http://www.applmathmech.cn

一维六方准晶非周期平面内中心开口裂纹的平面热弹性问题*

赵雪芬[1,2]，　卢绍楠[3]，　马园园[3]，　张保文[3]

（1. 宁夏大学 信息工程学院，银川 750021；
2. 宁夏大学新华学院，银川 750021；
3. 宁夏大学 数学统计学院，银川 750021）

摘要： 考虑裂纹内部介质的热传导率，研究了一维六方准晶非周期平面内含中心开口裂纹的平面热弹性问题。利用 Fourier 积分变换技术，得到了热应力、裂纹尖端处的热应力强度因子和应变能密度因子的封闭解。数值结果讨论了裂纹内部介质的热传导率、外载荷及声子场-相位子场耦合系数对热应力强度因子和应变能密度因子的影响。结果表明，声子场-相位子场耦合系数对裂纹扩展影响较大。当声子场载荷较小或热流密度较大时，裂纹不易扩展，热流密度在裂纹尖端处会出现集中热效应。随着裂纹内部介质热传导率的增大，热流密度逐渐增加而热应力强度因子逐渐减小。该文所得结果为准晶热力学性质的实际应用提供了理论依据，进而可用于优化准晶元器件的设计和制备。

关 键 词： 一维六方准晶；　中心开口裂纹；　Fourier 积分变换；　热应力强度因子；　应变能密度因子
中图分类号： O343.8　　**文献标志码：** A　　DOI：10.21656/1000-0887.440302

The Plane Thermoelastic Problem of a Central Opening Crack in the 1D Hexagonal Quasicrystal Non-Periodic Plane

ZHAO Xuefen[1,2]，　LU Shaonan[3]，　MA Yuanyuan[3]，　ZHANG Baowen[3]
（1. *School of Information Engineering*, *Ningxia University*, *Yinchuan* 750021, *P.R.China*;
2. *Xinhua College of Ningxia University*, *Yinchuan* 750021, *P.R.China*;
3. *School of Mathematics and Statistics*, *Ningxia University*, *Yinchuan* 750021, *P.R.China*）

Abstract: Considering the thermal conductivity of the medium inside the crack, the plane thermoelastic problem of the 1D hexagonal quasicrystal with a central open crack in an aperiodic plane, was studied. With the Fourier integral transformation technology, the closed form solutions of thermal stresses, thermal stress intensity factors and strain energy density factors were obtained. Numerical examples were used to analyze the effects of the thermal conductivity, the external load, and the phonon field-phason field coupling coefficient on the thermal stress intensity factor and the strain energy density factor at the crack tip. The results indicate that, the heat flux density gradually increases but the thermal stress intensity factor gradually decreases with the thermal conductivity. The phonon field-phason field coupling coefficient has a significant impact on the crack propa-

* **收稿日期：** 2023-10-10；**修订日期：** 2023-11-14
基金项目： 国家自然科学基金(12062021;62062057;12262033;12062022)
作者简介： 赵雪芬(1983—)，女，副教授，博士，硕士生导师(通讯作者. E-mail: snownfen@163.com).
引用格式： 赵雪芬，卢绍楠，马园园，张保文. 一维六方准晶非周期平面内中心开口裂纹的平面热弹性问题[J]. 应用数学和力学，2024，**45**(3)：303-317.

gation. When the phonon field load is relatively small or the heat flux density is relatively high, the crack is not easy to propagate. The heat flux density exhibits a concentration effect at the crack tip. The work provides a theoretical basis for the application of thermodynamic properties of quasicrystals, and the optimization of design and preparation of quasicrystal components.

Key words: 1D hexagonal quasicrystal; central opening crack; Fourier integral transformation; thermal stress intensity factor; strain energy density factor

0 引 言

1984 年,研究人员发现准晶体同时具有非晶体学旋转对称性和准周期平移对称性[1].准晶体中声子场和相位子场共存, 其中声子场相当于传统晶体的弹性场, 相位子场位移可以理解为对基本晶格的另外一种扰动.相位子相应的元激发,表现出扩散行为[2].由于材料的脆性,准晶对裂纹、孔洞、夹杂等其他缺陷很敏感[3].这些缺陷的存在显著影响了准晶的物理和力学性能.为此,许多学者对准晶断裂问题进行了大量的研究.Li 等[4]研究了一维六方准晶无限空间中半无限 Dugdale 裂纹尖端的塑性变形,估算了裂纹前缘塑性区的范围,并给出了扩展裂纹外的法向应力和裂纹表面位移.Yu 和 Guo[5]考虑了一维六方准晶带中Ⅲ型共线裂纹的奇异弹性场问题,通过第一类和第三类完全椭圆积分,导出了声子场和相位子场在裂纹尖端的应力强度因子.在考虑压电效应的作用下,卢绍楠等[6]基于复变函数方法研究了含界面共线裂纹的一维六方压电准晶双材料的断裂行为,求出了问题的精确解.

实验发现,传统方法制备的准晶仅在高温下稳定[7-8].为了提高材料的使用寿命,关于准晶热力学性能已广泛展开了实验和理论方面的研究[9-12].基于广义不连续位移法,Fan 等[13]研究了热效应下一维六方准晶非周期平面裂纹问题.利用广义势理论方法,Li 等[14]考虑了在一对均匀热流作用下无限大一维六方准晶中币形裂纹的反平面断裂问题,得到了温度、位移和应力的解析表达式.Zhang 等[15]借助于 Hankel 积分变换技术,分析了一维六方准晶涂层在热机械载荷作用下界面裂纹的三维问题,导出了界面位移和温度不连续的基本解.Guo 等[16]研究了含有导热椭圆孔的二维十次准晶的热弹性问题,考虑了椭圆孔的导热性,利用复变函数方法求得了应力的精确解.上述文献均假设缺陷处于闭合状态,其中文献[13-15]中裂纹是绝热的(热非渗透),文献[16]中缺陷是完全热导通的(热渗透).

工程实际中,当材料加载外载荷时,裂纹内部并不是完全闭合状态.张开的裂纹内部会充满具有导热性的介质(比如空气等).基于此,本文考虑裂纹内部介质热传导率,利用 Fourier 积分变换和叠加原理,研究了一维六方准晶非周期平面内含有中心开口裂纹的平面热弹性问题,得到了声子场和相位子场应力的解析解.数值算例分析热传导率、外载荷、热流密度和耦合效应对热应力强度因子和应变能密度因子的影响规律.本文得到的结论可作为设计和评估准晶高温材料的理论基础.

1 基本方程及问题描述

一维六方准晶中,若准周期方向取 z 轴,则周期平面为 xOy 面.假设缺陷沿 y 轴穿透材料,这时在垂直于周期方向平面内的弹性问题可视为非周期平面弹性问题.一维六方准晶体非周期平面弹性问题的平衡方程、变形几何方程、广义 Hooke 定律分别为[17-18]

$$\partial_j \sigma_{ij} = 0,\ \partial_j H_{ij} = 0, \tag{1a}$$

$$\varepsilon_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j),\ \omega_{ij} = \partial_j w_i, \tag{1b}$$

$$\begin{cases} \sigma_{xx} = C_{11}\varepsilon_{xx} + C_{13}\varepsilon_{zz} + R_1\omega_{zz} - \beta_1\theta, \\ \sigma_{zz} = C_{13}\varepsilon_{xx} + C_{33}\varepsilon_{zz} + R_2\omega_{zz} - \beta_3\theta, \\ \sigma_{xz} = \sigma_{zx} = 2C_{44}\varepsilon_{zx} + R_3\omega_{zx}, \\ H_{zz} = R_1\varepsilon_{xx} + R_2\varepsilon_{zz} + K_1\omega_{zz}, \\ H_{zx} = 2R_3\varepsilon_{zx} + K_2\omega_{zx}, \end{cases} \tag{1c}$$

这里, $i,j=x,z$, C_{11}, C_{13}, C_{33} 和 K_1, K_2 分别表示声子场和相位子场弹性常数, R_1, R_2, R_3 为声子场-相位子场耦合系数, σ_{ij}, H_{ij} 分别为声子场和相位子场应力分量, ε_{ij}, ω_{ij} 表示声子场和相位子场应变分量, u_i, w_i 代表声子场和相位子场位移分量, β_1, β_3 是热模量常数, θ 表示温度变化.

将式(1b)和(1c)代入式(1a),可得偏微分方程组：

$$\begin{cases} C_{11}\dfrac{\partial^2 u_x}{\partial x^2}+C_{44}\dfrac{\partial^2 u_x}{\partial z^2}+(C_{13}+C_{44})\dfrac{\partial^2 u_z}{\partial x\partial z}+(R_1+R_3)\dfrac{\partial^2 w_z}{\partial x\partial z}-\beta_1\dfrac{\partial\theta}{\partial x}=0, \\ C_{44}\dfrac{\partial^2 u_z}{\partial x^2}+C_{33}\dfrac{\partial^2 u_z}{\partial z^2}+R_3\dfrac{\partial^2 w_z}{\partial x^2}+R_2\dfrac{\partial^2 w_z}{\partial z^2}+(C_{13}+C_{44})\dfrac{\partial^2 u_x}{\partial x\partial z}-\beta_3\dfrac{\partial\theta}{\partial z}=0, \\ R_3\dfrac{\partial^2 u_z}{\partial x^2}+R_2\dfrac{\partial^2 u_z}{\partial z^2}+K_2\dfrac{\partial^2 w_z}{\partial x^2}+K_1\dfrac{\partial^2 w_z}{\partial z^2}+(R_1+R_3)\dfrac{\partial^2 u_x}{\partial x\partial z}=0. \end{cases} \tag{2}$$

根据 Fourier 热传导理论可知

$$q_x=-\lambda_x\frac{\partial\theta}{\partial x},\ q_z=-\lambda_z\frac{\partial\theta}{\partial z}, \tag{3}$$

$$\lambda^2\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial z^2}=0, \tag{4}$$

其中, q_x 和 q_z 代表热流密度, λ_x 和 λ_z 是一维六方准晶的热传导系数, $\lambda=\sqrt{\lambda_x/\lambda_z}$.

为了考虑开口裂纹内部介质热物理特性的影响,选择耦合边值条件[19-20]：

$$q_c=-\lambda_c\frac{\Delta\theta}{\Delta u}+\varepsilon q_0, \tag{5}$$

其中, ε 为考虑到实际情况的一个调节因子,是 q_0 的无穷小量, λ_c 是裂纹内部介质热传导率, Δu 是裂纹张开位移.当 $\lambda_c=0$, 表示裂纹内部是绝热的(热非渗透)；当 $\lambda_c\to\infty$, 表示裂纹内部是完全热导通的(热渗透)；$\lambda_c=0.024$ W/(m·K)表示 0 ℃时裂纹内部充满空气的热传导率.

如图 1 所示,考虑一维六方准晶体非周期平面内含一条中心开口裂纹,裂纹位于 x 轴上,长度为 $2c$. q_0, σ_0 和 h_0 分别代表无穷远处的热流密度、声子场载荷和相位子场载荷.

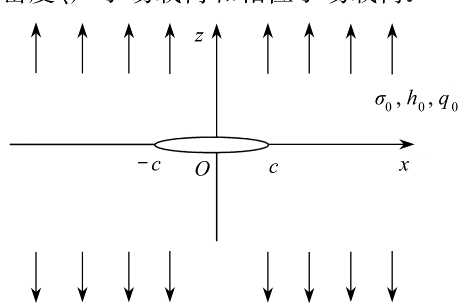


图 1　均匀的热流密度 q_0 和外载荷 σ_0, h_0 作用下的中心开口裂纹

Fig. 1　A central opening crack under uniform heat flux density q_0 and stress σ_0, h_0

假设裂纹内部充满介质,裂纹的边值条件如下：

$$q_z^+(x,0)=q_z^-(x,0)=-(q_0-q_c),\qquad |x|<c, \tag{6a}$$

$$\sigma_{zx}^+(x,0)=\sigma_{zx}^-(x,0)=0,\ H_{zx}^+(x,0)=H_{zx}^-(x,0)=0,\qquad |x|<c, \tag{6b}$$

$$\sigma_{zz}^+(x,0)=\sigma_{zz}^-(x,0)=-\sigma_0,\ H_{zz}^+(x,0)=H_{zz}^-(x,0)=-h_0,\qquad |x|<c, \tag{6c}$$

其中,上标“+”和“-”分别代表 $z>0$ 和 $z<0$ 平面的物理量.

在裂纹外连续边界条件可表示为

$$\sigma_{zx}^+(x,0)=\sigma_{zx}^-(x,0),\ \sigma_{zz}^+(x,0)=\sigma_{zz}^-(x,0),\qquad |x|>c, \tag{7a}$$

$$H_{zx}^+(x,0)=H_{zx}^-(x,0),\ H_{zz}^+(x,0)=H_{zz}^-(x,0),\qquad |x|>c, \tag{7b}$$

$$u_x^+(x,0)=u_x^-(x,0),\ u_z^+(x,0)=u_z^-(x,0),\ w_z^+(x,0)=w_z^-(x,0),\qquad |x|>c, \tag{7c}$$

$$\theta^+(x,0)=\theta^-(x,0),\ q_z^+(x,0)=q_z^-(x,0),\qquad |x|>c. \tag{7d}$$

2 温度场的全平面解

由于问题的对称性,仅讨论 $x > 0$ 部分.由式(4),利用 Fourier 积分变换, θ 可表示为

$$\theta(x,z) = \int_0^{+\infty} A^{\pm}(\xi)\mathrm{e}^{-\xi\delta^{\pm}\lambda z}\cos(\xi x)\mathrm{d}\xi, \tag{8}$$

其中 $A^{\pm}(\xi)$ 是未知的, $\delta^{\pm}=\pm 1$.再由式(3),得

$$q_x(x,z) = \lambda_x \int_0^{+\infty} \xi A^{\pm}(\xi)\mathrm{e}^{-\xi\delta^{\pm}\lambda z}\sin(\xi x)\mathrm{d}\xi, \tag{9a}$$

$$q_z(x,z) = \lambda_z\lambda \int_0^{+\infty} \xi\delta^{\pm} A^{\pm}(\xi)\mathrm{e}^{-\xi\delta^{\pm}\lambda z}\cos(\xi x)\mathrm{d}\xi. \tag{9b}$$

由于 $q_z(x,z)$ 在 x 轴上连续,可知

$$A^{+}(\xi) = -A^{-}(\xi).$$

根据边界条件(6a)和(7d),有

$$\int_0^{+\infty} \xi A^{+}(\xi)\cos(\xi x)\mathrm{d}\xi = -\frac{q_0 - q_c}{\lambda_z\lambda}, \qquad 0 < x < c, \tag{10a}$$

$$\int_0^{+\infty} A^{+}(\xi)\cos(\xi x)\mathrm{d}\xi = 0, \qquad x > c. \tag{10b}$$

式(10a)和(10b)为对偶积分方程组,其解为[21]

$$A^{+}(\xi) = -A^{-}(\xi) = -\frac{c\mathrm{J}_1(c\xi)}{\xi}\frac{q_0 - q_c}{\lambda\lambda_z}, \tag{11}$$

这里,$\mathrm{J}_1(\cdot)$是第一类 Bessel 函数.将式(11)代入式(8)、(9),得到温度场的积分表达式为

$$\theta^{\pm}(x,z) = \mp\frac{c(q_0 - q_c)}{\lambda\lambda_z}\int_0^{+\infty}\frac{\mathrm{J}_1(c\xi)}{\xi}\mathrm{e}^{-\xi\delta^{\pm}\lambda z}\cos(\xi x)\mathrm{d}\xi, \tag{12a}$$

$$q_x^{\pm}(x,z) = \mp\frac{c\lambda_x(q_0 - q_c)}{\lambda\lambda_z}\int_0^{+\infty}\mathrm{J}_1(c\xi)\mathrm{e}^{-\xi\delta^{\pm}\lambda z}\sin(\xi x)\mathrm{d}\xi, \tag{12b}$$

$$q_z^{\pm}(x,z) = -c(q_0 - q_c)\int_0^{+\infty}\mathrm{J}_1(c\xi)\mathrm{e}^{-\xi\delta^{\pm}\lambda z}\cos(\xi x)\mathrm{d}\xi. \tag{12c}$$

利用 Bessel 函数的积分结果[22],可得到温度场全平面的精确解:

$$\theta^{\pm}(x,z) = \mp\frac{q_0 - q_c}{\lambda\lambda_z}\left[\sqrt{l_2^2(\lambda) - x^2} \mp \lambda z\right], \tag{13a}$$

$$q_x^{\pm}(x,z) = \mp\lambda_x\frac{q_0 - q_c}{\lambda\lambda_z}\frac{l_1(\lambda)\sqrt{c^2 - l_1^2(\lambda)}}{l_2^2(\lambda) - l_1^2(\lambda)}, \tag{13b}$$

$$q_z^{\pm}(x,z) = -(q_0 - q_c)\left[1 - \frac{l_2(\lambda)\sqrt{l_2^2(\lambda) - c^2}}{l_2^2(\lambda) - l_1^2(\lambda)}\right], \tag{13c}$$

其中

$$l_{1,2}(\lambda) = \frac{\sqrt{(c + x)^2 + \lambda^2 z^2} \mp \sqrt{(c - x)^2 + \lambda^2 z^2}}{2}.$$

3 热应力的全平面解

为了求解热力荷载作用下的问题,需将 $u_x^{\pm}(x,z)$, $u_z^{\pm}(x,z)$ 和 $w_z^{\pm}(x,z)$ 分解为

$$u_x^{\pm}(x,z) = u_{x1}^{\pm}(x,z) + u_{x2}^{\pm}(x,z), \tag{14a}$$

$$u_z^{\pm}(x,z) = u_{z1}^{\pm}(x,z) + u_{z2}^{\pm}(x,z), \tag{14b}$$

$$w_z^{\pm}(x,z) = w_{z1}^{\pm}(x,z) + w_{z2}^{\pm}(x,z), \tag{14c}$$

其中, $u_{x1}^{\pm}(x,z)$, $u_{z1}^{\pm}(x,z)$ 和 $w_{z1}^{\pm}(x,z)$ 对应 $\theta^{\pm}(x,z)=0$ 时的通解, $u_{x2}^{\pm}(x,z)$, $u_{z2}^{\pm}(x,z)$ 和 $w_{z2}^{\pm}(x,z)$ 对应考虑热

载荷作用时的特解.

对式(2)进行 Fourier 积分变换,得到

$$u_{x1}^{\pm}(x,z)=\sum_{j=1}^{3}\int_{0}^{+\infty}A_j^{\pm}(\xi)\mathrm{e}^{-\xi\gamma_j z}\sin(\xi x)\mathrm{d}\xi, \tag{15a}$$

$$u_{z1}^{\pm}(x,z)=\sum_{j=1}^{3}\int_{0}^{+\infty}a_jA_j^{\pm}(\xi)\mathrm{e}^{-\xi\gamma_j z}\cos(\xi x)\mathrm{d}\xi, \tag{15b}$$

$$w_{z1}^{\pm}(x,z)=\sum_{j=1}^{3}\int_{0}^{+\infty}b_jA_j^{\pm}(\xi)\mathrm{e}^{-\xi\gamma_j z}\cos(\xi x)\mathrm{d}\xi, \tag{15c}$$

其中, $A_j^{\pm}(\xi)$ 为未知的, $\gamma_j(j=1,2,3)(\mathrm{Re}\,\gamma_j>0)$ 是特征方程 $e_1\gamma_j^6+e_2\gamma_j^4+e_3\gamma_j^2+e_4=0$ 的根,这里

$$a_j=n_2/n,\ b_j=-n_1/n,$$
$$n=[(C_{33}\gamma_j^2-C_{44})(R_1+R_3)+(R_3-R_2\gamma_j^2)(C_{13}+C_{44})]\gamma_j,$$
$$n_1=C_{33}C_{44}\gamma_j^4+C_{13}^2\gamma_j^2+2C_{13}C_{44}\gamma_j^2-C_{11}C_{33}\gamma_j^2+C_{11}C_{44},$$
$$n_2=C_{44}R_2\gamma_j^4+C_{13}R_1\gamma_j^2+C_{44}R_1\gamma_j^2-C_{11}R_2\gamma_j^2+C_{13}R_3\gamma_j^2+C_{11}R_3,$$
$$e_1=C_{44}(R_2^2-C_{33}K_1),$$
$$e_2=2C_{13}R_1R_2+2C_{13}R_2R_3-C_{11}R_2^2+2C_{44}R_1R_2-C_{13}^2K_1-2C_{13}C_{44}K_1+$$
$$C_{11}C_{33}K_1+C_{33}C_{44}K_2-C_{33}R_1^2-C_{33}R_3^2-2C_{33}R_1R_3,$$
$$e_3=2C_{11}R_2R_3-2C_{13}R_1R_3-2C_{13}R_3^2-C_{11}C_{44}K_1+C_{13}^2K_2+2C_{13}C_{44}K_2-C_{11}C_{33}K_2+C_{44}R_1^2,$$
$$e_4=C_{11}(C_{44}K_2-R_3^2).$$

类似地,基于 Fourier 积分变换可得到 $u_{x2}^{\pm}(x,z)$, $u_{z2}^{\pm}(x,z)$ 和 $w_{z2}^{\pm}(x,z)$ 的表达式:

$$u_{x2}^{\pm}(x,z)=\int_{0}^{+\infty}A^{*\pm}(\xi)\mathrm{e}^{-\xi\lambda z}\sin(\xi x)\mathrm{d}\xi, \tag{16a}$$

$$u_{z2}^{\pm}(x,z)=\int_{0}^{+\infty}B^{*\pm}(\xi)\mathrm{e}^{-\xi\lambda z}\cos(\xi x)\mathrm{d}\xi, \tag{16b}$$

$$w_{z2}^{\pm}(x,z)=\int_{0}^{+\infty}C^{*\pm}(\xi)\mathrm{e}^{-\xi\lambda z}\cos(\xi x)\mathrm{d}\xi, \tag{16c}$$

其中, $A^{*\pm}(\xi)$, $B^{*\pm}(\xi)$ 和 $C^{*\pm}(\xi)$ 是未知的.将式(16)代入式(2),可得

$$\begin{bmatrix}A^{*\pm}(\xi)\\B^{*\pm}(\xi)\\C^{*\pm}(\xi)\end{bmatrix}=\begin{bmatrix}N_1\\N_2\\N_3\end{bmatrix}\frac{A^{\pm}(\xi)}{\xi}, \tag{17}$$

其中

$$\begin{bmatrix}N_1\\N_2\\N_3\end{bmatrix}=\begin{bmatrix}C_{11}-C_{44}\lambda^2 & -(C_{13}+C_{44})\lambda & -(R_1+R_3)\lambda\\(C_{13}+C_{44})\lambda & C_{44}-C_{33}\lambda^2 & -(R_2\lambda^2-R_3)\\(R_1+R_3)\lambda & R_3-R_2\lambda^2 & -(K_1\lambda^2-K_2)\end{bmatrix}^{-1}\begin{bmatrix}\beta_1\\\beta_3\lambda\\0\end{bmatrix}.$$

从而,应力的通解可表示为

$$\sigma_{xx}^{\pm}(x,z)=\sum_{j=1}^{3}\int_{0}^{+\infty}(C_{11}-C_{13}\gamma_ja_j-R_1\gamma_jb_j)\xi A_j^{\pm}(\xi)\mathrm{e}^{-\xi\gamma_j z}\cos(\xi x)\mathrm{d}\xi+$$
$$(C_{11}N_1-C_{13}\lambda N_2-R_1\lambda N_3-\beta_1)\int_{0}^{+\infty}A^{\pm}(\xi)\mathrm{e}^{-\xi\lambda z}\cos(\xi x)\mathrm{d}\xi, \tag{18a}$$

$$\sigma_{zz}^{\pm}(x,z)=\sum_{j=1}^{3}\int_{0}^{+\infty}(C_{13}-C_{33}\gamma_ja_j-R_2\gamma_jb_j)\xi A_j^{\pm}(\xi)\mathrm{e}^{-\xi\gamma_j z}\cos(\xi x)\mathrm{d}\xi+$$
$$(C_{13}N_1-C_{33}\lambda N_2-R_2\lambda N_3-\beta_3)\int_{0}^{+\infty}A^{\pm}(\xi)\mathrm{e}^{-\xi\lambda z}\cos(\xi x)\mathrm{d}\xi, \tag{18b}$$

$$\sigma_{zx}^{\pm}(x,z)=-\sum_{j=1}^{3}\int_{0}^{+\infty}(C_{44}a_j+C_{44}\gamma_j+R_3b_j)\xi A_j^{\pm}(\xi)\mathrm{e}^{-\xi\gamma_j z}\sin(\xi x)\mathrm{d}\xi-$$

$$(C_{44}N_2 + C_{44}\lambda N_1 + R_3N_3)\int_0^{+\infty} A^{\pm}(\xi)\mathrm{e}^{-\xi\lambda z}\sin(\xi x)\mathrm{d}\xi, \tag{18c}$$

$$H_{zz}^{\pm}(x,z) = \sum_{j=1}^{3}\int_0^{+\infty}(R_1 - R_2\gamma_j a_j - K_1\gamma_j b_j)\xi A_j^{\pm}(\xi)\mathrm{e}^{-\xi\gamma_j z}\cos(\xi x)\mathrm{d}\xi +$$
$$(R_1N_1 - R_2\lambda N_2 - K_1\lambda N_3)\int_0^{+\infty} A^{\pm}(\xi)\mathrm{e}^{-\xi\lambda z}\cos(\xi x)\mathrm{d}\xi, \tag{18d}$$

$$H_{zx}^{\pm}(x,z) = -\sum_{j=1}^{3}\int_0^{+\infty}(R_3 a_j + R_3\gamma_j + K_2 b_j)\xi A_j^{\pm}(\xi)\mathrm{e}^{-\xi\gamma_j z}\sin(\xi x)\mathrm{d}\xi -$$
$$(R_3N_2 + R_3\lambda N_1 + K_2N_3)\int_0^{+\infty} A^{\pm}(\xi)\mathrm{e}^{-\xi\lambda z}\sin(\xi x)\mathrm{d}\xi. \tag{18e}$$

为得到式(18)的封闭解,可将问题等效为两个问题的叠加：第一个问题仅研究力作用,第二个问题仅讨论热载荷.首先,仅考虑力作用时的自由边界条件为

$$u_z^+(x,0) = u_z^-(x,0) = 0,\ w_z^+(x,0) = w_z^-(x,0) = 0, \qquad x > c, \tag{19a}$$

$$\sigma_{zx}^+(x,0) = \sigma_{zx}^-(x,0) = 0,\ H_{zx}^+(x,0) = H_{zx}^-(x,0) = 0, \qquad 0 < x < +\infty. \tag{19b}$$

结合式(18c)、(18e)和(19b),可以得到

$$A_j^+(\xi) = A_j^-(\xi), \tag{20a}$$

$$A_j^+(\xi) = \eta_j A_1^+(\xi), \tag{20b}$$

其中

$$\eta_1 = 1,\ \eta_2 = -\frac{(a_1b_3 + \gamma_1b_3) - (a_3b_1 + \gamma_3b_1)}{(a_2b_3 + \gamma_2b_3) - (a_3b_2 + \gamma_3b_2)},\ \eta_3 = \frac{(a_1b_2 + \gamma_1b_2) - (a_2b_1 + \gamma_2b_1)}{(a_2b_3 + \gamma_2b_3) - (a_3b_2 + \gamma_3b_2)}.$$

由式(6c)和(19a),可得

$$\int_0^{+\infty}\xi A_1^+(\xi)\cos(\xi x)\mathrm{d}\xi = \frac{\sigma_0}{\kappa_1} = \frac{h_0}{\kappa_1'}, \qquad 0 < x < c, \tag{21a}$$

$$\int_0^{+\infty}\xi A_1^+(\xi)\cos(\xi x)\mathrm{d}\xi = 0, \qquad x > c, \tag{21b}$$

其中

$$\kappa_1 = -(C_{13} - C_{33}\gamma_1a_1 - R_2\gamma_1b_1)\eta_1 - (C_{13} - C_{33}\gamma_2a_2 - R_2\gamma_2b_2)\eta_2 - (C_{13} - C_{33}\gamma_3a_3 - R_2\gamma_3b_3)\eta_3,$$
$$\kappa_1' = -(R_1 - R_2\gamma_1a_1 - K_1\gamma_1b_1)\eta_1 - (R_1 - R_2\gamma_2a_2 - K_1\gamma_2b_2)\eta_2 - (R_1 - R_2\gamma_3a_3 - K_1\gamma_3b_3)\eta_3.$$

式(21a)和(21b)是对偶积分方程,其解为[21]

$$A_1^+(\xi) = A_1^-(\xi) = \frac{c\mathrm{J}_1(c\xi)}{\xi}\frac{\sigma_0}{\kappa_1} = \frac{c\mathrm{J}_1(c\xi)}{\xi}\frac{h_0}{\kappa_1'}. \tag{22}$$

仅考虑热载荷 q_0 时,边界条件可表示为

$$\sigma_{zz}^+(x,0) = \sigma_{zz}^-(x,0) = 0,\ H_{zz}^+(x,0) = H_{zz}^-(x,0) = 0, \qquad 0 < x < +\infty, \tag{23a}$$

$$u_x^+(x,0) = u_x^-(x,0) = 0, \qquad 0 < x < +\infty. \tag{23b}$$

由式(18b)、(18d)和(23a),可得

$$\begin{cases} A_j^+(\xi) = -A_j^-(\xi), \\ \sum_{j=1}^{3}(C_{13} - C_{33}\gamma_j a_j - R_2\gamma_j b_j)A_j^+(\xi) = (C_{33}\lambda N_2 + R_2\lambda N_3 + \beta_3 - C_{13}N_1)\dfrac{A^+(\xi)}{\xi}, \\ \sum_{j=1}^{3}(R_1 - R_2\gamma_j a_j - K_1\gamma_j b_j)A_j^+(\xi) = (R_2\lambda N_2 + K_1\lambda N_3 - R_1N_1)\dfrac{A^+(\xi)}{\xi}. \end{cases} \tag{24}$$

由式(6b)和(23b),可得

$$\sum_{j=1}^{3}\int_0^{+\infty}(C_{44}a_j + C_{44}\gamma_j + R_3b_j)\xi A_j^{\pm}(\xi)\sin(\xi x)\mathrm{d}\xi +$$
$$\int_0^{+\infty}(C_{44}N_2 + C_{44}\lambda N_1 + R_3N_3)A^{\pm}(\xi)\sin(\xi x)\mathrm{d}\xi = 0, \qquad 0 < x < c, \tag{25a}$$

$$\int_0^{+\infty}\left[\sum_{j=1}^{3}A_j^+(\xi)+N_1\frac{A^+(\xi)}{\xi}\right]\sin(\xi x)\mathrm{d}\xi=0,\qquad x>c. \tag{25b}$$

为了求解上述对偶积分方程,引入函数

$$\varphi(x)=\frac{\partial}{\partial x}u_x(x,0),$$

即

$$\sum_{j=1}^{3}\xi A_j^+(\xi)+N_1A^+(\xi)=\frac{2}{\pi}\int_0^a\varphi(s)\cos(\xi s)\mathrm{d}s. \tag{26}$$

由式(25a)和(26),可得

$$A_1^+(\xi)\xi=\frac{M_2(\gamma_3b_3-\gamma_2b_2)+M_3(\gamma_2a_2-\gamma_3a_3)}{M_1}A^+(\xi)-\frac{M_4}{M_1}\frac{2}{\pi}\int_0^c\varphi(s)\cos(\xi s)\mathrm{d}s, \tag{27a}$$

$$A_2^+(\xi)\xi=-\frac{M_2(\gamma_3b_3-\gamma_1b_1)+M_3(\gamma_1a_1-\gamma_3a_3)}{M_1}A^+(\xi)+\frac{M_5}{M_1}\frac{2}{\pi}\int_0^c\varphi(s)\cos(\xi s)\mathrm{d}s, \tag{27b}$$

$$A_3^+(\xi)\xi=-\frac{M_2(\gamma_1b_1-\gamma_2b_2)+M_3(\gamma_2a_2-\gamma_1a_1)}{M_1}A^+(\xi)+\frac{M_6}{M_1}\frac{2}{\pi}\int_0^c\varphi(s)\cos(\xi s)\mathrm{d}s, \tag{27c}$$

其中

$$M_1=(C_{33}K_1-R_2^2)[\gamma_1\gamma_2(a_1b_2-a_2b_1)+\gamma_1\gamma_3(a_3b_1-a_1b_3)+\gamma_2\gamma_3(a_2b_3-a_3b_2)],$$

$$M_2=(C_{33}K_1-R_2^2)\lambda N_2+K_1\beta_3-C_{13}K_1N_1+R_1R_2N_1,$$

$$M_3=(C_{33}K_1-R_2^2)\lambda N_3-R_2\beta_3+C_{13}R_2N_1-C_{33}R_1N_1,$$

$$M_4=(C_{33}R_1-C_{13}R_2)(\gamma_2a_2-\gamma_3a_3)+(C_{13}K_1-R_1R_2)(\gamma_3b_3-\gamma_2b_2)+(C_{33}K_1-R_2^2)\gamma_2\gamma_3(a_3b_2-a_2b_3),$$

$$M_5=(C_{33}R_1-C_{13}R_2)(\gamma_1a_1-\gamma_3a_3)+(C_{13}K_1-R_1R_2)(\gamma_3b_3-\gamma_1b_1)+(C_{33}K_1-R_2^2)\gamma_1\gamma_3(a_3b_1-a_1b_3),$$

$$M_6=(C_{33}R_1-C_{13}R_2)(\gamma_2a_2-\gamma_1a_1)+(C_{13}K_1-R_1R_2)(\gamma_1b_1-\gamma_2b_2)+(C_{33}K_1-R_2^2)\gamma_1\gamma_2(a_1b_2-a_2b_1).$$

将式(27)代入式(26),可得

$$\frac{1}{\pi}\int_{-c}^{c}\frac{\varphi(s)}{x-s}\mathrm{d}s=\kappa_2\int_0^{+\infty}A^+(\xi)\sin(\xi x)\mathrm{d}\xi, \tag{28}$$

其中

$$\kappa_2=\frac{\kappa_{21}}{\kappa_{22}},$$

$$\kappa_{21}=(C_{44}a_2+C_{44}\gamma_2+R_3b_2)[M_2(\gamma_3b_3-\gamma_1b_1)+M_3(\gamma_1a_1-\gamma_3a_3)]+(C_{44}a_3+C_{44}\gamma_3+R_3b_3)[M_2(\gamma_1b_1-\gamma_2b_2)+M_3(\gamma_2a_2-\gamma_1a_1)]-(C_{44}N_2+C_{44}\lambda N_1+R_3N_3)M_1-(C_{44}a_1+C_{44}\gamma_1+R_3b_1)[M_2(\gamma_3b_3-\gamma_2b_2)+M_3(\gamma_2a_2-\gamma_3a_3)],$$

$$\kappa_{22}=M_5(C_{44}a_2+C_{44}\gamma_2+R_3b_2)+M_6(C_{44}a_3+C_{44}\gamma_3+R_3b_3)-M_4(C_{44}a_1+C_{44}\gamma_1+R_3b_1).$$

由式(11),式(28)可改写为

$$\frac{1}{\pi}\int_{-a}^{a}\frac{\varphi(s)}{x_1-s}\mathrm{d}s=-\frac{q_0-q_c}{\lambda\lambda_3}\kappa_2x_1, \tag{29}$$

其中

$$\int_0^{+\infty}\frac{\mathrm{J}_1(c\xi)}{\xi}\sin(\xi x)\mathrm{d}\xi=\frac{x}{c},\qquad 0<x<c.$$

式(29)是一个带 Cauchy 核的第一类奇异积分方程,其解为

$$\varphi(x)=\frac{(q_0-q_c)\kappa_2}{2\lambda\lambda_z\sqrt{(c^2-x^2)}}(2x^2-c^2). \tag{30}$$

代入式(26),有

$$\frac{2}{\pi}\int_0^c\varphi(s)\cos(\xi s)\,\mathrm{d}s=\frac{(q_0-q_c)\kappa_2}{\lambda\lambda_z}\left[\frac{c^2}{2}\mathrm{J}_0(c\xi)-\frac{c\mathrm{J}_1(c\xi)}{\xi}\right].$$

综上,可得热力荷载作用下应力场的全平面精确解:

$$\begin{aligned}\sigma_{xx}^{\pm}(x,z)=&\frac{\sigma_0}{\kappa_1}\sum_{j=1}^{3}(C_{11}-C_{13}\gamma_ja_j-R_1\gamma_jb_j)\eta_j\left[1-\frac{l_2(\gamma_j)\sqrt{l_2^2(\gamma_j)-c^2}}{l_2^2(\gamma_j)-l_1^2(\gamma_j)}\right]\mp\\&\frac{q_0-q_c}{\lambda\lambda_z}\sum_{j=1}^{3}(C_{11}-C_{13}\gamma_ja_j-R_1\gamma_jb_j)\varpi_j\left[\sqrt{l_2^2(\gamma_j)-x^2}\mp\gamma_jz\right]\mp\\&\frac{(q_0-q_c)\kappa_2c^2}{2\lambda\lambda_z}\sum_{j=1}^{3}(C_{11}-C_{13}\gamma_ja_j-R_1\gamma_jb_j)\chi_j\frac{\sqrt{l_2^2(\gamma_j)-x^2}}{l_2^2(\gamma_j)-l_1^2(\gamma_j)}\mp\\&\frac{q_0-q_c}{\lambda\lambda_z}(C_{11}N_1-C_{13}\lambda N_2-R_1\lambda N_3-\beta_1)\left[\sqrt{l_2^2(\lambda)-x^2}\mp\lambda z\right],\end{aligned} \tag{31a}$$

$$\begin{aligned}\sigma_{zz}^{\pm}(x,z)=&\frac{\sigma_0}{\kappa_1}\sum_{j=1}^{3}(C_{13}-C_{33}\gamma_ja_j-R_2\gamma_jb_j)\eta_j\left[1-\frac{l_2(\gamma_j)\sqrt{l_2^2(\gamma_j)-c^2}}{l_2^2(\gamma_j)-l_1^2(\gamma_j)}\right]\mp\\&\frac{q_0-q_c}{\lambda\lambda_z}\sum_{j=1}^{3}(C_{13}-C_{33}\gamma_ja_j-R_2\gamma_jb_j)\varpi_j\left[\sqrt{l_2^2(\gamma_j)-x^2}\mp\gamma_jz\right]\mp\\&\frac{(q_0-q_c)\kappa_2c^2}{2\lambda\lambda_z}\sum_{j=1}^{3}(C_{13}-C_{33}\gamma_ja_j-R_2\gamma_jb_j)\chi_j\frac{\sqrt{l_2^2(\gamma_j)-x^2}}{l_2^2(\gamma_j)-l_1^2(\gamma_j)}\mp\\&\frac{q_0-q_c}{\lambda\lambda_z}(C_{13}N_1-C_{33}\lambda N_2-R_2\lambda N_3-\beta_3)\left[\sqrt{l_2^2(\lambda)-x^2}\mp\lambda z\right],\end{aligned} \tag{31b}$$

$$\begin{aligned}H_{zz}^{\pm}(x,z)=&\frac{h_0}{\kappa_1'}\sum_{j=1}^{3}(R_1-R_2\gamma_ja_j-K_1\gamma_jb_j)\eta_j\left[1-\frac{l_2(\gamma_j)\sqrt{l_2^2(\gamma_j)-c^2}}{l_2^2(\gamma_j)-l_1^2(\gamma_j)}\right]\mp\\&\frac{q_0-q_c}{\lambda\lambda_z}\sum_{j=1}^{3}(R_1-R_2\gamma_ja_j-K_1\gamma_jb_j)\varpi_j\left[\sqrt{l_2^2(\gamma_j)-x^2}\mp\gamma_jz\right]\mp\\&\frac{(q_0-q_c)\kappa_2c^2}{2\lambda\lambda_z}\sum_{j=1}^{3}(R_1-R_2\gamma_ja_j-K_1\gamma_jb_j)\chi_j\frac{\sqrt{l_2^2(\gamma_j)-x^2}}{l_2^2(\gamma_j)-l_1^2(\gamma_j)}\mp\\&\frac{q_0-q_c}{\lambda\lambda_z}(R_1N_1-R_2\lambda N_2-K_1\lambda N_3)\left[\sqrt{l_2^2(\lambda)-x^2}\mp\lambda z\right],\end{aligned} \tag{31c}$$

$$\begin{aligned}\sigma_{zx}^{\pm}(x,z)=&\frac{q_0-q_c}{\lambda\lambda_z}\sum_{j=1}^{3}(C_{44}a_j+C_{44}\gamma_j+R_3b_j)\varpi_j\left[x-\sqrt{x^2-l_1^2(\gamma_j)}\right]+\\&\frac{q_0-q_c}{\lambda\lambda_z}(C_{44}N_2+C_{44}\lambda N_1+R_3N_3)\left[x-\sqrt{x^2-l_1^2(\lambda)}\right]+\\&\frac{(q_0-q_c)\kappa_2c^2}{2\lambda\lambda_z}\sum_{j=1}^{3}(C_{44}a_j+C_{44}\gamma_j+R_3b_j)\chi_j\frac{\sqrt{x^2-l_1^2(\gamma_j)}}{l_2^2(\gamma_j)-l_1^2(\gamma_j)}\mp\\&\frac{\sigma_0}{\kappa_1}\sum_{j=1}^{3}(C_{44}a_j+C_{44}\gamma_j+R_3b_j)\eta_j\left[\frac{l_1(\gamma_j)\sqrt{c^2-l_1^2(\gamma_j)}}{l_2^2(\gamma_j)-l_1^2(\gamma_j)}\right],\end{aligned} \tag{31d}$$

$$H_{zx}^{\pm}(x,z)=\frac{q_0-q_c}{\lambda\lambda_z}\sum_{j=1}^{3}(R_3a_j+R_3\gamma_j+K_2b_j)\varpi_j\left[x-\sqrt{x^2-l_1^2(\gamma_j)}\right]+$$

$$\frac{q_0-q_c}{\lambda\lambda_z}(R_3N_2+R_3\lambda N_1+K_2N_3)\left[x-\sqrt{x^2-l_1^2(\lambda)}\right]+$$
$$\frac{(q_0-q_c)\kappa_2c^2}{2\lambda\lambda_z}\sum_{j=1}^{3}(R_3a_j+R_3\gamma_j+K_2b_j)\chi_j\frac{\sqrt{x^2-l_1^2(\gamma_j)}}{l_2^2(\gamma_j)-l_1^2(\gamma_j)}\mp$$
$$\frac{h_0}{\kappa_1'}\sum_{j=1}^{3}(R_3a_j+R_3\gamma_j+K_2b_j)\eta_j\left[\frac{l_1(\gamma_j)\sqrt{c^2-l_1^2(\gamma_j)}}{l_2^2(\gamma_j)-l_1^2(\gamma_j)}\right], \tag{31e}$$

其中

$$l_{1,2}(\lambda)=\frac{\sqrt{(c+x)^2+\lambda^2z^2}\mp\sqrt{(c-x)^2+\lambda^2z^2}}{2},$$

$$l_{1,2}(\gamma_j)=\frac{\sqrt{(c+x)^2+\gamma_j^2z^2}\mp\sqrt{(c-x)^2+\gamma_j^2z^2}}{2},$$

$$\varpi_1=\frac{M_2(\gamma_3b_3-\gamma_2b_2)+M_3(\gamma_2a_2-\gamma_3a_3)-\kappa_2M_4}{M_1},$$

$$\varpi_2=-\frac{M_2(\gamma_3b_3-\gamma_1b_1)+M_3(\gamma_1a_1-\gamma_3a_3)-\kappa_2M_5}{M_1},$$

$$\varpi_3=-\frac{M_2(\gamma_1b_1-\gamma_2b_2)+M_3(\gamma_2a_2-\gamma_1a_1)-\kappa_2M_6}{M_1},$$

$$\chi_1=\frac{M_4}{M_1},\ \chi_2=-\frac{M_5}{M_1},\ \chi_3=-\frac{M_6}{M_1}.$$

对于 $z=0, x>c$, 有

$$\sigma_{xx}^{\pm}(x,0)=\frac{\sigma_0}{\kappa_1}\sum_{j=1}^{3}(C_{11}-C_{13}\gamma_ja_j-R_1\gamma_jb_j)\eta_j\left(1-\frac{x\sqrt{x^2-c^2}}{x^2-c^2}\right), \tag{32a}$$

$$\sigma_{zz}^{\pm}(x,0)=\frac{\sigma_0}{\kappa_1}\sum_{j=1}^{3}(C_{13}-C_{33}\gamma_ja_j-R_2\gamma_jb_j)\eta_j\left(1-\frac{x\sqrt{x^2-c^2}}{x^2-c^2}\right), \tag{32b}$$

$$H_{zz}^{\pm}(x,0)=\frac{h_0}{\kappa_1'}\sum_{j=1}^{3}(R_1-R_2\gamma_ja_j-K_1\gamma_jb_j)\eta_j\left(1-\frac{x\sqrt{x^2-c^2}}{x^2-c^2}\right), \tag{32c}$$

$$\sigma_{zx}^{\pm}(x,0)=\frac{q_0-q_c}{\lambda\lambda_z}\sum_{j=1}^{3}(C_{44}a_j+C_{44}\gamma_j+R_3b_j)\varpi_j(x-\sqrt{x^2-c^2})+$$
$$\frac{q_0-q_c}{\lambda\lambda_z}(C_{44}N_2+C_{44}\lambda N_1+R_3N_3)(x-\sqrt{x^2-c^2})+$$
$$\frac{(q_0-q_c)\kappa_2c^2}{2\lambda\lambda_z}\sum_{j=1}^{3}(C_{44}a_j+C_{44}\gamma_j+R_3b_j)\chi_j\frac{\sqrt{x^2-c^2}}{x^2-c^2}, \tag{32d}$$

$$H_{zx}^{\pm}(x,0)=\frac{q_0-q_c}{\lambda\lambda_z}\sum_{j=1}^{3}(R_3a_j+R_3\gamma_j+K_2b_j)\varpi_j(x-\sqrt{x^2-c^2})+$$
$$\frac{q_0-q_c}{\lambda\lambda_z}(R_3N_2+R_3\lambda N_1+K_2N_3)(x-\sqrt{x^2-c^2})+$$
$$\frac{(q_0-q_c)\kappa_2c^2}{2\lambda\lambda_z}\sum_{j=1}^{3}(R_3a_j+R_3\gamma_j+K_2b_j)\chi_j\frac{\sqrt{x^2-c^2}}{x^2-c^2}, \tag{32e}$$

这里

$$q_c=\frac{q_{c1}}{q_{c2}}+\varepsilon q_0,\ q_{c1}=\lambda_cq_0\kappa_1[(a_2b_3+\gamma_2b_3)-(a_3b_2+\gamma_3b_2)],$$

$$q_{c2} = \lambda\lambda_z\sigma_0[\gamma_1(a_3b_2 - a_2b_3) + \gamma_2(a_1b_3 - a_3b_1) + \gamma_3(a_2b_1 - a_1b_2)] + \lambda_c\kappa_1[(a_2b_3 + \gamma_2b_3) - (a_3b_2 + \gamma_3b_2)].$$

4 热应力强度因子

定义裂纹尖端热应力强度因子为[23]

$$K_\sigma = K_{\mathrm{I}}^\sigma + \mathrm{i}K_{\mathrm{II}}^\sigma = \lim_{x\to c^+}\sqrt{2\pi(x-c)}[\sigma_{zz}^\pm(x,0) + \mathrm{i}\sigma_{zx}^\pm(x,0)], \tag{33a}$$

$$K_H = K_{\mathrm{I}}^H + \mathrm{i}K_{\mathrm{II}}^H = \lim_{x\to c^+}\sqrt{2\pi(x-c)}[H_{zz}^\pm(x,0) + \mathrm{i}H_{zx}^\pm(x,0)], \tag{33b}$$

其中，K_{I}^σ 和 K_{II}^σ 分别表示声子场的Ⅰ型和Ⅱ型热应力强度因子，K_{I}^H 和 K_{II}^H 分别表示相位子场的Ⅰ型和Ⅱ型热应力强度因子.

将式(32)代入式(33)，可得

$$K_\sigma = \left[\frac{\sigma_0}{\kappa_1}\sum_{j=1}^{3}(C_{33}\gamma_ja_j - C_{13} + R_2\gamma_jb_j)\eta_j + \mathrm{i}\frac{(q_0-q_c)\kappa_2c}{2\lambda\lambda_z}\sum_{j=1}^{3}(C_{44}a_j + C_{44}\gamma_j + R_3b_j)\chi_j\right]\sqrt{\pi c}, \tag{34a}$$

$$K_H = \left[\frac{h_0}{\kappa_1'}\sum_{j=1}^{3}(R_2\gamma_ja_j - R_1 + K_1\gamma_jb_j)\eta_j + \mathrm{i}\frac{(q_0-q_c)\kappa_2c}{2\lambda\lambda_z}\sum_{j=1}^{3}(R_3a_j + R_3\gamma_j + K_2b_j)\chi_j\right]\sqrt{\pi c}. \tag{34b}$$

令裂纹尖端应变能密度因子 S 为[24]

$$S = r\frac{\mathrm{d}W}{\mathrm{d}V} = \frac{1}{4\pi}\left\{\frac{g_{21}}{g_{22}} + \frac{(K_\sigma)^2K_2 - 2K_\sigma K_HR_3 + (K_H)^2C_{44}}{C_{44}K_2 - R_3^2}\right\}, \tag{35}$$

其中，r 表示距离裂纹尖端的位移，$\mathrm{d}W/\mathrm{d}V$ 表示一维六方准晶材料单位体积的应变能，

$$\tilde{\kappa} = \frac{1}{\kappa_1}\sum_{j=1}^{3}(C_{11} - C_{13}\gamma_ja_j - R_1\gamma_jb_j)\eta_j,$$

$$g_{21} = \tilde{\kappa}^2(K_{\mathrm{I}}^\sigma)^2(R_2^2 - C_{33}K_1) + (K_{\mathrm{I}}^\sigma)^2(R_1^2 - C_{11}K_1) + (K_{\mathrm{I}}^H)^2(C_{13}^2 - C_{11}C_{33}) + 2\tilde{\kappa}(K_{\mathrm{I}}^\sigma)^2(C_{13}K_1 - R_1R_2) + 2K_{\mathrm{I}}^\sigma K_{\mathrm{I}}^H(C_{11}R_2 - C_{13}R_1) + 2\tilde{\kappa}K_{\mathrm{I}}^\sigma K_{\mathrm{I}}^H(C_{33}R_1 - C_{13}R_2),$$

$$g_{22} = [(C_{13}K_1 - R_1R_2)^2 - (C_{11}K_1 - R_1^2)(C_{33}K_1 - R_2^2)]/K_1.$$

当 $\lambda_c = 0$ 时，式(34)和(35)退化为

$$K_\sigma^{\mathrm{m}} = \left[-\frac{\sigma_0}{\kappa_1}\sum_{j=1}^{3}(C_{13} - C_{33}\gamma_ja_j - R_2\gamma_jb_j)\eta_j + \mathrm{i}\frac{q_0\kappa_2c}{2\lambda\lambda_z}\sum_{j=1}^{3}(C_{44}a_j + C_{44}\gamma_j + R_3b_j)\chi_j\right]\sqrt{\pi c}, \tag{36a}$$

$$K_H^{\mathrm{m}} = \left[-\frac{h_0}{\kappa_1'}\sum_{j=1}^{3}(R_1 - R_2\gamma_ja_j - K_1\gamma_jb_j)\eta_j + \mathrm{i}\frac{q_0\kappa_2c}{2\lambda\lambda_z}\sum_{j=1}^{3}(R_3a_j + R_3\gamma_j + K_2b_j)\chi_j\right]\sqrt{\pi c}, \tag{36b}$$

$$S_{\mathrm{m}} = r\frac{\mathrm{d}W}{\mathrm{d}V} = \frac{1}{4\pi}\left\{\frac{g_{21}}{g_{22}} + \frac{(K_\sigma^{\mathrm{m}})^2K_2 - 2K_\sigma^{\mathrm{m}}K_H^{\mathrm{m}}R_3 + (K_H^{\mathrm{m}})^2C_{44}}{C_{44}K_2 - R_3^2}\right\}. \tag{37}$$

5 数值结果

表 1 给出了一维六方准晶体的弹性常数.由式(21a)可知，h_0 与 σ_0 线性相关，因此只需讨论 σ_0 和 q_0 对 $K_\sigma/K_\sigma^{\mathrm{m}}$，$K_H/K_H^{\mathrm{m}}$ 和 S/S_{m} 的影响.

表 1 一维六方压电准晶体弹性常数[15]

Table 1 1D hexagonal piezoelectric quasicrystal elastic constants[15]

C_{11} /GPa	C_{12} /GPa	C_{13} /GPa	C_{33} /GPa	C_{44} /GPa	K_1 /GPa	K_2 /GPa
150	55	45	90	50	0.084	0.036
R_1 /GPa	R_2 /GPa	R_3 /GPa	β_1 /(MPa/K)	β_3 /(MPa/K)	λ_x /(W/(m·K))	λ_z /(W/(m·K))
−1.68	1.2	1.2	1.798	1.383	12.4	12.4

此外，取 $\varepsilon = 0.02$[20]，$\lambda_c = 0.024$ W/(m·K)，$q_0 = 20$ W/m^2，$\sigma_0 = 80$ MPa，$c = 0.1$ mm.

图 2 给出了 K_1, K_2, R_1, R_2, R_3 趋于 0 时，q_c/q_0，$K_\sigma/K_\sigma^{\mathrm{m}}$ 随 λ_c/λ_z 和 σ_0 的变化趋势，从图中可以看出本文的退化结果与文献[25]中的结论一致.

图 3 描述了 q_c/q_0,K_σ/K_σ^m,K_H/K_H^m 随 λ_c/λ_z 的变化趋势.由于声子场-相位子场多场耦合效应的影响,q_c/q_0,K_σ/K_σ^m,K_H/K_H^m 在 $0 \leqslant \lambda_c/\lambda_z \leqslant 0.2$ 时变化比较显著.当忽略多场耦合效应时,随着 λ_c/λ_z 的增加,K_σ/K_σ^m,K_H/K_H^m 逐渐减小.

图 4 给出了 q_c/q_0,K_σ/K_σ^m,K_H/K_H^m 随 σ_0 的变化趋势,由图中可以看出随着 σ_0 的增大,K_σ/K_σ^m,K_H/K_H^m 逐渐减小而 q_c/q_0 逐渐增大.

图 5、6 给出了 R_1,R_2,R_3 变化时,K_σ/K_σ^m,K_H/K_H^m 随 q_0 的变化趋势.

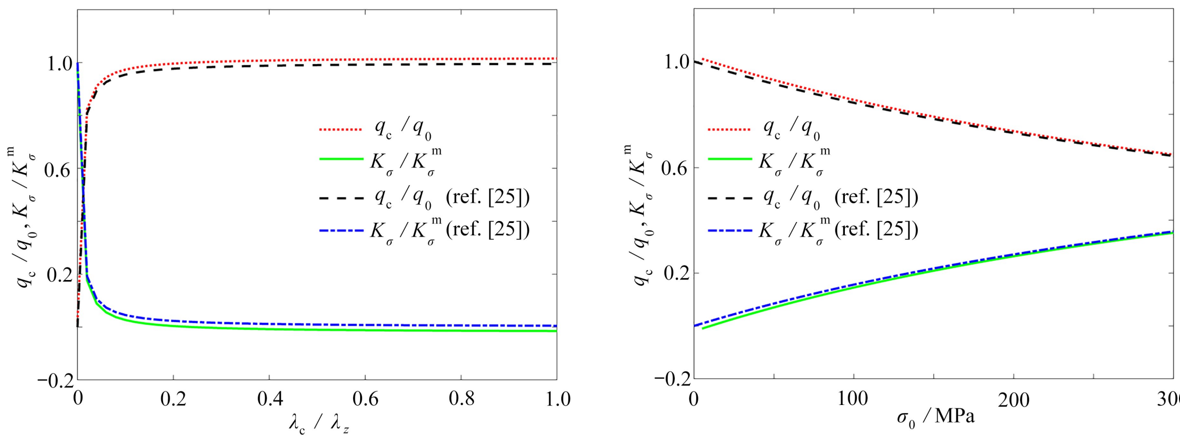


图 2　q_c/q_0, K_σ/K_σ^m 随 λ_c/λ_z 和 σ_0 的变化趋势

Fig. 2　Variations of q_c/q_0, K_σ/K_σ^m with λ_c/λ_z and σ_0

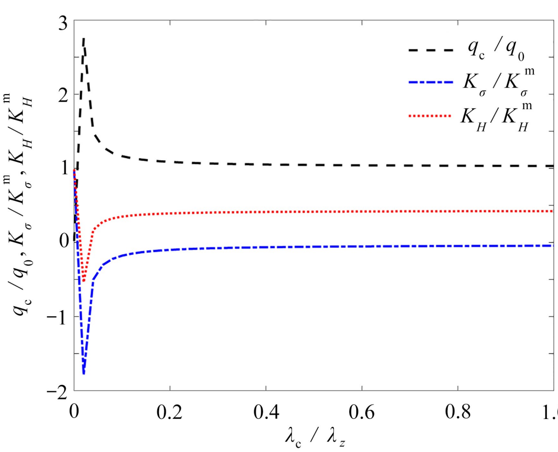


图 3　q_c/q_0, K_σ/K_σ^m 和 K_H/K_H^m 随 λ_c/λ_z 的变化趋势

Fig. 3　Variations of q_c/q_0, K_σ/K_σ^m and K_H/K_H^m with λ_c/λ_z

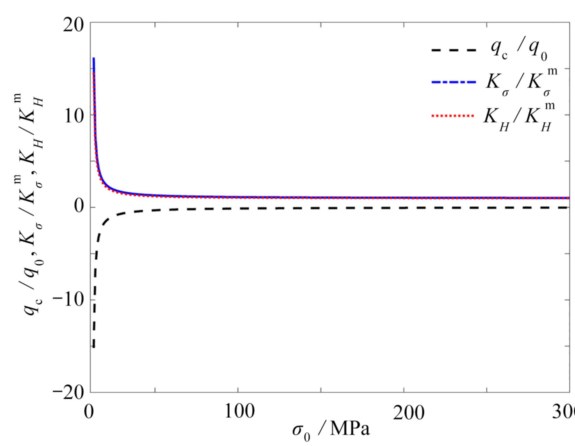
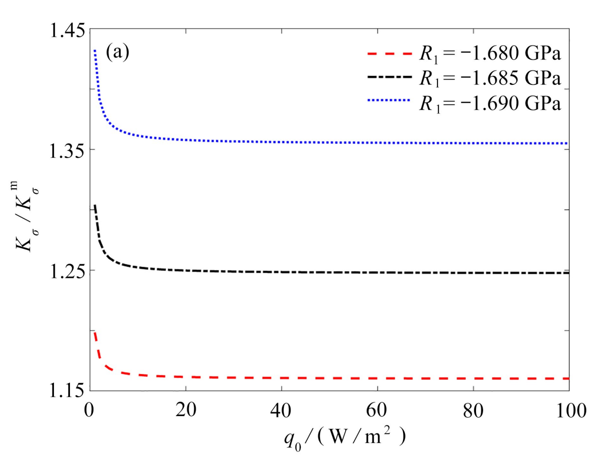


图 4　q_c/q_0, K_σ/K_σ^m 和 K_H/K_H^m 随 σ_0 的变化趋势

Fig. 4　Variations of q_c/q_0, K_σ/K_σ^m and K_H/K_H^m with σ_0



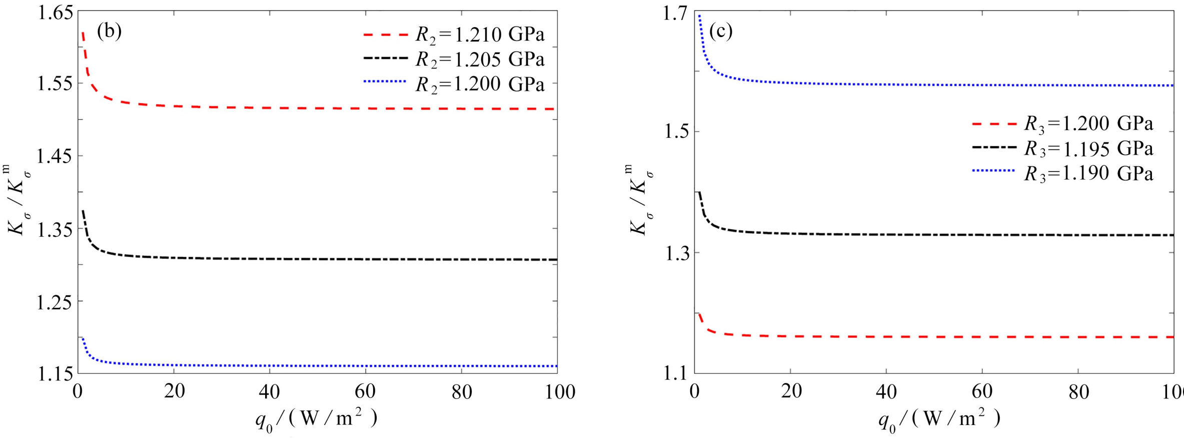


图 5　$K_\sigma/K_\sigma^{\mathrm{m}}$ 随 q_0 变化

Fig. 5　Variations of $K_\sigma/K_\sigma^{\mathrm{m}}$ with q_0

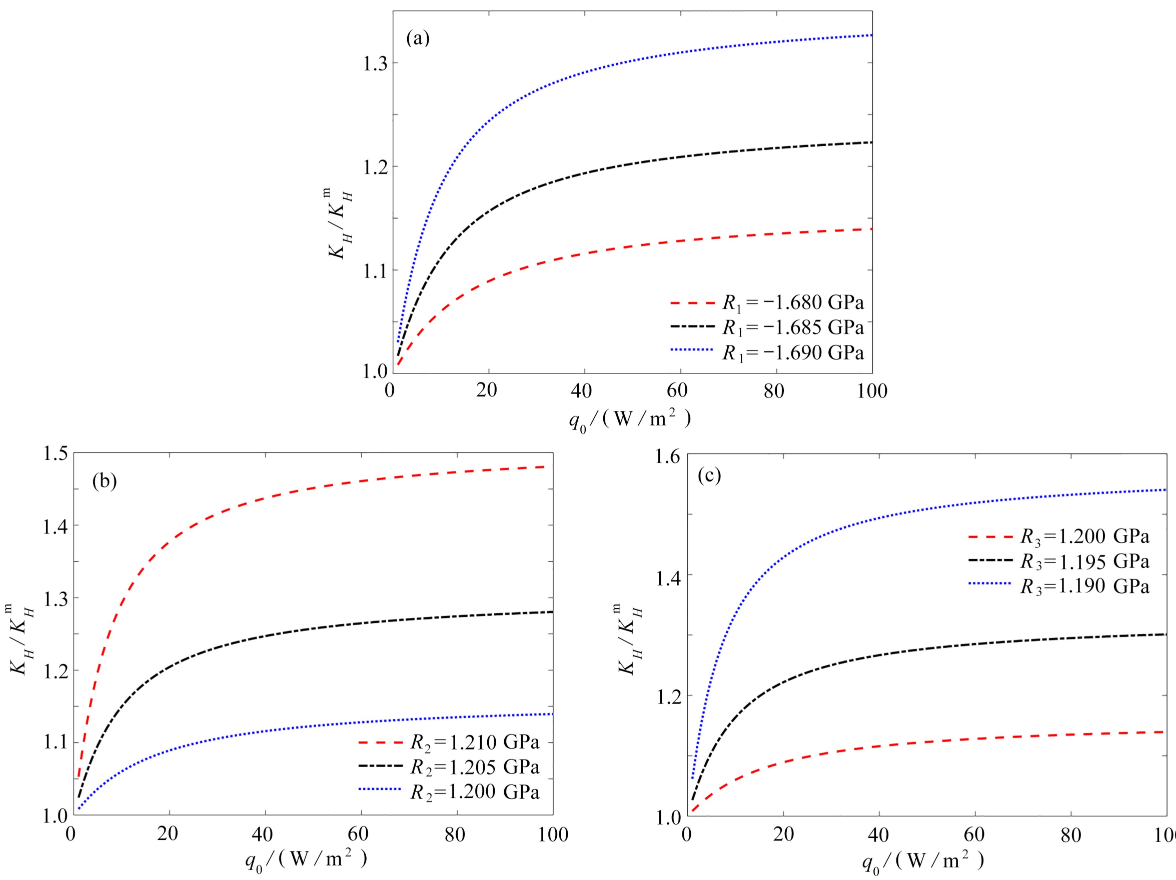


图 6　K_H/K_H^{m} 随 q_0 变化

Fig. 6　Variations of K_H/K_H^{m} with q_0

注　为了解释图中的颜色,读者可以参考本文的电子网页版本,后同.

由图 5、6 可以看出:随着 q_0 的增大, $K_\sigma/K_\sigma^{\mathrm{m}}$ 减小且变化趋势较小而 K_H/K_H^{m} 逐渐增大.当 R_1,R_3 增大时, $K_\sigma/K_\sigma^{\mathrm{m}}$,$K_H/K_H^{\mathrm{m}}$ 逐渐减小;当 R_2 增大时, $K_\sigma/K_\sigma^{\mathrm{m}}$,$K_H/K_H^{\mathrm{m}}$ 逐渐增大,说明 R_1,R_2,R_3 对裂纹的断裂影响较大.

图 7 给出了 σ_0 变化时, S/S_{m} 随 q_0 的变化趋势.可以看出:随着 σ_0 的增大 S/S_{m} 逐渐减小,当 q_0 增大时, S/S_{m} 缓慢增加.由文献[26]可知, S/S_{m} 越大材料的性能越好,可承受较大的外力.因此 σ_0 越小准晶材料越不容易断裂,而 q_0 较大时准晶材料不容易断裂,这也验证了准晶的耐高温性.

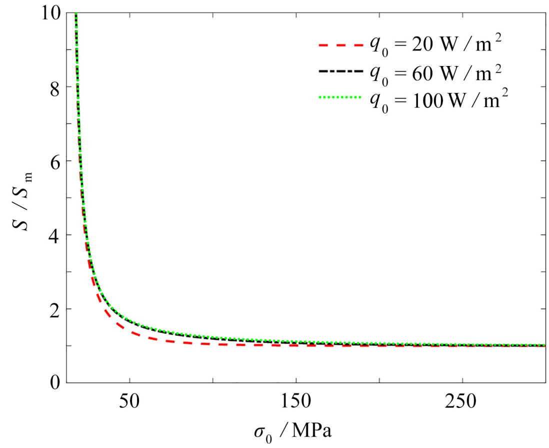


图 7　q_0 不同时, S/S_m 随 σ_0 变化

Fig. 7　Variations of S/S_m with σ_0 for different q_0 values

图 8 和图 9 给出了 q_z/q_0 的变化趋势.从图中可以看出：q_z/q_0 在裂纹尖端变化较为剧烈,在第一象限内随着 x 的增加, q_z/q_0 先增大后减小;随着 z 的增大, q_z/q_0 的变化趋势逐渐平缓.

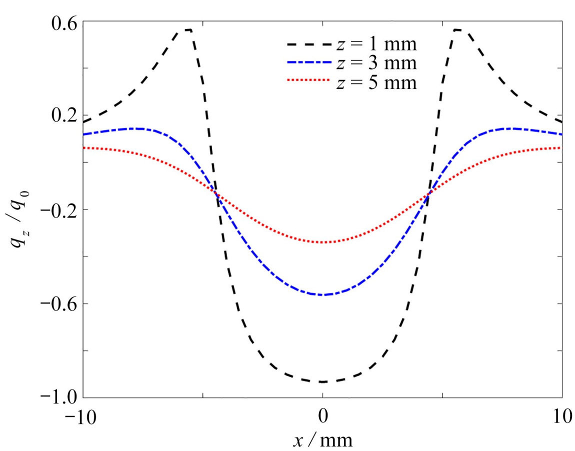


图 8　z 不同时, q_z/q_0 随 x 变化

Fig. 8　Variations of q_z/q_0 with x for different z values

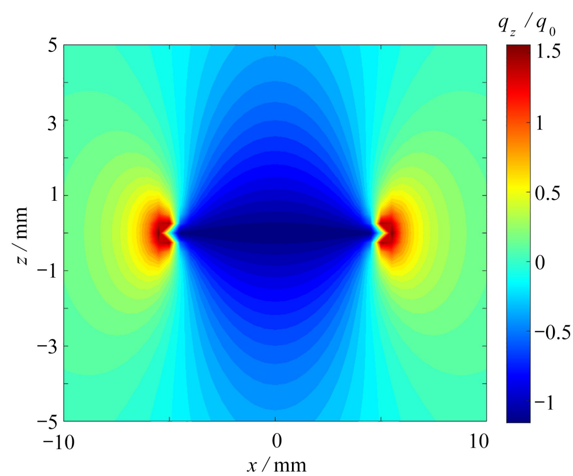


图 9　q_z/q_0 在 xOz 面上的变化趋势

Fig. 9　Variations of q_z/q_0 on plane xOz

6　结　　论

考虑裂纹内部介质具有热传导性,本文研究了一维六方准晶非周期平面内含中心开口裂纹的平面热弹性问题.利用 Fourier 积分变换技术,求得全平面热应力、热应力强度因子及应变能密度因子在裂纹尖端的封闭解.研究表明：

1）随着裂纹内部介质热传导率增大,热流密度逐渐增大,热应力强度因子逐渐减小.

2）声子场-相位子场多场耦合效应对裂纹扩展的影响较显著,当声子场载荷较小或施加的热流密度较大时裂纹不易扩展.

3）热流密度增加时相位子场热应力强度因子增大.在裂纹区域,随着 z 的增大,热流密度逐渐增大；在第一象限内随着 x 的增加,热流密度先增大后减小.

致谢　本文作者衷心感谢宁夏大学新华学院科学研究基金重点项目(23XHKY01)对本文的资助.

参考文献(References):

[1] SHECHTMAN D, BLECH I, GRATIAS D, et al. Metallic phase with long-range orientational order and no translational symmetry[J]. *Physical Review Letters*, 1984, **53**(20): 1951-1953.

[2] 范天佑. 准晶数学弹性理论及应用[M]. 北京: 北京理工大学出版社, 1999. (FAN Tianyou. *Mathematical Theory of Elasticity of Quasicrystals and Its Applications*[M]. Beijing: Beijing Institute of Technology Press, 1999. (in Chinese))

[3] ZHANG Z, URBAN K. Transmission electron microscope observations of dislocations and stacking faults in a decagonal Al-Cu-Co alloy[J]. *Philosophical Magazine Letters*, 1989, **60**(3): 97-102.

[4] LI P D, LI X Y, KANG G Z. Crack tip plasticity of a half-infinite Dugdale crack embedded in an infinite space of one-dimensional hexagonal quasicrystal[J]. *Mechanics Research Communications*, 2015, **70**: 72-78.

[5] YU J, GUO J H. Analytical solution for a 1D hexagonal quasicrystal strip with two collinear mode-Ⅲ cracks perpendicular to the strip boundaries[J]. *Crystals*, 2023, **13**(4): 661.

[6] 卢绍楠, 赵雪芬, 马园园. 一维六方压电准晶双材料界面共线裂纹问题[J]. 应用数学和力学, 2023, **44**(7): 809-824. (LU Shaonan, ZHAO Xuefen, MA Yuanyuan. Problem of interface collinear cracks between one dimensional hexagonal piezoelectric quasicrystal bimaterials[J]. *Applied Mathematics and Mechanics*, 2023, **44**(7): 809-824. (in Chinese))

[7] GRUSHKO B, HOLLAND-MORITZ D, HOLLAND-MORITZ D, et al. Transition between periodic and quasiperiodic structures in Al-Ni-Co[J]. *Journal Alloys and Compounds*, 1998, **280**(1): 215-230.

[8] HIRAGA K, OHSUNA T, SUN W, et al. The structural characteristics of Al-Co-Ni decagonal quasicrystals and crystalline approximants[J]. *Journal Alloys and Compounds*, 2002, **342**(1): 110-114.

[9] YAVAS B, LI M X, LEONARD H R, et al. Identifying experimental parameters for in situ TEM heating experiments on metastable microstructures: application to a quasicrystal-reinforced Al alloy[J]. *Microscopy and Microanalysis*, 2002, **28**(S1): 1840-1842.

[10] DANG H Y, QI D P, ZHAO M H, et al. The thermally induced interfacial behavior of a thin two-dimensional decagonal quasicrystal film[J]. *International Journal of Fracture*, 2023, **224**(1/2): 1-14.

[11] NGUYEN H V, DO N B, NGUYEN T H O, et al. Synthesis and magnetic properties of Al-Cu-Fe quasicrystals prepared by mechanical alloying and heat treatment[J]. *Journal of Materials Research*, 2023, **38**(3): 644-653.

[12] SLADEK J, SLADEK V, REPKA M, et al. Gradient theory of thermoelasticity for interface crack problems with a quasicrystal layer[J]. *International Journal of Solids and Structures*, 2023, **264**: 112097.

[13] FAN C Y, YUAN Y P, PAN Y B, et al. Analysis of cracks in one-dimensional hexagonal quasicrystals with the heat effect[J]. *International Journal of Solids and Structures*, 2017, **120**: 146-156.

[14] LI P D, LI X Y, KANG G Z. Axisymmetric thermo-elastic field in an infinite one-dimensional hexagonal quasicrystal space containing a penny-shaped crack under anti-symmetric uniform heat fluxes[J]. *Engineering Fracture Mechanics*, 2018, **190**: 74-92.

[15] ZHANG X, FAN C Y, LU C S, et al. Three-dimensional thermal fracture analysis of a one-dimensional hexagonal quasicrystal coating with interface cracks[J]. *Engineering Fracture Mechanics*, 2023, **277**: 1-20.

[16] GUO J H, YU J, XING Y M, et al. Thermoelastic analysis of a two-dimensional decagonal quasicrystal with a conductive elliptic hole[J]. *Acta Mechanica*, 2016, **227**(9): 2595-2607.

[17] DING D H, YANG W G, HU C Z. Generalized elasticity theory of quasicrystals[J]. *Physical Review B*, 1993, **48**: 7003-7010.

[18] LI L H, LIU G T. Icosahedral quasicrystals solids with an elliptic hole under uniform heat flow[J]. *Chinese Physics B*, 2014, **23**(5): 056101.

[19] ZHONG X C, LONG X Y, ZHANG L H. An extended thermal-medium crack model[J]. *Applied Mathematical Modelling*, 2018, **56**: 202-216.

[20] EDE A J. *An Introduction to Heat Transfer: Principles and Calculations*[M]. New York: Pergamon Press, 1967.

[21] 范天佑. 断裂理论基础[M]. 北京：科学出版社, 2006.(FAN Tianyou. *The Theoretical Basis of Fracture*[M]. Beijing: Science Press, 2006. (in Chinese))

[22] FABRIKANT V I. Computation of infinite integrals involving three Bessel functions by introduction of new formalism[J]. *ZAMM Journal of Applied Mathematics and Mechanics*, 2003, **83**(6): 363-374.

[23] FAN T Y. *Mathematical Theory of Elasticity of Quasicrystals and Its Applications*[M]. Beijing: Science Press, 2017.

[24] ZHONG X C, WU B, ZHANG K S. Thermally conducting collinear cracks engulfed by thermomechanical field in a material with orthotropy[J]. *Theoretical and Applied Fracture Mechanics*, 2013, **65**: 61-68.

[25] ZHONG X C, WU B. Thermoelastic analysis for an opening crack in an orthotropic material[J]. *International Journal of Fracture*, 2012, **173**(1): 49-55.

[26] BADALIANCE R. Application of strain energy density factor to fatigue crack growth analysis[J]. *Engineering Fracture Mechanics*, 1980, **13**(3): 657-666.