基于辛叠加方法的正交各向异性矩形悬臂薄板受迫振动解析解*

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（我刊编委李锐来稿）

摘要：　基于辛叠加方法研究了正交各向异性矩形悬臂薄板在谐载载荷作用下的受迫振动问题.首先从薄板受迫振动的基本方程出发，将问题导入到 Hamilton 体系，并将原问题拆分为若干子问题，然后在辛空间中利用分离变量和本征展开方法推导出子问题的解析解，最后通过叠加求解出悬臂薄板受迫振动的解析解.辛叠加方法的主要优点是经过逐步严格推导获得解析解，不需要对解的形式做任何假设，突破了传统半逆解法的限制.算例针对不同谐载载荷情况进行了数值计算，并将该文方法与有限元方法获得的结果进行比较，验证了该文方法的可靠性和精确性.

关　键　词：　辛叠加方法；　正交各向异性薄板；　解析解；　受迫振动

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Analytical Forced Vibration Solutions of Orthotropic Cantilever Rectangular Thin Plates With the Symplectic Superposition Method

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Abstract: The forced vibrations of orthotropic cantilever rectangular thin plates under harmonic loadings were investigated with the symplectic superposition method. The basic equations for the forced vibration of thin plates were introduced into the Hamiltonian system. The original problem was divided into some fundamental subproblems, and the analytical solutions of the subproblems were derived with the method of separation of variables and through eigenvector expansion in the symplectic space. The solution of the original problem was finally obtained by superposition. The main advantage of the symplectic superposition method is that the analyti-

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cal solution can be obtained by step-by-step rigorous derivation, without any assumptions on the form of the solution, which breaks through the limitations of traditional semi-inverse methods. The numerical results calculated corresponding to different harmonic loads were compared with those obtained via the finite element method to verify the reliability and accuracy of the proposed method.

Key words: symplectic superposition method; orthotropic thin plate; analytical solution; forced vibration

0 引　　言

板作为一种典型的结构元件,广泛应用于建筑、飞机、火箭、船舶等装备设施.与传统的各向同性板相比,各向异性板由于具有较高的比刚度和比强度,其应用更为广泛.对于许多在动载荷服役条件下的装备设施,板的动力问题成为其设计与研究的关键,如超高层建筑的楼面板抗震性能研究、桥梁板面在车辆荷载作用下的受迫振动等.

求解板振动的数值方法众多,例如有限元法(FEM)[1-3]、有限差分法[4]、边界元法[5-6]、无网格法[7-10]、有限条法[11-13]、等几何法[14]、Galerkin 法[15-16]等,它们无疑可以获得工程上可接受的板问题的数值解,但这并未动摇解析解的地位.由于能够精确反映各参量之间的关联,解析解可以作为检验各类数值解的基准,能为快速参数分析和优化提供有力的工具,也是高效指导实验设计重要的理论基础.然而,众所周知,由于解析求解难度较大,目前关于薄板受迫振动的解析解研究并不多见.除了经典的 Navier 法和 Lévy 法[17]之外,付宝连和李农[18-20]将功的互等法推广于求解简谐载荷下矩形板的受迫振动问题;Xing 和 Liu[21]使用分离变量法求解了正交各向异性薄板的自由振动问题;陈英杰等[22-23]应用混合变量最小原理求解了集中谐载作用下薄板和中厚板的受迫振动问题等.

针对许多解析方法仅适用于某些特定边界约束下的板壳问题这一缺憾,笔者等近年来提出了一种新的解析方法:辛叠加方法,能够广泛求解具有非 Lévy 型边界条件的板壳问题,包括弯曲[24-25]、振动[26-27]和屈曲[28-29]等.基于该方法,本文直接从正交各向异性薄板理论的基本方程出发,将薄板的受迫振动问题导入到 Hamilton 体系中,推导出了矩形悬臂薄板受迫振动问题的解析解,通过计算实例验证了所得解的正确性.本文所构造的 Hamilton 对偶方程形式简洁,求解方便.由于求解过程直接从薄板受迫振动问题的基本方程出发,通过逐步严格的数学推导求出问题的解析解,无需采用像传统半逆法等人为选定解的形式,由此使问题的求解更加严密.限于篇幅,本文只聚焦悬臂板这一类传统上公认较难处理的问题,但本文的求解思路对于其他任意边界条件下的板也都是适用的.

1 Hamiltion 体系的导入

如图 1 所示的一个正交各向异性矩形薄板,其中面位于平面 xOy 上,且板的正交主方向与坐标轴方向相同.规定板的长度为 a, 宽度为 b, 厚度为 h, 密度为 ρ_0.

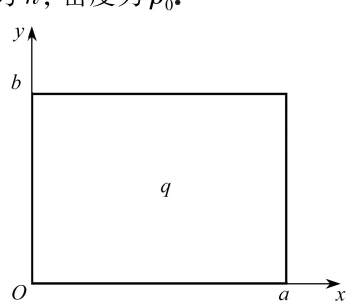


图 1 正交各向异性矩形薄板示意图

Fig. 1 Schematic diagram of an orthotropic rectangular thin plate

正交各向异性薄板受迫振动的控制方程为[30]

$$D_x \frac{\partial^4 \bar{w}(x,y,t)}{\partial x^4} + 2H \frac{\partial^4 \bar{w}(x,y,t)}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 \bar{w}(x,y,t)}{\partial y^4} - \rho \frac{\partial^2 \bar{w}(x,y,t)}{\partial t^2} = \bar{q}_t(x,y,t), \tag{1}$$

式中 D_x 和 D_y 分别为板在弹性主向的弯曲刚度; $H = D_1 + 2D_{xy}$ 为板的有效扭转刚度,其中 $D_1 = \upsilon_2 D_x = \upsilon_1 D_y$ 是通过 Poisson 比 υ_1 和 υ_2 来定义的; D_{xy} 为板的扭转刚度; $\rho = \rho_0 h$ 是板每单位面积的质量; $\bar{q}_t(x,y,t)$ 为板承受的动力荷载.

在简谐干扰力作用下,忽略阻尼,令

$$\bar{w}(x,y,t) = w(x,y)\sin(\omega t),\ \bar{q}_t(x,y,t) = q(x,y)\sin(\omega t). \tag{2}$$

将式(2)代入式(1),可得薄板稳态解振幅 $w(x,y)$ 满足的基本方程为

$$D_x \frac{\partial^4 w(x,y)}{\partial x^4} + 2H\frac{\partial^4 w(x,y)}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w(x,y)}{\partial y^4} + \rho\omega^2 w(x,y) = q(x,y), \tag{3}$$

式中 ω 是动力荷载的谐载频率.

由经典正交各向异性薄板理论可知,薄板受迫振动稳态解的平衡方程为

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0, \tag{4a}$$

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0, \tag{4b}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \rho\omega^2 w + q = 0, \tag{4c}$$

内力公式为

$$M_x = -\left(D_x \frac{\partial^2 w}{\partial x^2} + D_1 \frac{\partial^2 w}{\partial y^2}\right), \tag{5a}$$

$$M_y = -\left(D_y \frac{\partial^2 w}{\partial y^2} + D_1 \frac{\partial^2 w}{\partial x^2}\right), \tag{5b}$$

$$M_{xy} = M_{yx} = -2D_{xy}\frac{\partial^2 w}{\partial x \partial y}, \tag{5c}$$

$$Q_x = -\frac{\partial}{\partial x}\left(D_x \frac{\partial^2 w}{\partial x^2} + H\frac{\partial^2 w}{\partial y^2}\right), \tag{6a}$$

$$Q_y = -\frac{\partial}{\partial y}\left(D_y \frac{\partial^2 w}{\partial y^2} + H\frac{\partial^2 w}{\partial x^2}\right), \tag{6b}$$

$$V_x = Q_x + \frac{\partial M_{xy}}{\partial y}, \tag{7a}$$

$$V_y = Q_y + \frac{\partial M_{xy}}{\partial x}, \tag{7b}$$

其中 $M_x, M_y, M_{xy}, M_{yx}, Q_x, Q_y, V_x, V_y$ 分别为薄板的弯矩、扭矩、剪力和等效剪力.

由式(4c)、式(7a)和(7b)可得

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \rho\omega^2 w + q = 0. \tag{8}$$

引入

$$\frac{\partial w}{\partial y} = \theta, \tag{9}$$

则式(5b)可写成

$$\frac{\partial \theta}{\partial y} = \frac{\partial^2 w}{\partial y^2} = -\frac{D_1}{D_y}\frac{\partial^2 w}{\partial x^2} - \frac{M_y}{D_y}, \tag{10}$$

式(5c)可写成

$$M_{xy} = -2D_{xy}\frac{\partial \theta}{\partial x}. \tag{11}$$

由式(4c)、(5b)、(5c)、(6a)和式(7a)得

$$\frac{\partial V_y}{\partial y} = \left(\frac{D_x D_y - D_1^2}{D_y}\right)\frac{\partial^4 w}{\partial x^4} - \frac{D_1}{D_y}\frac{\partial^2 M_y}{\partial x^2} - \rho\omega^2 w - q. \tag{12}$$

由式(4b)、(7b)和式(11)得

$$\frac{\partial M_y}{\partial y} = V_y + 4D_{xy}\frac{\partial^2 \theta}{\partial x^2}. \tag{13}$$

令 $V_y = - T$, 则式(9)、(10)、(12)和式(13)可写成矩阵形式:

$$\frac{\partial \boldsymbol{Z}}{\partial y} = \boldsymbol{HZ} + \boldsymbol{f}, \tag{14}$$

其中 $\boldsymbol{Z} = [w,\theta,T,M_y]^{\mathrm{T}}$ 为板的状态向量, $\boldsymbol{f} = [0,0,q,0]^{\mathrm{T}}$ 为板的外力向量, $\boldsymbol{H} = \begin{bmatrix} \boldsymbol{F} & -\boldsymbol{G} \\ -\boldsymbol{Q} & -\boldsymbol{F}^{\mathrm{T}} \end{bmatrix}$, $\boldsymbol{F} = \begin{bmatrix} 0 & 1 \\ -(D_1/D_y)\partial^2/\partial x^2 & 0 \end{bmatrix}$, $\boldsymbol{G} = \begin{bmatrix} 0 & 0 \\ 0 & 1/D_y \end{bmatrix}$, $\boldsymbol{Q} = \begin{bmatrix} (D_x - D_1^2/D_y)\partial^4/\partial x^4 - \rho\omega^2 & 0 \\ 0 & -4D_{xy}\partial^2/\partial x^2 \end{bmatrix}$.矩阵 $\boldsymbol{H}$ 满足 $\boldsymbol{H}^{\mathrm{T}} = \boldsymbol{JHJ}$, 是一个 Hamilton 算子矩阵,其中 $\boldsymbol{J} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_2 \\ -\boldsymbol{I}_2 & \boldsymbol{0} \end{bmatrix}$ 为单位辛矩阵,这里 $\boldsymbol{I}_2$ 是二阶单位矩阵.式(14)即为正交各向异性薄板受迫振动稳态解问题的 Hamilton 对偶方程.

2 正交各向异性矩形悬臂薄板受迫振动问题求解

对于矩形悬臂薄板的受迫振动问题,令 $y = b$ 边固支而其他边自由,如图 2 所示,其中“F”代表自由,“C”代表固支,“SC”代表滑支(对应等效剪力和转角均为零的边界条件),“S”代表简支.该问题可以由以下三部分叠加而成,如图 2 所示.

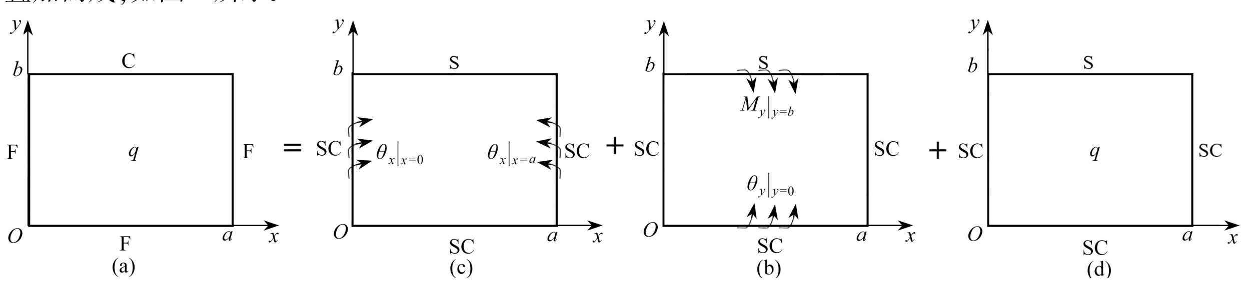


图 2 矩形悬臂薄板受迫振动问题的辛叠加示意图

Fig. 2 Symplectic superposition of the forced vibration of a cantilever rectangular thin plate

2.1 一边滑支对边简支矩形薄板的辛解析解

对于正交各向异性矩形悬臂薄板受迫振动问题,运用辛叠加方法,先求解第一类子问题——以一边滑支对边简支为基底的辛解析解,如图 2(b)所示.在方程(14)中交换 x 和 y,a 和 b 以及 D_x 和 D_y, 得到其 Hamilton 对偶方程:

$$\frac{\partial \boldsymbol{Z}'}{\partial x} = \boldsymbol{H}'\boldsymbol{Z}' + \boldsymbol{f}, \tag{15}$$

其中 $\boldsymbol{Z}' = [w,\theta_x,T_x,M_x]^{\mathrm{T}}$ 为板的状态向量, $\boldsymbol{f} = [0,0,q,0]^{\mathrm{T}}$ 为板的外力向量, $\boldsymbol{H}' = \begin{bmatrix} \boldsymbol{F}' & -\boldsymbol{G}' \\ -\boldsymbol{Q}' & -\boldsymbol{F}'^{\mathrm{T}} \end{bmatrix}$, $\boldsymbol{F}' = \begin{bmatrix} 0 & 1 \\ -(D_1/D_x)\partial^2/\partial y^2 & 0 \end{bmatrix}$, $\boldsymbol{G}' = \begin{bmatrix} 0 & 0 \\ 0 & 1/D_x \end{bmatrix}$, $\boldsymbol{Q}' = \begin{bmatrix} (D_y - D_1^2 D_x)\partial^4/\partial y^4 - \rho\omega^2 & 0 \\ 0 & -4D_{xy}\partial^2/\partial y^2 \end{bmatrix}$.

方程(15)的齐次方程为

$$\frac{\partial \boldsymbol{Z}'}{\partial x} = \boldsymbol{H}'\boldsymbol{Z}'. \tag{16}$$

在辛空间求解框架下,分离变量法有效[31],为此令

$$\boldsymbol{Z}' = \boldsymbol{Y}'(y)X'(x), \tag{17}$$

其中 $\boldsymbol{Y}'(y) = [w(y), \theta_x(y), T_x(y), M_x(y)]^{\mathrm{T}}$.

将式(17)代入式(16),得

$$\frac{\mathrm{d}X'(x)}{\mathrm{d}x} = \xi X'(x), \tag{18}$$

以及本征方程

$$\boldsymbol{H}'\boldsymbol{Y}'(y) = \xi \boldsymbol{Y}'(y), \tag{19}$$

其中 ξ 为方程的本征值, $\boldsymbol{Y}'(y)$ 为对应的本征向量.

与式(19)对应的特征方程为

$$\begin{vmatrix} -\xi & 1 & 0 & 0 \\ -\dfrac{D_1}{D_x}\lambda^2 & -\xi & 0 & -\dfrac{1}{D_x} \\ \dfrac{D_1^2 - D_x D_y}{D_x}\lambda^4 + \rho\omega^2 & 0 & -\xi & \dfrac{D_1}{D_x}\lambda^2 \\ 0 & 4D_{xy}\lambda^2 & -1 & -\xi \end{vmatrix} = \boldsymbol{0}. \tag{20}$$

展开式(20),注意 $H = D_1 + 2D_{xy}$, 得到

$$D_y\lambda^4 + 2H\xi^2\lambda^2 + D_y\xi^4 = \rho\omega^2. \tag{21}$$

令

$$R_y = \omega\sqrt{\rho/D_y}, \tag{22}$$

得到方程(21)的根为

$$\lambda_{1,2} = \pm\gamma_1 \mathrm{i},\ \lambda_{3,4} = \pm\gamma_2, \tag{23}$$

其中 $\gamma_1 = \sqrt{\sqrt{(D_{HX}^2 - D_{XY})\xi^4 + R_y^2} + D_{HY}\xi^2}$, $\gamma_2 = \sqrt{\sqrt{(D_{HX}^2 - D_{XY})\xi^4 + R_y^2} - D_{HY}\xi^2}$, $D_{HY} = H/D_y$, $D_{HX} = H/D_x$, $D_{XY} = D_x/D_y$.

由此,可以写出方程(19)中 $w(y)$ 的本征解为

$$w(y) = A'\cos(\gamma_1 y) + B'\sin(\gamma_1 y) + C'\cosh(\gamma_2 y) + F'\sinh(\gamma_2 y), \tag{24}$$

其中 A', B', C' 和 F' 为待定常数.

对于 $y = 0$ 边滑支、$y = b$ 边简支的矩形薄板,其 y 方向的边界条件为

$$\left.\frac{\partial w}{\partial y}\right|_{y=0} = V_y\big|_{y=0} = 0,\ w\big|_{y=b} = M_y\big|_{y=b} = 0. \tag{25}$$

将式(24)代入式(25)中,要满足式(24)中常数有非零值,令所得方程组的系数矩阵行列式为0,即得到

$$\cos(\gamma_1 b)\cosh(\gamma_2 b) = 0. \tag{26}$$

解得本征值为

$$\begin{cases} \xi_{\pm n1} = \pm\sqrt{D_{HX}\beta_n^2 - \sqrt{(D_{HX}^2 - D_{YX})\beta_n^4 + D_{YX}R_y^2}}, \\ \xi_{\pm n2} = \pm\sqrt{D_{HX}\beta_n^2 + \sqrt{(D_{HX}^2 - D_{YX})\beta_n^4 + D_{YX}R_y^2}}, \end{cases} \tag{27}$$

其中 $\beta_n = n\pi/(2b)$ $(n = 1, 3, 5, \cdots)$, $D_{YX} = D_y/D_x$. 与本征值 $\xi_{\pm n1}$ 和 $\xi_{\pm n2}$ 对应的本征向量为

$$\boldsymbol{Y}'_{\pm n1}(y) = [1, \xi_{\pm n1}, \xi_{\pm n1}[D_x\xi_{\pm n1}^2 - (D_1 + 4D_{xy})\beta_n^2], D_1\beta_n^2 - D_x\xi_{\pm n1}^2]^{\mathrm{T}}\cos(\beta_n y), \tag{28}$$

$$\boldsymbol{Y}'_{\pm n2}(y) = [1, \xi_{\pm n2}, \xi_{\pm n2}[D_x\xi_{\pm n2}^2 - (D_1 + 4D_{xy})\beta_n^2], D_1\beta_n^2 - D_x\xi_{\pm n2}^2]^{\mathrm{T}}\cos(\beta_n y), \tag{29}$$

其满足共轭辛正交关系[2].

非齐次方程(15)的解可写成

$$\boldsymbol{Z}' = \boldsymbol{Y}'(y)\boldsymbol{X}'(x), \tag{30}$$

其中 $\boldsymbol{Y}'(y)=[\boldsymbol{Y}'_{n1}(y),\boldsymbol{Y}'_{n2}(y),\cdots,\boldsymbol{Y}'_{-n1}(y),\boldsymbol{Y}'_{-n2}(y),\cdots]$，$\boldsymbol{X}'(x)=[X'_{n1}(x),X'_{n2}(x),\cdots,X'_{-n1}(x),X'_{-n2}(x),\cdots]^{\mathrm{T}}(n=1,3,5,\cdots)$.

将式(30)代入式(15)，得到

$$\boldsymbol{Y}'(y)\frac{\mathrm{d}\boldsymbol{X}'(x)}{\mathrm{d}x}=\boldsymbol{H}'\boldsymbol{Y}'(y)\boldsymbol{X}'(x)+\boldsymbol{f},\tag{31}$$

注意到

$$\boldsymbol{H}'\boldsymbol{Y}'(y)=\boldsymbol{Y}'(y)\boldsymbol{M}',\tag{32}$$

其中 $\boldsymbol{M}'=\mathrm{diag}(\xi_{n1},\xi_{n2},\cdots,\xi_{-n1},\xi_{-n2},\cdots)$，而式(31)中的非齐次项 $\boldsymbol{f}$ 可以按辛本征向量展开.令

$$\boldsymbol{f}=\boldsymbol{Y}'(y)\boldsymbol{G}',\tag{33}$$

其中 $\boldsymbol{G}'=[g'_{n1},g'_{n2},\cdots,g'_{-n1},g'_{-n2},\cdots]^{\mathrm{T}}$ 为展开系数矩阵，各元素可通过对式(33)两边同时左乘 $(\boldsymbol{Y}'(y))^{\mathrm{T}}\boldsymbol{J}$ 并关于 y 从0到 b 积分，由共轭辛正交关系求出.将式(32)和式(33)代入式(31)得到

$$\frac{\mathrm{d}\boldsymbol{X}'(x)}{\mathrm{d}x}=\boldsymbol{M}'\boldsymbol{X}'(x)+\boldsymbol{G}'.\tag{34}$$

展开上式，即可求出 $X'_{n1}(x)$，$X'_{n2}(x)$，$X'_{-n1}(x)$ 和 $X'_{-n2}(x)$.

对于承受均布谐载幅值为 $q(x,y)=p$ 的薄板，可得

$$\begin{cases}X'_{n1}(x)=A'_{n1}\mathrm{e}^{\xi_{n1}x}-\dfrac{2p(-1)^{n+1}}{n\pi D_x\xi_{n1}^2(\xi_{n1}^2-\xi_{n2}^2)},\\ X'_{n2}(x)=B'_{n1}\mathrm{e}^{\xi_{n2}x}-\dfrac{2p(-1)^{n+1}}{n\pi D_x\xi_{n2}^2(\xi_{n2}^2-\xi_{n1}^2)},\\ X'_{-n1}(x)=C'_{n1}\mathrm{e}^{-\xi_{n1}x}-\dfrac{2p(-1)^{n+1}}{n\pi D_x\xi_{n1}^2(\xi_{n1}^2-\xi_{n2}^2)},\\ X'_{-n2}(x)=D'_{n1}\mathrm{e}^{-\xi_{n2}x}-\dfrac{2p(-1)^{n+1}}{n\pi D_x\xi_{n2}^2(\xi_{n2}^2-\xi_{n1}^2)}.\end{cases}\tag{35}$$

对于在 (x_0,y_0) 处作用一幅值为 p_0 的集中谐载的板，可得

$$\begin{cases}X'_{n1}(x)=A'_{n2}\mathrm{e}^{\xi_{n1}x}+\dfrac{\mathrm{e}^{\xi_{n1}(x-x_0)}p_0\mathrm{H}(x-x_0)}{bD_x\xi_{n1}(\xi_{n1}^2-\xi_{n2}^2)}\cos(\beta_ny_0),\\ X'_{n2}(x)=B'_{n2}\mathrm{e}^{\xi_{n2}x}+\dfrac{\mathrm{e}^{\xi_{n2}(x-x_0)}p_0\mathrm{H}(x-x_0)}{bD_x\xi_{n2}(\xi_{n2}^2-\xi_{n1}^2)}\cos(\beta_ny_0),\\ X'_{-n1}(x)=C'_{n2}\mathrm{e}^{-\xi_{n1}x}-\dfrac{\mathrm{e}^{-\xi_{n1}(x-x_0)}p_0\mathrm{H}(x-x_0)}{bD_x\xi_{n1}(\xi_{n1}^2-\xi_{n2}^2)}\cos(\beta_ny_0),\\ X'_{-n2}(x)=D'_{n2}\mathrm{e}^{-\xi_{n2}x}-\dfrac{\mathrm{e}^{-\xi_{n2}(x-x_0)}p_0\mathrm{H}(x-x_0)}{bD_x\xi_{n2}(\xi_{n2}^2-\xi_{n1}^2)}\cos(\beta_ny_0),\end{cases}\tag{36}$$

其中 $\mathrm{H}(x-x_0)$ 为单位阶跃函数，A'_{n1}，B'_{n1}，C'_{n1}，D'_{n1} 及 A'_{n2}，B'_{n2}，C'_{n2}，D'_{n2} 为待求常数，由板在 x 方向的边界条件决定.

于是，由状态向量表示的一边滑支对边简支矩形薄板的受迫振动的解可以写成

$$\boldsymbol{Z}'=\sum_{n=1,3,5,\cdots}^{\infty}[X'_{n1}(x)\boldsymbol{Y}'_{n1}(y)+X'_{n2}(x)\boldsymbol{Y}'_{n2}(y)+X'_{-n1}(x)\boldsymbol{Y}'_{-n1}(y)+X'_{-n2}(x)\boldsymbol{Y}'_{-n2}(y)],\tag{37}$$

其中 $X'_{n1}(x)$，$X'_{n2}(x)$，$X'_{-n1}(x)$，$X'_{-n2}(x)$，$\boldsymbol{Y}'_{n1}(y)$，$\boldsymbol{Y}'_{n2}(y)$，$\boldsymbol{Y}'_{-n1}(y)$ 和 $\boldsymbol{Y}'_{-n2}(y)$ 等已在式(28)、(29)、(35)或式(36)中解析给出.

以均布谐载为例，对于 $y=0$ 边滑支和 $y=b$ 边简支而 $x=0$ 边和 $x=a$ 边滑支的板，x 方向的边界条件为

$$\left.\frac{\partial w}{\partial x}\right|_{x=0,a}=V_x\big|_{x=0,a}=0.\tag{38}$$

将状态向量(37)中的结果代入式(38),求出常数 $A'_{n1}, B'_{n1}, C'_{n1}$ 和 D'_{n1}, 最终可以得到均布谐载作用下一边滑支对边简支正交各向异性矩形薄板受迫振动问题的辛解析解:

$$w_1(x,y) = -\frac{2p}{D_x b}\sum_{n=1,3,5,\cdots}^{\infty}\frac{(-1)^{n+1}\cos(\beta_n y)}{\beta_n(\xi_{n1}^2-\xi_{n2}^2)}\left(\frac{1}{\xi_{n1}^2}-\frac{1}{\xi_{n2}^2}\right). \tag{39}$$

对于第二个子问题,如图 2(c)所示,在 $x=0$ 滑支边和 $x=a$ 滑支边强加待定转角,于是 x 方向的边界条件为

$$V_x\big|_{x=0,a}=0,\ \frac{\partial w}{\partial x}\bigg|_{x=0}=\sum_{n=1,3,5,\cdots}^{\infty}G_n\cos(\beta_n y),\ \frac{\partial w}{\partial x}\bigg|_{x=a}=\sum_{n=1,3,5,\cdots}^{\infty}H_n\cos(\beta_n y), \tag{40}$$

其中 G_n 和 H_n 是待定展开系数.

在式(35)中令 $p=0$, 连同状态向量(37)中的结果代入式(40),即可求得由 G_n 和 H_n 表达的子问题二的解为

$$\begin{aligned} w_2(x,y) = {} & \sum_{n=1,3,5,\cdots}^{\infty}\frac{\cos(\beta_n y)}{D_x\xi_{n1}\xi_{n2}(\xi_{n1}^2-\xi_{n2}^2)}\operatorname{csch}(\xi_{n1}a)\operatorname{csch}(\xi_{n2}a)\ \times \\ & \{[\xi_{n1}\zeta_{nx1}\sinh(\xi_{n1}a)\cosh(\xi_{n2}(a-x))-\xi_{n2}\zeta_{nx2}\sinh(\xi_{n2}a)\cosh(\xi_{n1}(a-x))]G_n - \\ & [\xi_{n1}\zeta_{nx1}\sinh(\xi_{n1}a)\cosh(\xi_{n2}x)-\xi_{n2}\zeta_{nx2}\sinh(\xi_{n2}a)\cosh(\xi_{n1}x)]H_n\}, \end{aligned} \tag{41}$$

其中 $\zeta_{nx1}=(D_1+4D_{xy})\beta_n^2-D_x\xi_{n1}^2, \zeta_{nx2}=(D_1+4D_{xy})\beta_n^2-D_x\xi_{n2}^2(n=1,3,5,\cdots)$.

2.2　对边滑支矩形薄板的辛解析解

正交各向异性矩形悬臂薄板受迫振动问题求解的第三个子问题为以对边滑支为基底的薄板,如图 2(d)所示,方程(14)为其 Hamilton 对偶方程,对应的齐次方程为

$$\frac{\partial \boldsymbol{Z}}{\partial y}=\boldsymbol{HZ}. \tag{42}$$

令 $\boldsymbol{Z}=\boldsymbol{X}(x)Y(y)$, 其中 $\boldsymbol{X}(x)=[w(x),\theta_y(x),T_y(x),M_y(x)]^{\mathrm{T}}$, 可得到

$$\frac{\mathrm{d}Y(y)}{\mathrm{d}y}=\mu Y(y), \tag{43}$$

以及本征方程

$$\boldsymbol{HX}(x)=\mu\boldsymbol{X}(x), \tag{44}$$

其中 μ 为方程的本征值, $\boldsymbol{X}(x)$ 为对应的本征向量.同上一小节的思路,由本征方程(44)可给出

$$w(x)=A\cos(\alpha_1 x)+B\sin(\alpha_1 x)+C\cosh(\alpha_2 x)+F\sinh(\alpha_2 x), \tag{45}$$

其中 $\alpha_1=\sqrt{\sqrt{(D_{HX}^2-D_{YX})\mu^4+R_x^2}+D_{HX}\mu^2}, \alpha_2=\sqrt{\sqrt{(D_{HX}^2-D_{YX})\mu^4+R_x^2}-D_{HX}\mu^2}, R_x=\omega\sqrt{\rho/D_x}, A,B,C$ 和 F 为待定常数.

对于 $x=0$ 边和 $x=a$ 边滑支的矩形薄板,其 x 方向的边界条件为

$$\frac{\partial w}{\partial x}\bigg|_{x=0,a}=V_x\big|_{x=0,a}=0. \tag{46}$$

将式(45)代入式(46)中,要满足式(45)中待定常数有非零值,令所得方程组的系数矩阵行列式为 0,即得到

$$\sin(\alpha_1 a)\sinh(\alpha_2 a)=0. \tag{47}$$

解得本征值为

$$\begin{cases}\mu_{\pm m1}=\pm\sqrt{D_{HY}\alpha_m^2-\sqrt{(D_{HY}^2-D_{XY})\alpha_m^4+D_{XY}R_x^2}},\\ \mu_{\pm m2}=\pm\sqrt{D_{HY}\alpha_m^2+\sqrt{(D_{HY}^2-D_{XY})\alpha_m^4+D_{XY}R_x^2}},\end{cases} \tag{48}$$

其中 $\alpha_m=m\pi/a(m=0,1,2,3,\cdots)$.对应的本征向量为

$$\begin{cases}\boldsymbol{X}_{\pm m1}(x)=[1,\mu_{\pm m1},\mu_{\pm m1}[D_y\mu_{\pm m1}^2-(D_1+4D_{xy})\alpha_m^2],D_1\alpha_m^2-D_y\mu_{\pm m1}^2]^{\mathrm{T}}\cos(\alpha_m x),\\ \boldsymbol{X}_{\pm m2}(x)=[1,\mu_{\pm m2},\mu_{\pm m2}[D_y\mu_{\pm m2}^2-(D_1+4D_{xy})\alpha_m^2],D_1\alpha_m^2-D_y\mu_{\pm m2}^2]^{\mathrm{T}}\cos(\alpha_m x).\end{cases} \tag{49}$$

由共轭辛正交关系,状态向量可以表示为

$$\boldsymbol{Z} = \sum_{n=0,1,2,\cdots}^{\infty} [A_m e^{\mu_{m1}y}\boldsymbol{X}_{m1}(x) + B_m e^{\mu_{m2}y}\boldsymbol{X}_{m2}(x) + C_m e^{-\mu_{m1}y}\boldsymbol{X}_{-m1}(x) + D_m e^{-\mu_{m2}y}\boldsymbol{X}_{-m2}(x)], \tag{50}$$

其中 A_m, B_m, C_m 和 D_m 为待求常数.

在 $y=0$ 滑支边强加待定转角,在 $y=b$ 简支边强加待定弯矩,于是对应的边界条件为

$$V_y|_{y=0} = 0,\ \left.\frac{\partial w}{\partial y}\right|_{y=0} = \sum_{m=0,1,2,\cdots}^{\infty} E_m\cos(\alpha_m x),\ w|_{y=b} = 0,\ M_y|_{y=b} = \sum_{m=0,1,2,\cdots}^{\infty} F_m\cos(\alpha_m x), \tag{51}$$

其中 E_m 和 F_m 是待定展开系数.

将式(50)代入到式(51),可得到由 E_m 和 F_m 表达的子问题三的解:

$$w_3(x,y) = \sum_{m=0,1,2,\cdots}^{\infty} \frac{\cos(\alpha_m x)}{D_y\mu_{m1}\mu_{m2}(\mu_{m1}^2 - \mu_{m2}^2)}\operatorname{sech}(\mu_{m1}b)\operatorname{sech}(\mu_{m2}b) \times \{\{\mu_{m1}\zeta_{my1}\cosh(\mu_{m1}b)\sinh[\mu_{m2}(b-y)] - \mu_{m2}\zeta_{my2}\cosh(\mu_{m2}b)\sinh[\mu_{m1}(b-y)]\}E_m + \mu_{m1}\mu_{m2}[\cosh(\mu_{m1}b)\cosh(\mu_{m2}y) - \cosh(\mu_{m2}b)\cosh(\mu_{m1}y)]F_m\}, \tag{52}$$

其中 $\zeta_{my1} = (D_1 + 4D_{xy})\alpha_m^2 - D_y\mu_{m1}^2$, $\zeta_{my2} = (D_1 + 4D_{xy})\alpha_m^2 - D_y\mu_{m2}^2 (m = 0,1,2,\cdots)$.

2.3 正交各向异性矩形悬臂薄板受迫振动问题的辛解析解

以上两小节求解出了正交各向异性矩形悬臂薄板受迫振动问题的三个子问题的解,则悬臂薄板的挠度解为

$$w(x,y) = w_1(x,y) + w_2(x,y) + w_3(x,y). \tag{53}$$

对于如图 2(a)所示的悬臂薄板,其真实边界条件为

$$M_x|_{x=0,a} = V_x|_{x=0,a} = 0, \tag{54}$$

$$M_y|_{y=0} = V_y|_{y=0} = 0,\ w|_{y=b} = \theta_y|_{y=b} = 0. \tag{55}$$

在悬臂薄板的自由边界 $x=0$ 和 $x=a$ 处,子问题解之和已满足等效剪力 $V_x|_{x=0,a}=0$ 的条件,还需满足弯矩 M_x 为零,即

$$M_x|_{x=0,a} = \left.\left(D_x\frac{\partial^2 w}{\partial x^2} + D_1\frac{\partial^2 w}{\partial y^2}\right)\right|_{x=0,a} = \sum_{i=1}^{3}\left.\left(D_x\frac{\partial^2 w_i}{\partial x^2} + D_1\frac{\partial^2 w_i}{\partial y^2}\right)\right|_{x=0,a} = 0. \tag{56}$$

在悬臂薄板的自由边界 $y=0$ 处,子问题解之和已满足等效剪力 $V_y|_{y=0}=0$ 的条件,还需满足弯矩 M_y 为零,即

$$M_y|_{y=0} = \left.\left(D_y\frac{\partial^2 w}{\partial y^2} + D_1\frac{\partial^2 w}{\partial x^2}\right)\right|_{y=0} = \sum_{i=1}^{3}\left.\left(D_y\frac{\partial^2 w_i}{\partial y^2} + D_1\frac{\partial^2 w_i}{\partial x^2}\right)\right|_{y=0} = 0. \tag{57}$$

在悬臂薄板的固支边界 $y=b$ 处,子问题解之和已满足挠度 $w|_{y=b}=0$ 的条件,还需满足转角 θ_y 为零,即

$$\theta_y|_{y=b} = \left.\frac{\partial w}{\partial y}\right|_{y=b} = \sum_{i=1}^{3}\left.\frac{\partial w_i}{\partial y}\right|_{y=b} = 0. \tag{58}$$

将式(39)、(41)和式(52)代入到式(53),然后将其挠度代入到式(56)、(57)和式(58),即可联立求出常数 E_m, F_m, G_n 和 H_n,再将它们代回式(41)和式(52),连同式(39)进行求和,即得到均布谐载作用下正交各向异性矩形悬臂薄板的解析解.需要指出,由式(56)、(57)和式(58)得到的是四组无穷联立方程组,理论上取无穷个常数时可得到精确结果,而实际当中必须取有限组进行计算,例如可令式(39)、(41)中 $n = 1,3,5,\cdots,N$,式(52)中 $m = 0,1,2,\cdots,N$ 进行求解,其中 N 为求和上限,根据解的精度需要确定.

集中谐载作用下薄板受迫振动的解析解可完全按照上述求解思路得到.

3 数值算例与讨论

为了验证本文求解的正确性,本节给出几种正交各向异性矩形悬臂薄板受迫振动解析解的数值算例:

① 悬臂薄板承受均布谐载 $q(x,y)=p$ 的作用;

② 悬臂薄板在中心点承受幅值为 p_0 的集中谐载作用;

③ 悬臂薄板在自由边上一点 $(a/4,0)$ 承受幅值为 p_0 的集中谐载作用.

表 1 和表 2 给出了正交各向异性矩形悬臂薄板($H = 0.5D_x, D_y = 0.5D_x$)在均布和集中谐载作用下的无量纲挠度和弯矩解的收敛性分析.结果表明,挠度解收敛速度比弯矩解快,对于均布谐载的情况,N 取 100 可收敛到四位小数,对于集中谐载的情况,N 取 150 可收敛到四位小数.

表 1　均布谐载下正交各向异性悬臂板的无量纲挠度和弯矩收敛性分析

Table 1　Convergence of non-dimensional deflections and bending moments of an orthotropic cantilever plate under a uniformly distributed harmonic load

	ω/ω_{11}	N									
		10	20	30	40	50	60	70	80	90	100
$\bar{w}_p(0,0)$	0.3	0.282 1	**0.282 0**	0.282 0	0.282 0	0.282 0	0.282 0	0.282 0	0.282 0	0.282 0	0.282 0
	0.5	0.343 0	0.342 9	0.342 9	**0.342 8**	0.342 8	0.342 8	0.342 8	0.342 8	0.342 8	0.342 8
	0.8	0.718 4	0.718 0	**0.717 9**	0.717 9	0.717 9	0.717 9	0.717 9	0.717 9	0.717 9	0.717 9
	1.1	1.236 5	1.237 5	1.237 7	1.237 7	**1.237 8**	1.237 8	1.237 8	1.237 8	1.237 8	1.237 8
$\bar{w}_p(0,0.4b)$	0.3	0.130 5	**0.130 4**	0.130 4	0.130 4	0.130 4	0.130 4	0.130 4	0.130 4	0.130 4	0.130 4
	0.5	0.157 8	**0.157 7**	0.157 7	0.157 7	0.157 7	0.157 7	0.157 7	0.157 7	0.157 7	0.157 7
	0.8	0.326 1	0.325 9	**0.325 8**	0.325 8	0.325 8	0.325 8	0.325 8	0.325 8	0.325 8	0.325 8
	1.1	0.550 0	0.550 4	**0.550 5**	0.550 5	0.550 5	0.550 5	0.550 5	0.550 5	0.550 5	0.550 5
$\bar{M}_y(0.5a,b)$	0.3	−0.603 0	−0.578 9	−0.593 2	−0.583 2	−0.590 7	−0.584 3	−0.584 3	**−0.584 4**	−0.584 4	−0.584 4
	0.5	−0.720 4	−0.691 5	−0.708 5	−0.696 5	−0.705 6	−0.697 9	−0.697 9	**−0.698 0**	−0.698 0	−0.698 0
	0.8	−1.443 6	−1.384 5	−1.418 7	−1.394 7	−1.412 7	−1.397 4	−1.397 6	−1.397 7	**−1.397 8**	−1.397 8
	1.1	−2.319 1	−2.225 1	−2.281 0	−2.242 8	−2.271 5	−2.247 4	−2.247 7	−2.247 9	**−2.248 1**	−2.248 1

表 2　中心点处作用集中谐载的正交各向异性悬臂板的无量纲挠度和弯矩收敛性分析

Table 2　Convergence of non-dimensional deflections and bending moments of an orthotropic cantilever plate under a concentrated harmonic load at the center

	ω/ω_{11}	N									
		10	30	50	70	90	110	120	130	140	150
$\bar{w}_{p0}(0,0)$	0.3	0.237 3	0.238 3	**0.238 4**	0.238 4	0.238 4	0.238 4	0.238 4	0.238 4	0.238 4	0.238 4
	0.5	0.290 5	0.291 8	**0.291 9**	0.291 9	0.291 9	0.291 9	0.291 9	0.291 9	0.291 9	0.291 9
	0.8	0.619 2	0.621 4	0.621 7	0.621 7	0.621 7	**0.621 8**	0.621 8	0.621 8	0.621 8	0.621 8
	1.1	1.092 4	1.098 0	1.098 5	1.098 6	**1.098 7**	1.098 7	1.098 7	1.098 7	1.098 7	1.098 7
$\bar{w}_{p0}(0,0.4b)$	0.3	0.114 8	0.115 1	**0.115 2**	0.115 2	0.115 2	0.115 2	0.115 2	0.115 2	0.115 2	0.115 2
	0.5	0.138 7	0.139 1	**0.139 2**	0.139 2	0.139 2	0.139 2	0.139 2	0.139 2	0.139 2	0.139 2
	0.8	0.286 1	0.287 0	**0.287 1**	0.287 1	0.287 1	0.287 1	0.287 1	0.287 1	0.287 1	0.287 1
	1.1	0.480 8	0.483 3	0.483 6	0.483 6	**0.483 7**	0.483 7	0.483 7	0.483 7	0.483 7	0.483 7
$\bar{M}_{y0}(0.5a,b)$	0.3	−0.630 6	−0.621 5	−0.619 1	−0.613 3	−0.613 4	−0.613 4	**−0.613 5**	−0.613 5	−0.613 5	−0.613 5
	0.5	−0.733 8	−0.723 2	−0.720 5	−0.713 6	−0.713 7	**−0.713 8**	−0.713 8	−0.713 8	−0.713 8	−0.713 8
	0.8	−1.367 9	−1.348 5	−1.343 4	−1.329 9	−1.330 1	−1.330 3	−1.330 4	−1.330 3	**−1.330 4**	−1.330 4
	1.1	−1.924 8	−1.902 7	−1.895 5	−1.875 1	−1.875 5	−1.875 7	−1.876 0	−1.875 9	**−1.876 0**	−1.876 0

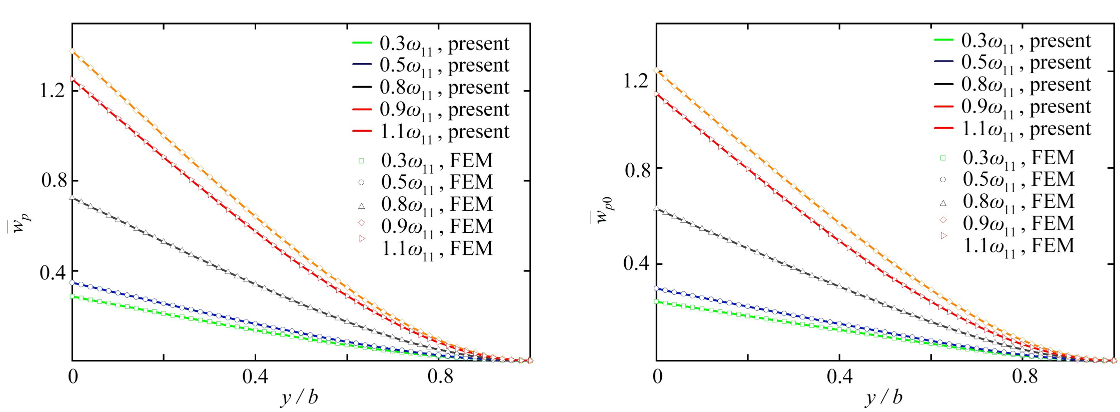
表 3 给出了各向同性矩形悬臂薄板($\nu = 1/6, b = a$)在均布谐载和集中谐载作用下的无量纲挠度和弯矩解.其中均布谐载对应的无量纲挠度 $\bar{w}_p = wqa^4/D$(D 为板的弯曲刚度),无量纲弯矩 $\bar{M}_y = M_y qa^2$,集中谐载对应的无量纲挠度 $\bar{w}_{p0} = wqa^2/D$,无量纲弯矩 $\bar{M}_{y0} = M_y q$,表中 ω_{11} 是由精细有限元分析(采用 ABAQUS 软件中 S4R 单元,$a/100$ 的网格)得到的基频结果.通过与精细有限元分析的收敛结果进行对比可见,本文求解方法得到的结果与有限元结果吻合得很好,从而证明了本文方法的准确性.本文结果与文献[20]利用功的互等定理法计算出的部分结果存在一定差异,原因可能是文献中并未说明薄板的基频大小,导致本文的基频和文献中的基频存在差异,进而产生外载频率和响应的差异.

表 3 各向同性悬臂板的无量纲挠度和弯矩

Table 3 Non-dimensional deflections and bending moments of an isotropic cantilever plate

	ω/ω_{11}	method	$\bar{w}_p(\bar{w}_{p0})$, $x=0$			$\bar{w}_p(\bar{w}_{p0})$, $y=0$			$\bar{M}_y(\bar{M}_{y0})$, $y=b$		
			$y=0$	$y=0.1b$	$y=0.3b$	$x=0.1a$	$x=0.3a$	$x=0.5a$	$x=0.1a$	$x=0.3a$	$x=0.5a$
uniformly distributed harmonic load	0.3	present	0.137 9	0.119 3	0.082 4	0.047 8	0.138 3	0.138 8	0.139 0	−0.548 4	−0.562 5
		FEM	0.137 9	0.119 3	0.082 4	0.047 8	0.138 3	0.138 8	0.139 0	−0.538 0	−0.554 5
		ref. [20]	0.142 2	0.123 0	0.084 8	0.049 1	0.142 5	0.142 9	0.143 1	−0.581 1	−0.569 7
	0.5	present	0.167 7	0.144 9	0.099 9	0.057 7	0.168 1	0.168 8	0.169 0	−0.653 4	−0.671 1
		FEM	0.167 7	0.144 9	0.099 9	0.057 7	0.168 1	0.168 8	0.169 1	−0.641 3	−0.662 0
		ref. [20]	0.174 2	0.150 5	0.102 5	0.059 7	0.174 5	0.175 0	0.175 2	−0.696 8	−0.684 7
	0.8	present	0.350 9	0.302 7	0.207 3	0.118 6	0.351 9	0.353 5	0.354 0	−1.298 9	−1.339 5
		FEM	0.350 9	0.302 7	0.207 3	0.118 6	0.351 9	0.353 5	0.354 0	−1.276 1	−1.322 4
		ref. [20]	0.381 6	0.329 0	0.225 0	0.128 5	0.382 5	0.383 8	0.384 2	−1.448 1	−1.430 7
	1.1	present	0.605 5	0.520 7	0.353 3	0.199 2	0.607 4	0.610 4	0.611 4	−2.067 1	−2.145 7
		FEM	0.605 8	0.521 0	0.353 5	0.199 3	0.607 7	0.610 6	0.611 7	−2.035 0	−2.123 1
concentrated harmonic load at the center	0.3	present	0.114 1	0.099 4	0.070 0	0.041 4	0.115 7	0.118 2	0.119 3	−0.492 3	−0.581 7
		FEM	0.114 0	0.099 4	0.070 0	0.041 4	0.115 6	0.118 2	0.119 3	−0.483 5	−0.572 9
		ref. [20]	0.116 3	0.101 3	0.071 3	0.042 1	0.117 8	0.120 4	0.121 5	−0.451 1	−0.568 7
	0.5	present	0.140 1	0.121 8	0.085 3	0.050 1	0.141 8	0.144 5	0.145 6	−0.584 4	−0.677 1
		FEM	0.140 0	0.121 8	0.085 3	0.050 1	0.141 7	0.144 5	0.145 6	−0.574 1	−0.667 3
		ref. [20]	0.143 8	0.125 0	0.087 4	0.051 3	0.145 4	0.148 1	0.149 2	−0.536 8	−0.664 7
	0.8	present	0.300 4	0.259 8	0.179 3	0.103 4	0.302 5	0.306 0	0.307 5	−1.149 9	−1.262 9
		FEM	0.300 3	0.259 8	0.179 3	0.103 4	0.302 5	0.306 0	0.307 4	−1.130 2	−1.246 2
		ref. [20]	0.322 1	0.278 4	0.191 9	0.110 5	0.324 1	0.327 5	0.328 9	−1.090 0	−1.283 4
	1.1	present	0.536 6	0.460 7	0.311 3	0.174 7	0.537 0	0.537 4	0.537 5	−1.794 1	−1.784 9
		FEM	0.536 9	0.461 0	0.311 4	0.174 8	0.537 3	0.537 6	0.537 6	−1.766 0	−1.767 2

表 4—9 给出了正交各向异性矩形悬臂薄板（$H=0.5D_x$）在均布谐载或集中谐载下的无量纲挠度和弯矩解，其中均布谐载对应的无量纲挠度 $\bar{w}_p=wqa^4/D_x$，无量纲弯矩 $\bar{M}_y=M_yqa^2$，集中谐载对应的无量纲挠度 $\bar{w}_{p0}=wqa^2/D_x$，无量纲弯矩 $\bar{M}_{y0}=M_yq$．本文方法得到的结果均与有限元结果吻合得很好．另外，表 10 给出了正交各向异性矩形悬臂薄板（$H=0.5D_x, D_y=0.5D_x, b=a$）在均布谐载和集中谐载下谐载频率 $\omega=0.8\omega_{11}$ 时的挠度和弯矩云图，同样可以看出本文方法得到的结果与有限元吻合良好．



（a）均布谐载

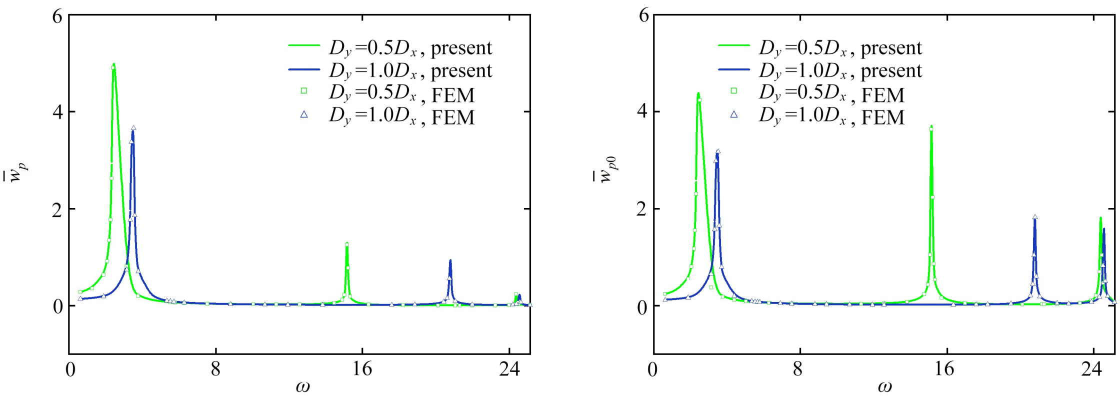
（a）Uniformly distributed harmonic loads

（b）集中谐载

（b）Concentrated harmonic loads

图 3 不同谐载频率下正交各向异性悬臂板 $x=a/2$ 处的无量纲挠度沿 y 轴变化曲线

Fig. 3 Non-dimensional deflections along the y axis at $x=a/2$ of an orthotropic cantilever plate under different frequencies of harmonic loads



（a）均布谐载

（a）Uniformly distributed harmonic loads

（b）集中谐载

（b）Concentrated harmonic loads

图 4　正交各向异性悬臂板在不同弯曲刚度下的频率响应曲线

Fig. 4　Frequency response curves of orthotropic cantilever plates with different flexural rigidities

注　为了解释图中的颜色,读者可以参考本文的电子网页版本,后同.

众所周知,当动力载荷 $\bar{q}_t(x,y,t)$ 的频率 ω 接近薄板的固有频率 ω_{11} 时,板的挠度将变得非常大,这通常被称为共振现象.图 3 给出了各激励频率下板在 $x = a/2$ 处的挠度分布曲线, 图 4 给出了不同弯曲刚度下正交各向异性悬臂薄板（$H = 0.5D_x$, $b = a$）在点（$a/2$,0）处的频率响应曲线, 可观察出激励频率对矩形薄板受迫振动的影响, 和理论预期相符. 同时将本文求解结果和精细有限元模拟结果进行直观比较, 发现两者吻合良好.

表 4　均布谐载下正交各向异性悬臂板的无量纲挠度和弯矩（$b = a$）

Table 4　Non-dimensional deflections and bending moments of orthotropic cantilever plates under uniformly distributed harmonic loads, with $b = a$

D_y/D_x	ω/ω_{11}	method	$\bar{w}_p$, $x = 0$			$\bar{w}_p$, $y = 0$			$\bar{M}_y$, $y = b$		
			$y = 0$	$y = 0.4b$	$y = 0.8b$	$x = 0.1a$	$x = 0.3a$	$x = 0.5a$	$x = 0.1a$	$x = 0.3a$	$x = 0.5a$
0.5	0.3	present	0.282 0	0.130 4	0.016 3	0.282 8	0.283 9	0.284 4	−0.541 8	−0.585 8	−0.584 4
		FEM	0.282 0	0.130 4	0.016 3	0.282 8	0.284 0	0.284 4	−0.524 1	−0.577 5	−0.580 9
	0.5	present	0.342 8	0.157 7	0.019 5	0.343 8	0.345 3	0.345 8	−0.644 5	−0.699 2	−0.698 1
		FEM	0.342 8	0.157 7	0.019 5	0.343 8	0.345 3	0.345 8	−0.623 6	−0.689 6	−0.694 0
	0.8	present	0.717 9	0.325 8	0.039 0	0.720 1	0.723 5	0.724 7	−1.276 4	−1.398 2	−1.397 8
		FEM	0.717 5	0.325 6	0.039 0	0.719 7	0.723 1	0.724 3	−1.235 7	−1.379 5	−1.390 3
	1.1	present	1.237 8	0.550 5	0.062 7	1.242 0	1.248 5	1.251 0	−2.015 1	−2.243 2	−2.248 1
		FEM	1.239 0	0.551 0	0.062 7	1.243 2	1.249 7	1.252 2	−1.957 3	−2.220 2	−2.242 7
1.0	0.3	present	0.141 1	0.064 6	0.007 7	0.141 6	0.142 4	0.142 7	−0.527 7	−0.586 8	−0.591 8
		FEM	0.141 1	0.064 6	0.007 7	0.141 6	0.142 4	0.142 7	−0.523 8	−0.580 7	−0.583 4
	0.5	present	0.171 5	0.078 1	0.009 1	0.172 2	0.173 1	0.173 5	−0.627 3	−0.700 6	−0.706 9
		FEM	0.171 5	0.078 1	0.009 1	0.172 1	0.173 1	0.173 5	−0.622 8	−0.693 5	−0.697 2
	0.8	present	0.359 1	0.161 1	0.018 2	0.360 5	0.362 8	0.363 6	−1.240 6	−1.401 7	−1.416 5
		FEM	0.359 2	0.161 1	0.018 2	0.360 6	0.362 8	0.363 7	−1.233 1	−1.389 1	−1.398 8
	1.1	present	0.619 9	0.272 2	0.029 0	0.622 7	0.627 1	0.628 8	−1.956 6	−2.254 3	−2.283 9
		FEM	0.620 0	0.272 2	0.029 0	0.622 8	0.627 2	0.628 9	−1.948 2	−2.237 9	−2.259 7

表 5 均布谐载下正交各向异性悬臂板的无量纲挠度和弯矩（$b = 2a$）

Table 5 Non-dimensional deflections and bending moments of orthotropic cantilever plates under uniformly distributed harmonic loads, with $b = 2a$

D_y/D_x	ω/ω_{11}	method	$\bar{w}_p$, $x = 0$			$\bar{w}_p$, $y = 0$			$\bar{M}_y$, $y = b$		
			$y = 0$	$y = 0.4b$	$y = 0.8b$	$x = 0.1a$	$x = 0.3a$	$x = 0.5a$	$x = 0.1a$	$x = 0.3a$	$x = 0.5a$
0.5	0.3	present	4.615 4	2.159 0	0.282 4	4.616 6	4.618 5	4.619 1	−1.974 6	−2.402 5	−2.481 2
		FEM	4.606 3	2.154 8	0.281 9	4.607 5	4.609 3	4.610 0	−2.008 7	−2.385 6	−2.433 1
	0.5	present	5.610 5	2.611 4	0.337 9	5.612 0	5.614 4	5.615 3	−2.347 3	−2.866 4	−2.962 1
		FEM	5.616 9	2.614 2	0.338 3	5.618 4	5.620 7	5.621 5	−2.395 1	−2.854 6	−2.913 1
	0.8	present	11.753 5	5.403 1	0.680 3	11.757 1	11.762 6	11.764 6	−4.645 4	−5.727 6	−5.928 0
		FEM	11.754 3	5.403 3	0.680 3	11.757 9	11.763 5	11.765 5	−4.739 4	−5.700 9	−5.827 5
	1.1	present	20.257 0	9.139 5	1.102 1	20.264 0	20.274 9	20.278 9	−7.316 8	−9.167 9	−9.513 3
		FEM	20.526 7	9.262 1	1.117 2	20.534 0	20.545 3	20.549 5	−7.585 9	−9.262 4	−9.492 9
1.0	0.3	present	2.320 6	1.080 4	0.134 7	2.321 1	2.321 8	2.322 0	−2.052 6	−2.441 8	−2.469 3
		FEM	2.320 9	1.080 5	0.134 7	2.321 5	2.322 2	2.322 5	−1.986 0	−2.415 4	−2.469 5
	0.5	present	2.820 3	1.306 3	0.161 0	2.821 0	2.821 9	2.822 2	−2.438 1	−2.913 1	−2.948 3
		FEM	2.818 8	1.305 6	0.160 9	2.819 5	2.820 5	2.820 8	−2.358 3	−2.880 3	−2.947 0
	0.8	present	5.905 2	2.700 3	0.323 1	5.906 8	5.909 0	5.909 8	−4.814 4	−5.819 3	−5.902 5
		FEM	5.915 0	2.704 7	0.323 6	5.916 6	5.919 0	5.919 9	−4.669 6	−5.768 4	−5.914 1
	1.1	present	10.204 8	4.577 2	0.522 5	10.208 1	10.212 7	10.214 3	−7.581 3	−9.343 8	−9.511 6
		FEM	10.147 8	4.551 3	0.519 4	10.151 2	10.156 1	10.158 1	−7.308 0	−9.201 4	−9.465 7

表 6 中心点处作用集中谐载的正交各向异性悬臂板的无量纲挠度和弯矩（$b = a$）

Table 6 Non-dimensional deflections and bending moments of orthotropic cantilever plates under concentrated harmonic loads at the center, with $b = a$

D_y/D_x	ω/ω_{11}	method	$\bar{w}_{p0}$, $x = 0$			$\bar{w}_{p0}$, $y = 0$			$\bar{M}_{y0}$, $y = b$		
			$y = 0$	$y = 0.4b$	$y = 0.8b$	$x = 0.1a$	$x = 0.3a$	$x = 0.5a$	$x = 0.1a$	$x = 0.3a$	$x = 0.5a$
0.5	0.3	present	0.238 4	0.115 2	0.014 7	0.240 1	0.242 7	0.243 7	−0.505 7	−0.594 5	−0.613 5
		FEM	0.239 8	0.116 3	0.015 0	0.241 2	0.243 3	0.243 7	−0.497 0	−0.589 5	−0.608 3
	0.5	present	0.291 9	0.139 2	0.017 5	0.293 8	0.296 7	0.297 7	−0.596 3	−0.694 7	−0.713 8
		FEM	0.293 3	0.140 4	0.017 8	0.294 9	0.297 3	0.297 7	−0.585 0	−0.688 6	−0.708 3
	0.8	present	0.621 8	0.287 1	0.034 7	0.624 7	0.629 2	0.630 9	−1.152 9	−1.310 4	−1.330 4
		FEM	0.623 3	0.288 3	0.035 0	0.625 9	0.629 9	0.630 9	−1.125 0	−1.297 1	−1.322 2
	1.1	present	1.098 7	0.483 7	0.054 7	1.101 4	1.105 6	1.107 2	−1.740 2	−1.891 4	−1.876 0
		FEM	1.097 1	0.482 3	0.054 4	1.100 1	1.105 0	1.107 3	−1.679 9	−1.866 6	−1.871 2
1.0	0.3	present	0.119 6	0.055 1	0.006 2	0.121 0	0.123 2	0.124 0	−0.466 2	−0.608 5	−0.650 0
		FEM	0.120 8	0.056 0	0.006 4	0.122 0	0.123 8	0.124 1	−0.472 5	−0.606 9	−0.640 4
	0.5	present	0.146 5	0.067 0	0.007 5	0.148 1	0.150 5	0.151 3	−0.554 7	−0.709 6	−0.752 4
		FEM	0.147 8	0.068 0	0.007 7	0.149 1	0.151 1	0.151 4	−0.560 9	−0.707 4	−0.741 6
	0.8	present	0.312 7	0.140 6	0.015 6	0.315 0	0.318 5	0.319 7	−1.098 9	−1.331 8	−1.382 0
		FEM	0.314 3	0.141 7	0.015 8	0.316 3	0.319 2	0.319 8	−1.103 3	−1.325 1	−1.363 9
	1.1	present	0.554 6	0.243 2	0.026 3	0.556 0	0.558 4	0.559 4	−1.733 2	−1.905 5	−1.894 0
		FEM	0.552 2	0.241 6	0.026 0	0.554 1	0.557 3	0.559 4	−1.708 7	−1.884 1	−1.874 8

表 7　中心点处作用集中谐载的正交各向异性悬臂板的无量纲挠度和弯矩($b = 2a$)

Table 7　Non-dimensional deflections and bending moments of orthotropic cantilever plates under concentrated harmonic loads at the center, with $b = 2a$

D_y/D_x	ω/ω_{11}	method	$\bar{w}_{p0}$, $x = 0$			$\bar{w}_{p0}$, $y = 0$			$\bar{M}_{y0}$, $y = b$		
			$y = 0$	$y = 0.4b$	$y = 0.8b$	$x = 0.1a$	$x = 0.3a$	$x = 0.5a$	$x = 0.1a$	$x = 0.3a$	$x = 0.5a$
0.5	0.3	present	1.937 8	0.980 7	0.139 4	1.938 2	1.938 6	1.938 8	−0.983 3	−1.185 2	−1.221 7
		FEM	1.934 2	0.979 0	0.139 2	1.934 5	1.935 0	1.935 1	−1.000 2	−1.177 4	−1.198 6
	0.5	present	2.370 1	1.177 5	0.163 6	2.370 5	2.371 2	2.371 5	−1.146 1	−1.387 7	−1.431 5
		FEM	2.373 6	1.179 1	0.163 8	2.374 1	2.374 7	2.375 0	−1.169 0	−1.382 2	−1.408 2
	0.8	present	5.039 4	2.391 4	0.312 6	5.040 8	5.042 8	5.043 5	−2.147 3	−2.633 3	−2.722 3
		FEM	5.042 4	2.392 7	0.312 7	5.043 8	5.045 8	5.046 6	−2.190 6	−2.622 1	−2.677 7
	1.1	present	8.874 9	3.928 8	0.461 8	8.878 2	8.883 3	8.885 2	−3.051 2	−3.838 5	−3.985 4
		FEM	8.824 1	3.905 7	0.458 9	8.827 4	8.832 4	8.834 2	−3.097 0	−3.799 0	−3.896 9
1.0	0.3	present	0.978 4	0.490 4	0.066 4	0.978 5	0.978 5	0.978 5	−1.020 8	−1.205 4	−1.217 4
		FEM	0.978 6	0.490 5	0.066 4	0.978 6	0.978 7	0.978 7	−0.988 0	−1.192 4	−1.217 5
	0.5	present	1.196 1	0.589 0	0.077 9	1.196 3	1.196 4	1.196 4	−1.189 6	−1.411 7	−1.427 1
		FEM	1.195 5	0.588 7	0.077 8	1.195 6	1.195 7	1.195 8	−1.151 0	−1.395 9	−1.426 4
	0.8	present	2.540 8	1.197 0	0.148 6	2.541 3	2.542 0	2.542 2	−2.227 0	−2.680 6	−2.717 1
		FEM	2.545 7	1.199 2	0.148 9	2.546 3	2.547 0	2.547 3	−2.161 1	−2.657 6	−2.722 6
	1.1	present	4.483 7	1.975 6	0.219 9	4.485 2	4.487 5	4.488 3	−3.171 7	−3.927 2	−4.001 6
		FEM	4.459 3	1.964 5	0.218 6	4.461 0	4.463 4	4.464 4	−3.059 2	−3.867 5	−3.981 4

表 8　($a/4$,0)处作用集中谐载的正交各向异性悬臂板的无量纲挠度和弯矩($b = a$)

Table 8　Non-dimensional deflections and bending moments of orthotropic cantilever plates under concentrated harmonic loads at ($a/4$,0), with $b = a$

D_y/D_x	ω/ω_{11}	method	$\bar{w}_{p0}$, $x = 0$			$\bar{w}_{p0}$, $y = 0$			$\bar{M}_{y0}$, $y = b$		
			$y = 0$	$y = 0.4b$	$y = 0.8b$	$x = 0.1a$	$x = 0.3a$	$x = 0.5a$	$x = 0.1a$	$x = 0.3a$	$x = 0.5a$
0.5	0.3	present	0.897 5	0.390 6	0.046 0	0.878 7	0.831 3	0.758 7	−1.399 6	−1.379 7	−1.235 3
		FEM	0.907 3	0.394 0	0.046 4	0.887 1	0.835 6	0.762 2	−1.384 1	−1.370 2	−1.232 8
	0.5	present	1.058 0	0.463 1	0.054 5	1.038 9	0.990 6	0.915 9	−1.671 8	−1.677 9	−1.525 7
		FEM	1.068 3	0.466 7	0.055 0	1.047 8	0.995 3	0.920 3	−1.652 2	−1.663 6	−1.522 5
	0.8	present	2.032 0	0.901 2	0.105 8	2.013 8	1.965 8	1.884 4	−3.344 3	−3.484 5	−3.318 1
		FEM	2.044 6	0.905 9	0.106 4	2.024 9	1.972 4	1.894 0	−3.260 5	−3.452 4	−3.310 8
	1.1	present	2.962 7	1.334 0	0.152 7	3.001 2	3.072 0	3.163 9	−5.033 3	−5.829 0	−6.038 9
		FEM	2.958 2	1.332 7	0.152 5	2.998 4	3.085 1	3.182 2	−4.878 6	−5.767 2	−6.024 8
1.0	0.3	present	0.543 1	0.236 5	0.026 5	0.519 4	0.460 6	0.382 2	−1.624 4	−1.525 2	−1.255 8
		FEM	0.555 5	0.240 8	0.027 0	0.529 8	0.467 3	0.383 4	−1.641 7	−1.524 5	−1.245 4
	0.5	present	0.632 3	0.276 9	0.031 2	0.606 9	0.543 9	0.461 2	−1.926 3	−1.839 7	−1.550 3
		FEM	0.645 5	0.281 6	0.031 7	0.618 0	0.551 4	0.462 7	−1.945 1	−1.838 0	−1.537 1
	0.8	present	1.144 1	0.506 7	0.057 0	1.114 4	1.043 5	0.947 7	−3.687 3	−3.715 0	−3.364 2
		FEM	1.164 5	0.514 5	0.057 9	1.132 2	1.054 5	0.951 8	−3.666 5	−3.699 4	−3.338 2
	1.1	present	1.276 3	0.557 2	0.057 1	1.332 5	1.453 3	1.590 3	−4.208 2	−5.501 4	−6.133 2
		FEM	1.264 1	0.552 6	0.056 4	1.323 9	1.453 9	1.600 2	−4.137 6	−5.445 4	−6.072 5

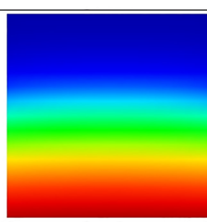
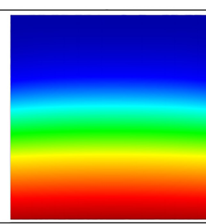
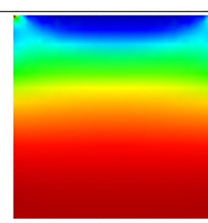
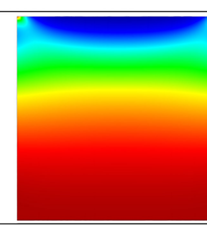
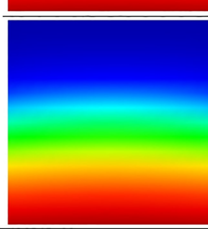
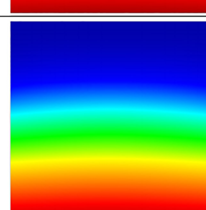
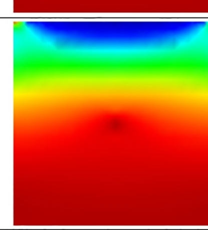
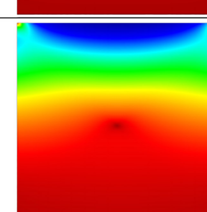
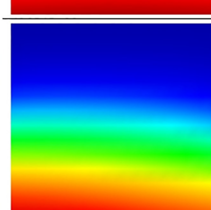
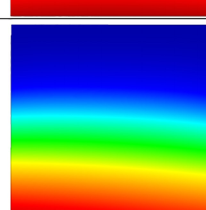
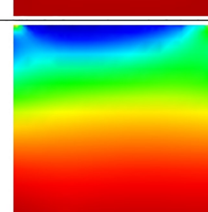
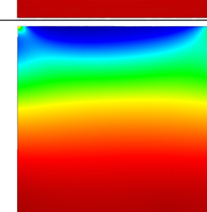
表 9 $(a/4,0)$ 处作用集中谐载的正交各向异性悬臂板的无量纲挠度和弯矩 $(b=2a)$

Table 9 Non-dimensional deflections and bending moments of orthotropic cantilever plates under concentrated harmonic loads at $(a/4,0)$, with $b=2a$

D_y/D_x	ω/ω_{11}	method	$\bar{w}_{p0}$, $x=0$			$\bar{w}_{p0}$, $y=0$			$\bar{M}_{y0}$, $y=b$		
			$y=0$	$y=0.4b$	$y=0.8b$	$x=0.1a$	$x=0.3a$	$x=0.5a$	$x=0.1a$	$x=0.3a$	$x=0.5a$
0.5	0.3	present	6.504 9	2.821 9	0.340 9	6.445 4	6.316 0	6.164 4	−2.284 7	−2.648 6	−2.596 4
		FEM	6.513 0	2.822 7	0.340 8	6.453 1	6.322 1	6.167 5	−2.330 2	−2.634 0	−2.554 8
	0.5	present	7.784 6	3.403 4	0.412 3	7.724 8	7.594 9	7.565 5	−2.764 4	−3.242 9	−3.210 4
		FEM	7.813 0	3.413 4	0.413 3	7.752 9	7.621 3	7.465 6	−2.827 4	−3.234 8	−3.168 1
	0.8	present	15.669 6	6.986 2	0.851 7	15.610 9	15.481 9	15.393 4	−5.716 1	−6.909 8	−7.006 0
		FEM	15.702 4	6.998 3	0.853 1	15.643 3	15.512 5	15.355 2	−5.842 0	−6.887 8	−6.903 6
	1.1	present	25.350 4	11.650 1	1.432 2	25.425 1	25.579 9	25.609 7	−9.617 3	−12.187 7	−12.795 1
		FEM	25.238 0	11.601 5	1.426 2	25.313 2	25.469 9	25.640 0	−9.777 2	−12.088 8	−12.533 9
1.0	0.3	present	3.572 0	1.558 5	0.183 2	3.486 9	3.306 3	3.090 6	−2.634 7	−2.827 9	−2.597 5
		FEM	3.586 2	1.561 9	0.183 5	3.500 2	3.316 4	3.109 3	−2.550 5	−2.810 1	−2.603 6
	0.5	present	4.224 1	1.854 6	0.218 1	4.137 2	3.952 7	3.730 5	−3.145 4	−3.439 0	−3.209 9
		FEM	4.237 2	1.857 6	0.218 3	4.149 3	3.961 6	3.750 3	−3.044 6	−3.414 2	−3.215 1
	0.8	present	8.211 3	3.660 0	0.429 0	8.120 2	7.926 2	7.700 7	−6.236 8	−7.185 1	−6.995 2
		FEM	8.245 7	3.672 5	0.430 4	8.153 5	7.956 4	7.734 4	−6.057 4	−7.143 7	−7.022 5
	1.1	present	12.410 5	5.649 1	0.650 7	12.517 1	12.738 7	12.967 4	−9.583 4	−12.238 8	−12.794 6
		FEM	12.353 7	5.626 1	0.648 1	12.461 7	12.687 2	12.932 3	−9.266 6	−12.068 7	−12.757 3

表 10 正交各向异性悬臂板的挠度和弯矩分布云图

Table 10 Contour plots of deflections and bending moments of the orthotropic cantilever plate

	$\bar{w}_p$ ($\bar{w}_{p0}$)		$\bar{M}_y$	
	present	FEM	present	FEM
uniformly distributed harmonic load				
concentrated harmonic load at the center				
concentrated harmonic load at $(a/4, 0)$				

4 结　　论

本文基于辛叠加方法求解了正交各向异性矩形悬臂薄板受迫振动问题的解析解.板的力学问题的主要任务之一就是求解满足给定边界条件下的高阶偏微分控制方程,传统的半逆解法需要提前假设解的形式,而

本文方法则是从板的基本方程出发求解,无需任何假设,逐步严格地推导出问题的解.通过所得解析解,本文给出了较大量的数值结果,可作为检验各类求解方法的对比基准;同时,量化地考查了谐载频率对薄板受迫振动特性的影响规律,可为工程师开展相关结构设计提供理论依据.本文方法还可进一步扩展至考虑各种复杂边界条件和组合载荷的板壳问题,因而在更多受迫振动问题的研究中具有较大的应用潜力.

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