sup>利用一阶剪切变形梁理论和微分求积的方法求解了两端固支四层梁的自由振动频率.基于 Timoshenko 梁理论、线性 zigzag 函数和 Reissner 混合变分原理,贺丹和杨万里<sup>[4]</sup>研究了二层、三层、四层和五层简支梁中轴向位移、挠度、正应力、剪应力和前 5 阶振动频率的变化规律.但一阶剪切变形梁理论会出现剪切自锁问题,需引入剪切修正因子.高阶剪切变形梁理论可充分考虑剪切变形的影响,同时不出现剪切自锁问题.惠维维等<sup>[5]</sup>采用抛物线型剪切变形理论和 Ritz 法开展了复合材料层合梁的模态分析,并揭示了铺层方式和铺设角度对频率分布的影响.Nguyen 等<sup>[6]</sup>运用高阶剪切变形理论、三角级数和 Lagrange 方程方法研究了多种边界条件下,复合梁中位移、应力、振动频率和屈曲荷载的变化规律.但高阶剪切变形梁理论求解相对复杂,降低计算效率.同时,学者们开发了锯齿(zig-zag)理论和有限元方法求解复合梁的力学响应.Kapurja 等<sup>[7]</sup>采用 zig-zag 理论给出了复合梁中挠度、应力、自振频率、振型和屈曲应变的解,并将其与一阶和三阶剪切变形理论的结果进行对比.杨胜奇等<sup>[8]</sup>基于新型锯齿理论模型分析了三层、四层、八层、二十五层、五十层和一百层复合梁的静力弯曲响应.陈玲俐等<sup>[9]</sup>在 ABAQUS 中采用 Shell 单元建立了以碳纤维/环氧树脂为构成材料的简支复合梁有限元力学模型,并展示了其前 6 阶振型,同时分析了铺层方式、角度与数目对自由振动响应的影响.基于四阶 zig-zag 理论,Garg 和 Chalak<sup>[10]</sup>提出了三节点有限元模型并分析了复合梁中位移、应力、自振频率和各阶振型的分布规律.为了求解更为复杂的多段梁、悬臂 L 型细长梁以及 Euler-Bernoulli 双曲梁的动力响应,学者们分别采用了谱几何法<sup>[11]</sup>、微分求积法<sup>[12]</sup>和 Laplace 变换法<sup>[13]</sup>.由以上分析可知,需开发一种不出现剪切自锁现象的精确高效计算复合梁自由振动频率的数值算法.

比例边界有限元方法(SBFEM)求解梁结构的振动频率时仅需离散平行于  $x$  轴的纵向线,将梁简化为一维模型,并且位移沿梁厚方向的变化规律可解析表示.采用线性谱单元对梁进行网格划分,并且仅以  $x$  和  $z$  方向的线位移为基本未知量.通过比例边界坐标和虚功原理将弹性力学偏微分基本方程简化为易于求解的二阶常微分控制方程,对位移沿梁厚方向的变化模式不采取任何假设并且不需要剪切修正因子.利用对偶变量方法,二阶方程进一步降阶为一阶常微分矩阵方程,其通解为解析的矩阵指数函数.运用 Padé 级数求解,可得高精度的动力刚度方程.作为一种半解析的数值算法,比例边界有限元方法引起了众多学者的关注,现已在层状地基<sup>[14-16]</sup>、板<sup>[17-19]</sup>、黏弹性<sup>[20]</sup>和断裂<sup>[21]</sup>等领域取得了一系列研究成果,同时用于计算开孔板<sup>[22]</sup>、复合压电板<sup>[23]</sup>、层状磁电弹板<sup>[24]</sup>、功能梯度板<sup>[25]</sup>、弹性梁<sup>[26]</sup>以及壳<sup>[27]</sup>等结构的自由振动频率.

本文将比例边界有限元方法进一步拓展用于计算复合梁的自由振动频率.仅采用两个谱单元对梁进行离散,提高了计算效率.通过 Padé 级数,首先得到了任意梁层对应于转角和位移自由度的动力刚度矩阵;其

次,运用转换矩阵实现自由度变换,得到了对应于线性位移自由度的动力刚度矩阵;最后,根据自由度匹配原则,将各梁层的动力刚度矩阵组装,得到了复合梁的整体刚度和质量矩阵,通过求解特征值方程进而得到了复合梁的自由振动频率。

## 1 复合梁的控制方程

本节将推导复合梁的控制方程.复合梁的长度为  $l$ , 横截面的宽度为  $b$ , 总厚度为  $t$ , 如图 1 所示.运用比例边界有限元方法计算复合梁的自由振动频率时,仅以  $x$  和  $z$  方向的线位移  $u_x = u_x(x, z)$  和  $u_z = u_z(x, z)$  为基本未知量.为了便于分析,我们将弹性位移组合为向量  $\mathbf{u} = \mathbf{u}(x, z) = [u_z \quad u_x]^T$ .

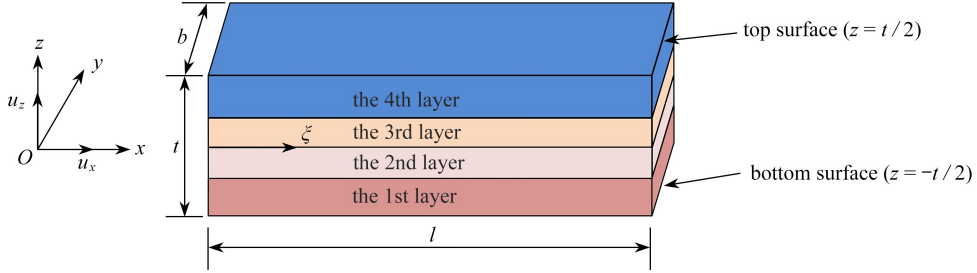


图 1 四层复合梁示意图

Fig. 1 The 4 layered composite beam

以复合梁中的第  $i$  个梁层为研究对象,其几何方程表示为

$$\varepsilon_{zz}^i = \frac{\partial u_z^i}{\partial z}, \quad \varepsilon_{xx}^i = \frac{\partial u_x^i}{\partial x}, \quad \gamma_{xz}^i = \frac{\partial u_z^i}{\partial x} + \frac{\partial u_x^i}{\partial z}, \quad (1)$$

式中,  $u_z^i$  和  $u_x^i$  为第  $i$  个梁层  $z$  和  $x$  方向的弹性位移,  $\varepsilon_{zz}^i$  和  $\varepsilon_{xx}^i$  为第  $i$  个梁层的正应变,  $\gamma_{xz}^i$  为第  $i$  个梁层的剪应变。

引入微分算子  $\mathbf{L}(z, x)$ :

$$\mathbf{L}(z, x) = \begin{bmatrix} \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \end{bmatrix}^T. \quad (2)$$

式(1)简化为

$$\boldsymbol{\varepsilon}^i = [\varepsilon_{zz}^i \quad \varepsilon_{xx}^i \quad \gamma_{xz}^i]^T = \mathbf{L}(z, x) \mathbf{u}^i, \quad (3)$$

其中  $\mathbf{u}^i = [u_z^i \quad u_x^i]^T$ .

第  $i$  个梁层中应力与应变的关系为

$$\boldsymbol{\sigma}^i = [\sigma_{zz}^i \quad \sigma_{xx}^i \quad \tau_{xz}^i]^T = \begin{bmatrix} D_{33}^i & D_{13}^i & 0 \\ D_{13}^i & D_{11}^i & 0 \\ 0 & 0 & D_{66}^i \end{bmatrix} \boldsymbol{\varepsilon}^i = \mathbf{D}^i \boldsymbol{\varepsilon}^i, \quad (4)$$

式中  $\mathbf{D}^i$  为第  $i$  个梁层对应的弹性矩阵。

第  $i$  个梁层无外力作用时,其动力平衡方程表示为

$$\frac{\partial \sigma_{xx}^i}{\partial x} + \frac{\partial \tau_{xz}^i}{\partial z} + \omega^2 \rho^i u_x^i = 0, \quad (5)$$

$$\frac{\partial \tau_{xz}^i}{\partial x} + \frac{\partial \sigma_{zz}^i}{\partial z} + \omega^2 \rho^i u_z^i = 0, \quad (6)$$

式中  $\rho^i$  为第  $i$  个梁层的密度。

利用微分算子  $\mathbf{L}(z, x)$ , 第  $i$  个梁层的平衡方程(5)、(6)简化为

$$\mathbf{L}^T(z, x)\boldsymbol{\sigma}^i + \omega^2 \rho^i \mathbf{u}^i = \mathbf{0}, \tag{7}$$

比例边界有限元方法中引入局部坐标  $\xi$ , 且仅采用两个谱单元对梁进行离散, 四阶谱单元如图 2 所示. 谱单元中积分点与网格节点重合, 可使系数矩阵简化为对角或块对角矩阵, 减少存储量, 提高计算效率. 梁中任意位置处的横坐标可通过节点向量  $\mathbf{x} = \{x_1 \ x_2 \ \dots \ x_m\}$  和 Lagrange 形函数  $\mathbf{N}^* = \mathbf{N}^*(\xi) = [N_1(\xi) \ N_2(\xi) \ \dots \ N_m(\xi)]$  插值得到

$$x(\xi) = \mathbf{N}^* \mathbf{x}, \tag{8}$$

其中参数  $m$  为网格节点数目, 四阶谱单元时  $m = 5$ .

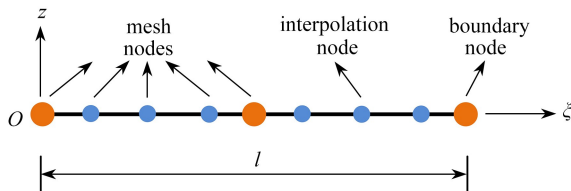


图 2 四阶谱单元

Fig. 2 The 4th order spectral element

式(8)中插值形函数  $N_k(\xi)$  ( $k = 1, 2, 3, \dots, m$ ) 的表达式为

$$N_k(\xi) = \frac{(\xi - \xi_1)(\xi - \xi_2) \dots (\xi - \xi_{k-1})(\xi - \xi_{k+1}) \dots (\xi - \xi_{m-1})(\xi - \xi_m)}{(\xi_k - \xi_1)(\xi_k - \xi_2) \dots (\xi_k - \xi_{k-1})(\xi_k - \xi_{k+1}) \dots (\xi_k - \xi_{m-1})(\xi_k - \xi_m)}, \tag{9}$$

式中  $\xi_k$  为节点  $x_k$  对应的局部坐标.

利用式(8), 比例边界坐标  $\xi$  和整体坐标  $x$  之间的变换式为

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x} \frac{\partial x}{\partial \xi} = \frac{\partial}{\partial x} x_{, \xi} = \mathbf{J}(\xi) \frac{\partial}{\partial x}, \tag{10}$$

$$\frac{\partial}{\partial x} = \frac{1}{|\mathbf{J}(\xi)|} \frac{\partial}{\partial \xi} = \frac{1}{|\mathbf{J}|} \frac{\partial}{\partial \xi}, \tag{11}$$

式(10)中下标“,”表示对变量进行偏导运算,  $x_{, \xi} = \frac{\partial x}{\partial \xi}$  表示  $x$  对局部坐标  $\xi$  的一阶偏导数.

基于式(11), 局部坐标系下微分算子  $\mathbf{L}(z, \xi)$  表示为

$$\mathbf{L}(z, \xi) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial}{\partial z} + \frac{1}{|\mathbf{J}|} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial \xi} = \mathbf{b}^1 \frac{\partial}{\partial z} + \mathbf{b}^2 \frac{\partial}{\partial \xi}. \tag{12}$$

基于式(8)中的形函数, 第  $i$  个梁层中任意位置处的位移  $\{u^i(z, \xi)\}$  表示为

$$\mathbf{u}^i(z, \xi) = \begin{bmatrix} \mathbf{N}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{N}^* \end{bmatrix} \begin{bmatrix} \mathbf{u}_z^i(z) \\ \mathbf{u}_x^i(z) \end{bmatrix} = \mathbf{N} \mathbf{u}^i(z). \tag{13}$$

将式(13)分别代入式(3)和(4), 并结合式(12)的微分运算关系, 分别得到式(14)和(15):

$$\boldsymbol{\varepsilon}^i = \mathbf{b}^1 \mathbf{N} \mathbf{u}^i(z)_{, z} + \mathbf{b}^2 \mathbf{N}_{, \xi} \mathbf{u}^i(z) = \mathbf{B}^1 \mathbf{u}^i(z)_{, z} + \mathbf{B}^2 \mathbf{u}^i(z), \tag{14}$$

$$\boldsymbol{\sigma}^i = \mathbf{D}^i (\mathbf{B}^1 \mathbf{u}^i(z)_{, z} + \mathbf{B}^2 \mathbf{u}^i(z)), \tag{15}$$

其中  $\mathbf{u}^i(z)_{, z}$  表示  $\mathbf{u}^i(z)$  对坐标  $z$  的一阶偏导数,  $\mathbf{N}_{, \xi}$  表示形函数矩阵  $\mathbf{N}$  对坐标  $\xi$  的一阶偏导数.

运用虚功原理推导第  $i$  个梁层的比例边界有限元控制方程:

$$\delta W^i - \delta U^i = 0, \tag{16}$$

其中  $\delta W^i = \int_l (\delta \mathbf{u}^i)^T \mathbf{T}^i dx + \int_V (\delta \mathbf{u}^i)^T \omega^2 \rho^i \mathbf{u}^i dV$  为第  $i$  个梁层的外力虚功,  $\delta U^i = \int_V (\delta \boldsymbol{\varepsilon}^i)^T \boldsymbol{\sigma}^i dV$  为其内力虚功.

根据梁层边界条件, 推导得到第  $i$  个梁层自由振动问题的控制方程(具体推导过程从略, 可参考文献[17])如下:

$$\mathbf{E}^{i0} \mathbf{u}^i(z)_{, zz} + ((\mathbf{E}^{i1})^T - \mathbf{E}^{i1}) \mathbf{u}^i(z)_{, z} + (\omega^2 \mathbf{M}^{i0} - \mathbf{E}^{i2}) \mathbf{u}^i(z) = \mathbf{0}, \tag{17}$$

其中  $\mathbf{u}^i(z)_{,zz}$  表示  $\mathbf{u}^i(z)$  对坐标  $z$  的二阶偏导数.

式(17)中系数矩阵  $\mathbf{E}^{i0}$ ,  $\mathbf{E}^{i1}$ ,  $\mathbf{E}^{i2}$  和  $\mathbf{M}^{i0}$  的表达式分别为

$$\mathbf{E}^{i0} = \int_{-1}^1 (\mathbf{B}^1)^T \mathbf{D}^i \mathbf{B}^1 |J| b d\xi, \quad (18)$$

$$\mathbf{E}^{i1} = \int_{-1}^1 (\mathbf{B}^2)^T \mathbf{D}^i \mathbf{B}^1 |J| b d\xi, \quad (19)$$

$$\mathbf{E}^{i2} = \int_{-1}^1 (\mathbf{B}^2)^T \mathbf{D}^i \mathbf{B}^2 |J| b d\xi, \quad (20)$$

$$\mathbf{M}^{i0} = \int_{-1}^1 \mathbf{N}^T \rho^i \mathbf{N} |J| b d\xi. \quad (21)$$

由于梁层被划分为两个单元,可分别求解每个单元对应的系数矩阵,然后仿照有限元方法将单元系数矩阵组装,最终得到控制方程中的整体系数矩阵.从式(17)可以看出,本文算法具有与 Galerkin 法和有限元等方法相同的能力,实现了将弹性梁的偏微分方程简化为二阶常微分方程,降低了问题的求解难度.

## 2 复合梁的整体刚度和质量矩阵

通过变量  $\mathbf{X}^i(z) = [\mathbf{u}^i(z) \quad \mathbf{q}^i(z)]^T$ , 式(17)进一步简化为一阶常微分方程:

$$\mathbf{X}^i(z)_{,z} = - \begin{bmatrix} (\mathbf{E}^{i0})^{-1} (\mathbf{E}^{i1})^T & - (\mathbf{E}^{i0})^{-1} \\ \omega^2 \mathbf{M}^{i0} - \mathbf{E}^{i2} + \mathbf{E}^{i1} (\mathbf{E}^{i0})^{-1} (\mathbf{E}^{i1})^T & - \mathbf{E}^{i1} (\mathbf{E}^{i0})^{-1} \end{bmatrix} \mathbf{X}^i(z) = - \mathbf{Z}^i \mathbf{X}^i(z), \quad (22)$$

其中作为  $\mathbf{u}^i(z)$  对偶变量的  $\mathbf{q}^i(z)$  是第  $i$  个梁层的内部节点力向量,

$$\mathbf{q}^i(z) = \mathbf{E}^{i0} \mathbf{u}^i(z)_{,z} + (\mathbf{E}^{i1})^T \mathbf{u}^i(z). \quad (23)$$

方程(22)的通解为解析的矩阵指数函数:

$$\mathbf{X}^i(z) = e^{-\mathbf{Z}^i z} \mathbf{c}^i, \quad (24)$$

式中  $\mathbf{c}^i$  为积分常数向量,从后续推导可知由梁层底面处的位移和节点力确定.

将厚度为  $t_i$  的第  $i$  个梁层顶面和底面处的竖向坐标  $z = t_i$  和  $z = 0$  代入式(24),可得

$$\begin{Bmatrix} \mathbf{u}^i(0) \\ \mathbf{q}^i(0) \end{Bmatrix} = \begin{Bmatrix} \mathbf{u}_B^i \\ \mathbf{q}_B^i \end{Bmatrix} = \begin{Bmatrix} \mathbf{u}_B^i \\ -\mathbf{F}_B^i \end{Bmatrix} = \begin{Bmatrix} \mathbf{c}_1^i \\ \mathbf{c}_2^i \end{Bmatrix}, \quad (25)$$

$$\begin{Bmatrix} \mathbf{u}^i(t_i) \\ \mathbf{q}^i(t_i) \end{Bmatrix} = \begin{Bmatrix} \mathbf{u}_T^i \\ \mathbf{q}_T^i \end{Bmatrix} = \begin{Bmatrix} \mathbf{u}_T^i \\ \mathbf{F}_T^i \end{Bmatrix} = e^{-\mathbf{Z}^i t_i} \begin{Bmatrix} \mathbf{c}_1^i \\ \mathbf{c}_2^i \end{Bmatrix}. \quad (26)$$

式(25)和(26)中梁层顶面位移  $\mathbf{u}_T^i$  和底面位移  $\mathbf{u}_B^i$  的表达式为  $\mathbf{u}_T^i = [ \{ \mathbf{u}_{T_z}^i \} \quad \{ \mathbf{u}_{T_x}^i \} ]^T$  和  $\mathbf{u}_B^i = [ \{ \mathbf{u}_{B_z}^i \} \quad \{ \mathbf{u}_{B_x}^i \} ]^T$ ;  $\mathbf{F}_T^i = [ \{ \mathbf{F}_{T_z}^i \} \quad \{ \mathbf{F}_{T_x}^i \} ]^T$  和  $\mathbf{F}_B^i = [ \{ \mathbf{F}_{B_z}^i \} \quad \{ \mathbf{F}_{B_x}^i \} ]^T$  分别为第  $i$  个梁层顶面和底面处由外荷载产生的节点力向量.

本文选用二阶 Padé 级数求解式(26)中的矩阵指数函数<sup>[18]</sup>:

$$e^{-\mathbf{Z}^i t_i} \approx \mathbf{Q}^i(t_i)^{-1} \mathbf{P}^i(t_i) = (\mathbf{Q}^i)^{-1} \mathbf{P}^i, \quad (27)$$

$$\mathbf{Q}^i = \begin{bmatrix} \mathbf{Q}_{11}^i & \mathbf{Q}_{12}^i \\ \mathbf{Q}_{21}^i & \mathbf{Q}_{22}^i \end{bmatrix} = \mathbf{I} + \frac{1}{2} t_i \mathbf{Z}^i + t_i^2 \mathbf{V}^i, \quad (28)$$

$$\mathbf{P}^i = \begin{bmatrix} \mathbf{P}_{11}^i & \mathbf{P}_{12}^i \\ \mathbf{P}_{21}^i & \mathbf{P}_{22}^i \end{bmatrix} = \mathbf{I} - \frac{1}{2} t_i \mathbf{Z}^i + t_i^2 \mathbf{V}^i, \quad (29)$$

其中  $\mathbf{V}^i = \frac{1}{12} (\mathbf{Z}^i)^2$ .

根据式(25)–(27),第  $i$  个梁层顶面和底面处位移与外力的关系为

$$\begin{bmatrix} \mathbf{P}_{11}^i & -\mathbf{Q}_{11}^i \\ \mathbf{P}_{21}^i & -\mathbf{Q}_{21}^i \end{bmatrix} \begin{Bmatrix} \mathbf{u}_B^i \\ \mathbf{u}_T^i \end{Bmatrix} = \begin{bmatrix} \mathbf{P}_{12}^i & \mathbf{Q}_{12}^i \\ \mathbf{P}_{22}^i & \mathbf{Q}_{22}^i \end{bmatrix} \begin{Bmatrix} \mathbf{F}_B^i \\ \mathbf{F}_T^i \end{Bmatrix}. \quad (30)$$

为了便于刚度和质量矩阵的推导,将自由度  $[\mathbf{u}_B^i \quad \mathbf{u}_T^i]^T$  变换为转角和位移自由度  $[\boldsymbol{\theta}^i \quad \bar{\mathbf{u}}^i]^T$ ,如图 3 所示。

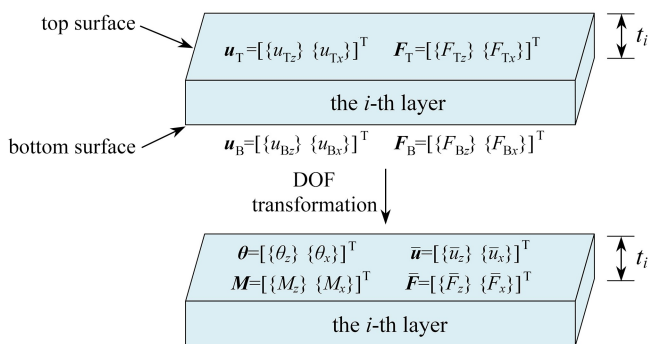


图 3 梁层模型图

Fig. 3 The model for the beam layer

变换矩阵  $\mathbf{T}_r^i$  和其逆的转置表达式为

$$\mathbf{T}_r^i = \begin{bmatrix} -\frac{t_i}{2} \mathbf{I} & \mathbf{I} \\ \frac{t_i}{2} \mathbf{I} & \mathbf{I} \end{bmatrix}, \quad (31)$$

$$(\mathbf{T}_r^i)^{-T} = \begin{bmatrix} -\frac{1}{t_i} \mathbf{I} & \frac{1}{2} \mathbf{I} \\ \frac{1}{t_i} \mathbf{I} & \frac{1}{2} \mathbf{I} \end{bmatrix}, \quad (32)$$

式中  $\mathbf{I}$  为单位矩阵。

利用式(31),位移向量  $[\mathbf{u}_B^i \quad \mathbf{u}_T^i]^T$  与转角和位移  $[\boldsymbol{\theta}^i \quad \bar{\mathbf{u}}^i]^T$  的关系式为

$$\begin{bmatrix} \frac{1}{t_i}(\mathbf{u}_T^i - \mathbf{u}_B^i) \\ \frac{1}{2}(\mathbf{u}_T^i + \mathbf{u}_B^i) \end{bmatrix} = \begin{bmatrix} \frac{1}{t_i}(\{u_{Tx}^i\} - \{u_{Bx}^i\}) \\ \frac{1}{2}(\{u_{Tx}^i\} + \{u_{Bx}^i\}) \\ \frac{1}{t_i}(\{u_{Tz}^i\} - \{u_{Bz}^i\}) \\ \frac{1}{2}(\{u_{Tz}^i\} + \{u_{Bz}^i\}) \end{bmatrix} = \begin{bmatrix} \{\theta_x^i\} \\ \{\theta_z^i\} \\ \{\bar{u}_x^i\} \\ \{\bar{u}_z^i\} \end{bmatrix} = [\boldsymbol{\theta}^i \quad \bar{\mathbf{u}}^i]^T = (\mathbf{T}_r^i)^{-1} \begin{bmatrix} \mathbf{u}_B^i \\ \mathbf{u}_T^i \end{bmatrix}. \quad (33)$$

同时对应于转角和位移自由度的荷载向量表示为

$$\begin{bmatrix} \frac{t_i}{2}(\mathbf{F}_T^i - \mathbf{F}_B^i) \\ \mathbf{F}_T^i + \mathbf{F}_B^i \end{bmatrix} = \begin{bmatrix} \frac{t_i}{2}(\{F_{Tx}^i\} - \{F_{Bx}^i\}) \\ \{F_{Tx}^i\} + \{F_{Bx}^i\} \\ \frac{t_i}{2}(\{F_{Tz}^i\} - \{F_{Bz}^i\}) \\ \{F_{Tz}^i\} + \{F_{Bz}^i\} \end{bmatrix} = \begin{bmatrix} \{M_x^i\} \\ \{M_z^i\} \\ \{\bar{F}_x^i\} \\ \{\bar{F}_z^i\} \end{bmatrix} = [\bar{\mathbf{M}}^i \quad \bar{\mathbf{F}}^i]^T = (\mathbf{T}_r^i)^T \begin{bmatrix} \mathbf{F}_B^i \\ \mathbf{F}_T^i \end{bmatrix}. \quad (34)$$

自由度变换后经推导得到第 *i* 个梁层的动力刚度方程(详细推导过程可参考文献[26]):

$$(\mathbf{K}^i - \omega^2 \mathbf{M}^i) \begin{bmatrix} \boldsymbol{\theta}^i \\ \bar{\mathbf{u}}^i \end{bmatrix} = \mathbf{S}^i(\omega) \begin{bmatrix} \boldsymbol{\theta}^i \\ \bar{\mathbf{u}}^i \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{M}}^i \\ \bar{\mathbf{F}}^i \end{bmatrix}, \quad (35)$$



表 1 梁两端处的约束情况

Table 1 The constraint conditions at ends of the beam

constraint condition	$x = 0$	$x = l$
SS	$u_z(0, z) = 0$	$u_z(l, z) = 0$
CS	$u_z(0, z) = u_x(0, z) = 0$	$u_z(l, z) = 0$
CC	$u_z(0, z) = u_x(0, z) = 0$	$u_z(l, z) = u_x(l, z) = 0$
CF	$u_z(0, z) = u_x(0, z) = 0$	free

### 3.1 (0°/90°/0°/90°) 四层梁

本例分析(0°/90°/0°/90°)四层简支梁自振频率的变化规律,如图 4 所示.复合梁中的 0°和 90°为纤维铺设角度,即材料纤维坐标系(1-2-3)中 1 主轴与直角坐标系中  $x$  轴的夹角,如图 5 中的角度  $\alpha$ ,文献[28]给出了纤维坐标系与直角坐标系下材料参数的变换关系.梁的总厚度为  $t$ ,每个梁层厚度相同,宽度与厚度相同即  $b = t$ ,选用 4 种长厚比  $l/t = 100, 20, 10$  和 5.梁的材料参数为  $E_1 = 181 \text{ GPa}$ ,  $E_2 = E_3 = 10.3 \text{ GPa}$ ,  $G_{12} = G_{13} = 7.17 \text{ GPa}$ ,  $G_{23} = 2.87 \text{ GPa}$ ,  $\nu_{12} = \nu_{13} = 0.25$ ,  $\nu_{23} = 0.33$ ,  $\rho = 1\,578 \text{ kg/m}^3$ .梁被离散为两个单元,不同阶次的谱单元如图 6 所示,2 至 7 阶谱单元对应的节点数目分别为 5, 7, 9, 11, 13 和 15.自振频率的计算结果按照式  $\omega_0 = (\omega l^2/t) \sqrt{\rho/Y_0}$  进行无量纲化,其中  $\rho = 1\,578 \text{ kg/m}^3$ ,  $Y_0 = 10.3 \text{ GPa}$ .表 2 和图 7 将比例边界有限元的频率计算结果与二维弹性解<sup>[7]</sup>进行对比,可以看出各种长厚比下,本文结果与弹性解均能完全吻合.并且由表 2 可知,随着谱单元的阶次由 2 阶升高到 7 阶,即离散节点数由 5 升至 15,比例边界有限元解快速收敛于弹性解,同时 2 至 7 阶谱单元的计算时间分别为 0.120 1 s, 0.132 7 s, 0.140 9 s, 0.159 2 s, 0.164 0 s 和 0.289 9 s,从而验证了比例边界有限元法可准确高效计算复合梁的自由振动频率.

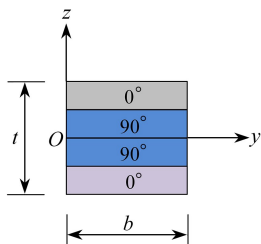


图 4 (0°/90°/0°/90°) 四层复合梁

Fig. 4 The 4-layered (0°/90°/0°/90°) beam

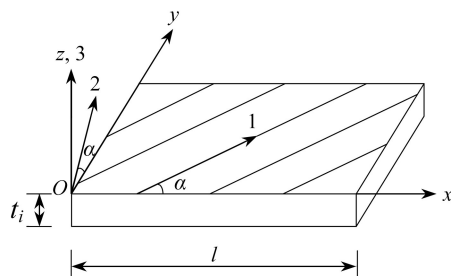


图 5 材料纤维坐标系(1-2-3)与直角坐标系

Fig. 5 The fiber-matrix coordinate system (1-2-3) and the Cartesian coordinate system

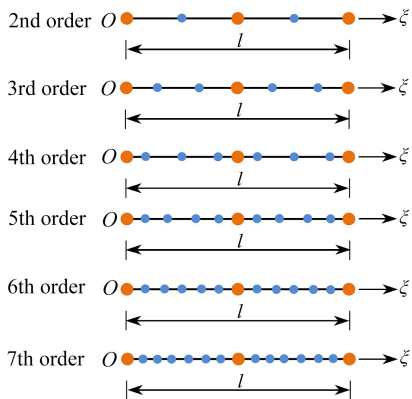


图 6 梁各阶谱单元

Fig. 6 The different orders of spectral elements

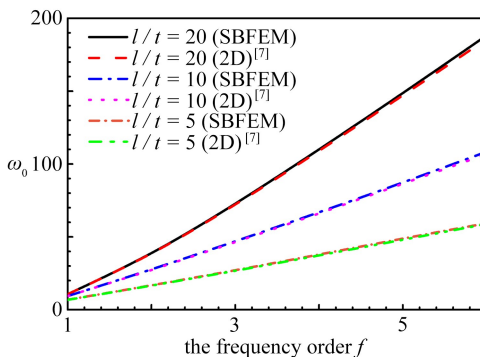


图 7 (0°/90°/0°/90°) 四层复合梁的振动频率

Fig. 7 The vibration frequencies of the 4-layered (0°/90°/0°/90°) composite beam



表2  $l/t = 100$  时( $0^\circ/90^\circ/0^\circ/90^\circ$ )四层梁的振动频率

Table 2 The natural frequencies of the 4-layered ( $0^\circ/90^\circ/0^\circ/90^\circ$ ) beam with  $l/t = 100$

element order	1st frequency	2nd frequency	3rd frequency	4th frequency	5th frequency	6th frequency
2nd order	12.309 2	44.215 7	505.889 6	956.705 8	2 118.235 8	2 708.157 8
3rd order	11.252 5	48.945 4	115.329 1	184.133 7	958.889 6	1 005.616 1
4th order	11.212 9	44.560 2	101.600 6	211.074 6	328.678 5	431.467 9
5th order	11.212 9	44.571 8	99.420 0	173.882 4	280.551 6	498.252 5
6th order	11.212 9	44.561 7	99.217 6	174.345 3	269.073 7	380.223 2
7th order	11.212 9	44.561 7	99.206 1	173.834 0	267.054 3	377.217 1
2D <sup>[7]</sup>	11.193 0	44.477 0	98.988 0	173.390 0	266.010 0	374.910 0
error $\delta / \%$	0.177 8	0.190 5	0.220 4	0.256 1	0.392 6	0.615 4

### 3.2 三层简支夹层梁

本例求解三层简支夹层梁的自由振动频率,如图8所示.夹层梁面层和核心层的厚度分别为  $0.1t$  和  $0.8t$ , 面层材料参数为  $E_1 = 131.1 \text{ GPa}$ ,  $E_2 = E_3 = 6.9 \text{ GPa}$ ,  $G_{12} = 3.588 \text{ GPa}$ ,  $G_{13} = 3.088 \text{ GPa}$ ,  $G_{23} = 2.332 \text{ GPa}$ ,  $\nu_{12} = \nu_{13} = 0.32$ ,  $\nu_{23} = 0.49$ ,  $\rho = 1\,000 \text{ kg/m}^3$ ;核心层材料参数为  $E_1 = 0.220 \text{ MPa}$ ,  $E_2 = 0.220 \text{ MPa}$ ,  $E_3 = 2\,760 \text{ MPa}$ ,  $G_{12} = 16.56 \text{ MPa}$ ,  $G_{13} = 545.1 \text{ MPa}$ ,  $G_{23} = 455.4 \text{ MPa}$ ,  $\nu_{12} = 0.99$ ,  $\nu_{13} = 0.000\,03$ ,  $\nu_{23} = 0.000\,03$ ,  $\rho = 70 \text{ kg/m}^3$ .横截面宽度与总厚度相等,并考虑4种长厚比  $l/t = 100, 20, 10$  和  $5$  的影响.频率根据式  $\omega_0 = (\omega l^2/t) \sqrt{\rho/Y_0}$  表示为无量纲的形式,其中  $\rho = 1\,000 \text{ kg/m}^3$ ,  $Y_0 = 6.9 \text{ GPa}$ .将利用比例边界有限元方法计算得到的前六阶频率与文献[7]的参考解进行对比,如图9所示.计算结果表明,比例边界有限元算法具有较高的精度,对于薄梁  $l/t = 100$  和中厚梁  $l/t = 5$  比例边界有限元解均与参考解基本一致,说明比例边界有限元方法具有广泛适用性,可精确求解薄梁和厚梁的固有频率.

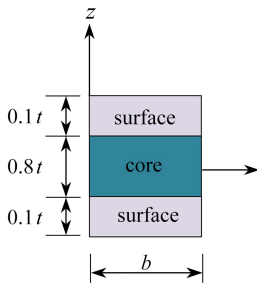


图8 三层夹层梁

Fig. 8 The 3-layered sandwich beam

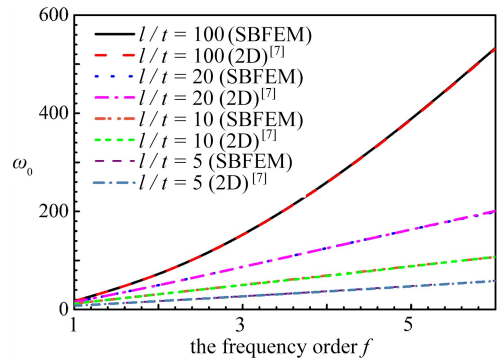


图9 三层简支夹层梁振动频率

Fig. 9 The eigenfrequencies of the 3-layered SS sandwich beam

### 3.3 ( $0^\circ/90^\circ/0^\circ$ ) 三层梁

本例研究不同边界约束下( $0^\circ/90^\circ/0^\circ$ )三层梁自由振动频率的分布规律,如图10所示.3个梁层的厚度均为  $t/3$ , 选择3种长厚比为  $l/t = 50, 30$  和  $20$ .梁层的材料参数为  $E_1 = 40E_2$ ,  $E_2 = E_3$ ,  $G_{12} = G_{13} = 0.6E_2$ ,  $G_{23} = 0.5E_2$ ,  $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$ .振动频率按照  $\omega_0 = (\omega l^2/t) \sqrt{\rho/E_2}$  进行无量纲化,比例边界有限元的计算结果见表3,表中还列出了文献[6]利用高阶剪切变形理论得到的结果.由表3可知,比例边界有限元解与文献解吻合较好,并且3种边界条件下( $0^\circ/90^\circ/0^\circ$ )复合梁的自振频率均会随着长厚比的增加而增大.

### 3.4 ( $0^\circ/90^\circ/0^\circ/90^\circ/0^\circ/90^\circ/0^\circ/90^\circ/0^\circ/90^\circ$ ) 十层梁

为了进一步展示比例边界有限元算法对复杂梁结构的适用性,本例以文献[29]中的( $0^\circ/90^\circ/0^\circ/90^\circ/0^\circ/90^\circ/0^\circ/90^\circ/0^\circ/90^\circ$ )十层梁为研究对象.梁的边界条件分别为 SS、CS、CC 和 CF,其相应的约束情况见表1.每个梁层的厚度均相同,且宽度  $b$  与总厚度  $t$  相等,分析两种长厚比  $l/t = 5$  和  $10$  下振动频率的变化规律.

梁层的材料参数为  $E_1/E_2 = 40, E_2 = E_3, G_{12} = G_{13} = 0.6E_2, G_{23} = 0.5E_2, \nu_{12} = \nu_{23} = \nu_{13} = 0.25$ . 比例边界有限元方法计算得到的自振频率见表 4, 其中频率根据式  $\omega_0 = (\omega l^2/t) \sqrt{\rho/E_2}$  表示为无量纲的形式. 由表可知本文解与文献[29]的解基本一致, 从而比例边界有限元算法求解十层复合梁自由振动频率的适用性和准确性得到了验证. 从表中可以看出, 相同长厚比时, 两端固支梁的频率值最大, 然后依次为一端固支一端简支梁、简支梁和悬臂梁. 同时将图 7 与表 4 进行对比可知, 当  $l/t$  和边界约束相同时, 十层梁的振动频率要大于四层梁的固有频率.

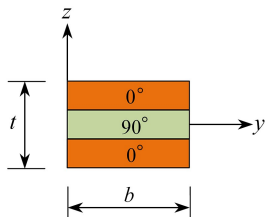


图 10 (0°/90°/0°) 三层复合梁

Fig. 10 The 3-layered (0°/90°/0°) composite beam

表 3 三层梁(0°/90°/0°)振动频率

Table 3 Eigensolutions of the (0°/90°/0°) beam

$l/t$		SS	CF	CC
50	SBFEM	17.480	6.270	37.808
	ref. [6]	17.462	6.267	37.679
	error $\delta$ / %	0.107	0.053	0.343
30	SBFEM	17.103	6.205	34.511
	ref. [6]	17.055	6.198	34.268
	error $\delta$ / %	0.281	0.119	0.709
20	SBFEM	16.432	6.084	29.982
	ref. [6]	16.338	6.070	29.695
	error $\delta$ / %	0.5780	0.234	0.967

表 4 十层梁的振动频率

Table 4 The eigenvalues of the 10-layered beam

$l/t$		SS	CS	CC	CF
10	SBFEM	10.862	14.058	17.245	4.184
	ref. [29]	10.893	14.221	17.580	4.197
	error $\delta$ / %	-0.288	-1.147	-1.905	-0.316
5	SBFEM	8.118	9.199	10.541	3.497
	ref. [29]	8.156	9.356	10.784	3.527
	error $\delta$ / %	-0.467	-1.675	-2.250	-0.856

### 3.5 阶梯型悬臂梁

为充分展示本文算法的准确性, 以图 11 中的阶梯型悬臂梁为例, 将比例边界有限元方法求解得到的振动频率与实验实测值进行对比. 为了便于阶梯梁的网格划分, 将其分为四个部分, 因此图 11 中的阶梯梁可视为由部分 1、2、3 和 4 组成的复合梁. 阶梯梁的弹性参数为  $E = 150 \text{ GPa}, \nu = 0.3, \rho = 7\,634.4 \text{ kg/m}^3$ , 梁截面的宽度  $b = 3 \text{ cm}$ . 阶梯型悬臂梁前 4 阶自振频率的计算结果如表 5 所示, 表中同时展示了实测值<sup>[30]</sup>和复合单元法 (composite element method, CEM)<sup>[30]</sup>的计算结果. 由表 5 可知, 与复合单元法的结果相比, 比例边界有限元解更加接近实验实测值, 从而本文算法的正确性再次得到了验证.

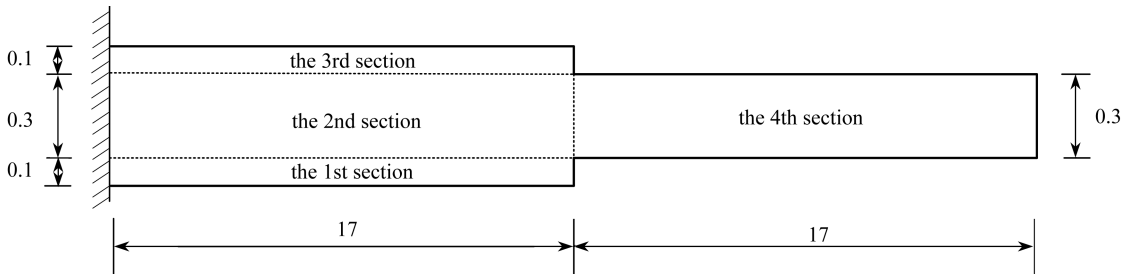


图 11 阶梯型悬臂梁(单位: cm)

Fig. 11 The step-shaped cantilever beam (unit: mm)

表 5 阶梯型悬臂梁的振动频率

Table 5 Vibration frequencies of the step-shaped cantilever beam

	SBFEM	experiment <sup>[30]</sup>	CEM <sup>[30]</sup>
1st frequency	36.224 9	39.06	36.595
2nd frequency	146.580 7	138.67	148.13
3rd frequency	428.683 8	417.48	433.75
4th frequency	788.257 6	776.37	798.28

## 4 结 论

本文基于比例边界有限元方法,提出了一种高精度计算复合梁自由振动频率的数值算法.首先,利用比例边界坐标和虚功原理,将弹性力学偏微分方程简化为易于求解的二阶常微分方程;运用由内部节点力和位移组成的对偶变量,二阶方程进一步降阶为一阶常微分矩阵方程,其解为解析的矩阵指数函数;然后,采用 Padé 级数方法求解矩阵指数函数,得到任意梁层的动力刚度方程;最后,根据相邻梁层交界面处的自由度匹配原则,将各梁层组装,得到复合梁的整体刚度和质量矩阵,求解特征方程,进而得到复合梁的自由振动频率.与其他方法类似,本文算法对复合梁的层数和边界条件均没有限制,对位移沿梁厚方向的变化模式不采取任何假设并且不需要剪切修正因子.通过三层、四层、十层复合梁和阶梯型悬臂梁固有频率的计算,本文算法的精确性、高效性和快速收敛性得到了验证.同时计算结果表明,长厚比、边界条件和梁层数目对复合梁振动频率的分布有重要影响,因此设计复合梁结构时,需予以重点关注.

## 参考文献(References):

- [1] SAYYAD A S, GHUGAL Y M. Bending, buckling and free vibration of laminated composite and sandwich beams: a critical review of literature[J]. *Composite Structures*, 2017, **171**: 486-504.
- [2] 杨坤, 张玮, 杜度. 复合材料夹层结构动力学特性研究进展[J]. *玻璃钢/复合材料*, 2019, **9**: 110-118. (YANG Kun, ZHANG Wei, DU Du. The research progress of dynamic characteristics of the composite sandwich structure[J]. *Fibre Reinforced Plastics/Composites*, 2019, **9**: 110-118. (in Chinese))
- [3] 宋丽红, 陈殿云, 张传敏. 层合梁自由振动的微分求积分析[J]. *河南科技大学学报(自然科学版)*, 2005, **26**(2): 89-92. (SONG Lihong, CHEN Dianyun, ZHANG Chuanmin. Free vibration analysis of laminated beam by differential quadrature[J]. *Journal of Henan University of Science and Technology (Natural Science)*, 2005, **26**(2): 89-92. (in Chinese))
- [4] 贺丹, 杨万里. 基于广义变分原理和锯齿理论的高精度层合梁模型[J]. *宇航总体技术*, 2017, **1**(2): 26-32. (HE Dan, YANG Wanli. A high-accuracy composite laminated beam model based on generalized variational principle and Zigzag theory[J]. *Astronautical Systems Engineering Technology*, 2017, **1**(2): 26-32. (in Chinese))
- [5] 惠维维, 韩宾, 张钱城, 等. 基于一种简化剪切变形理论的层合梁自由振动分析[J]. *应用力学学报*, 2017, **34**(6): 1067-1071. (HUI Weiwei, HAN Bin, ZHANG Qiancheng, et al. Free vibration analysis of laminated composite beams based on a simplified shear deformation theory[J]. *Chinese Journal of Applied Mechanics*,

- 2017, **34**(6): 1067-1071. (in Chinese))
- [6] NGUYEN T K, NGUYEN N D, VO T P, et al. Trigonometric-series solution for analysis of laminated composite beams[J]. *Composite Structures*, 2017, **160**: 142-151.
- [7] KAPURIA S, DUMIR P C, JAIN N K. Assessment of zigzag theory for static loading, buckling, free and forced response of composite and sandwich beams[J]. *Composite Structures*, 2004, **64**(3/4): 317-327.
- [8] 杨胜奇, 张永存, 刘书田. 一种准确预测层合梁结构层间剪应力的新锯齿理论[J]. 航空学报, 2019, **40**(11): 223028. (YANG Shengqi, ZHANG Yongcun, LIU Shutian. A new zig-zag theory for accurately predicting inter-laminar shear stress of laminated beam structures[J]. *Acta Aeronautica et Astronautica Sinica*, 2019, **40**(11): 223028. (in Chinese))
- [9] 陈玲俐, 赵晓昱, 张鑫, 等. 复合材料简支梁的模式分析[J]. 农业装备与车辆工程, 2020, **58**(4): 128-130. (CHEN Lingli, ZHAO Xiaoyu, ZHANG Xin, et al. Modal analysis of composite simply supported beam[J]. *Agricultural Equipment & Vehicle Engineering*, 2020, **58**(4): 128-130. (in Chinese))
- [10] GARG A, CHALAK H D. Novel higher-order zigzag theory for analysis of laminated sandwich beams[J]. *Proceedings of the Institution of Mechanical Engineers (Part L): Journal of Materials; Design and Applications*, 2021, **235**(1): 176-194.
- [11] 鲍四元, 周静, 陆健炜. 任意弹性边界的多段梁自由振动研究[J]. 应用数学和力学, 2020, **41**(9): 985-993. (BAO Siyuan, ZHOU Jing, LU Jianwei. Free vibration of multi-segment beams with arbitrary boundary conditions[J]. *Applied Mathematics and Mechanics*, 2020, **41**(9): 985-993. (in Chinese))
- [12] 李智超, 郝育新. 微分求积法求解悬臂 L 梁固有振动特性研究[J]. 应用数学和力学, 2023, **44**(5): 525-534. (LI Zhichao, HAO Yuxin. Study on natural vibration characteristics of L-shaped cantilever beams with the differential quadrature method[J]. *Applied Mathematics and Mechanics*, 2023, **44**(5): 525-534. (in Chinese))
- [13] 赵翔, 孟诗瑶. 基于 Green 函数分析 Euler-Bernoulli 双曲梁系统的受迫振动[J]. 应用数学和力学, 2023, **44**(2): 168-177. (ZHAO Xiang, MENG Shiyao. Forced vibration analysis of Euler-Bernoulli double-beam systems by means of Green's functions[J]. *Applied Mathematics and Mechanics*, 2023, **44**(2): 168-177. (in Chinese))
- [14] 韩泽军, 林皋. 三维层状地基动力刚度矩阵连分式算法[J]. 大连理工大学学报, 2012, **52**(6): 862-869. (HAN Zejun, LIN Gao. A continued-fraction algorithm for dynamic stiffness matrix of foundation located or embedded in three-dimensional layered subgrade[J]. *Journal of Dalian University of Technology*, 2012, **52**(6): 862-869. (in Chinese))
- [15] 韩泽军, 林皋, 钟红. 改进的比例边界有限元法求解层状地基动力刚度矩阵[J]. 水电能源科学, 2012, **30**(7): 100-104. (HAN Zejun, LIN Gao, ZHONG Hong. Modified scaled boundary finite element solution for dynamic stiffness matrix of laminar foundation[J]. *Water Resources and Power*, 2012, **30**(7): 100-104. (in Chinese))
- [16] 张海廷, 杨林青, 郭芳. 基于 SBFEM 的层状地基埋置管道动力响应求解与分析[J]. 岩土力学, 2019, **40**(7): 2713-2722. (ZHANG Haiting, YANG Linqing, GUO Fang. Solution and analysis of dynamic response for rigid pipe in multi-layered soil based on SBFEM[J]. *Rock and Soil Mechanics*, 2019, **40**(7): 2713-2722. (in Chinese))
- [17] MAN H, SONG C, GAO W, et al. A unified 3D-based technique for plate bending analysis using scaled boundary finite element method[J]. *International Journal for Numerical Methods in Engineering*, 2012, **91**(5): 491-515.
- [18] MAN H, SONG C, XIANG T, et al. High-order plate bending analysis based on the scaled boundary finite element method[J]. *International Journal for Numerical Methods in Engineering*, 2013, **95**(4): 331-360.
- [19] 林皋, 张鹏冲. 板结构计算模型的新发展[J]. 计算力学学报, 2019, **36**(4): 429-440. (LIN Gao, ZHANG Pengchong. New development of the computational model for the analysis of laminated plate structures[J]. *Chinese Journal of Computational Mechanics*, 2019, **36**(4): 429-440. (in Chinese))
- [20] 何宜谦, 王霄腾, 祝雪峰, 等. 求解粘弹性问题的时域自适应等几何比例边界有限元法[J]. 工程力学, 2020, **37**(2): 23-33. (HE Yiqian, WANG Xiaoteng, ZHU Xuefeng, et al. A temporally piecewise adaptive isogeometric

- SBFEM for viscoelastic problems[J]. *Engineering Mechanics*, 2020, **37**(2): 23-33. (in Chinese))
- [21] 钟红, 贺帅, 李红军. 基于比例边界有限元的温度断裂研究[J]. 水电与抽水蓄能, 2017, **3**(3): 66-70. (ZHONG Hong, HE Shuai, LI Hongjun. Thermal fracture analysis based on the polygon scaled boundary finite element method[J]. *Hydropower and Pumped Storage*, 2017, **3**(3): 66-70. (in Chinese))
- [22] SUN X, ZHANG P C, QIAO H, et al. High-order free vibration analysis of elastic plates with multiple cutouts [J]. *Archive of Applied Mechanics*, 2021, **91**: 1837-1858.
- [23] ZHANG P C, QI C Z, FANG H Y, et al. Bending and free vibration analysis of laminated piezoelectric composite plates[J]. *Structural Engineering and Mechanics*, 2020, **75**(6): 747-769.
- [24] ZHANG P C, QI C Z, FANG H Y, et al. Semi-analytical analysis of static and dynamic responses for laminated magneto-electro-elastic plates[J]. *Composite Structures*, 2019, **222**: 110933.
- [25] XIANG T S, NATARAJAN S, MAN H, et al. Free vibration and mechanical buckling of plates with in-plane material inhomogeneity: a three dimensional consistent approach[J]. *Composite Structures*, 2014, **118**: 634-642.
- [26] LI J H, SHI Z Y, NING S W. A two-dimensional consistent approach for static and dynamic analyses of uniform beams[J]. *Engineering Analysis With Boundary Elements*, 2017, **82**: 1-16.
- [27] LI J H, SHI Z Y, LIU L. A scaled boundary finite element method for static and dynamic analyses of cylindrical shells[J]. *Engineering Analysis With Boundary Elements*, 2019, **98**: 217-231.
- [28] KANT T, SWAMINATHAN K. Analytical solutions for the static analysis of laminated composite and sandwich plates based on a higher order refined theory[J]. *Composite Structures*, 2002, **56**(4): 329-344.
- [29] KHDEIR A A, REDDY J N. Free vibration of cross-ply laminated beams with arbitrary boundary conditions [J]. *International Journal of Engineering Science*, 1994, **32**(12): 1971-1980.
- [30] 谢瑾荣, 周翠英, 程晔, 等. 一种求解多阶梯悬臂梁自由振动问题的新方法 with 实验验证[J]. 工程抗震与加固改造, 2012, **34**(4): 52-55. (XIE Jinrong, ZHOU Cuiying, CHENG Ye, et al. A new method for solving free vibration of cantilever beam with multiple steps and its experimental validation[J]. *Earthquake Resistant Engineering and Retrofitting*, 2012, **34**(4): 52-55. (in Chinese))