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一类具有转点的右端不连续奇摄动边值问题*

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摘要： 研究了一类具有转点的右端不连续二阶半线性奇摄动边值问题解的渐近性.首先,在间断处将原问题分为左右两个问题,通过修正左问题退化问题的正则化方程,提高了左问题渐近解的精度,并利用 Nagumo 定理证明了左问题光滑解的存在性.其次,证明了右问题具有空间对照结构的解,并通过在间断点的光滑缝接,得到了原问题的渐近解.最后,通过一个算例验证了结果的正确性.

关　键　词： 奇摄动；　边值问题；　转点；　右端不连续；　空间对照；　渐近解

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A Class of Right-Hand Discontinuous Singularly Perturbed Boundary Value Problems With Turning Points

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Abstract: The asymptotic behavior of solutions to a class of right-hand discontinuous 2nd-order semilinear singularly perturbed boundary value problems with turning points was studied. Firstly, the original problem was divided into left and right problems at the discontinuity, the accuracy of the asymptotic solution to the left problem was improved through modification of the regularization equation for the left problem degradation problem, and the existence of the smooth solution to the left problem was proved by means of the Nagumo theorem. Secondly, the solution to the right problem was proved to have a spatial contrast structure, and the asymptotic solution to the original problem was obtained through smooth joints at the discontinuity points. Finally, the correctness of the results was verified by an example.

Key words: singular perturbation; boundary value problem; turning point; discontinuous right-hand side; spatial contrast structure; asymptotic solution estimation

0　引　　言

奇摄动理论和方法一直受到国内外学者们的广泛关注[1-9].如果一个或多个系数,如对流项、反应项、源项或边界条件是不连续的,则这类问题的解显示出强或弱的内层,这取决于奇异扰动参数的大小和系数的性

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质.奇摄动问题通常出现在应用数学的许多分支中,例如:流体力学中的边界层、固体力学中的边缘层、电气应用中的蒙皮层、流体和固体力学中的激波层、量子力学中的过渡点以及数学中的 Stokes 线和面.

从 Ackerberg、O'Malley[10] 关于转点问题研究的工作开始,近些年来,许多学者对转点问题进行了研究. Sharma 等[11] 总结了从 1970 年到 2011 年用于求解奇异摄动转点和内层问题的渐近和数值方法,这些研究主要集中于二阶线性奇摄动微分方程.Karali 等[12] 研究了在稳定性交换情况下,奇异摄动问题中的共振现象. Kumar[13] 研究了一类具有转点的奇摄动时滞微分方程,证明了转点的存在会导致边界层或内部层的出现.

Butuzov 等[14] 首次系统性地提出了低阶奇摄动转点问题的解决方案.在文献[15]的基础上,对于带转点的奇摄动 Neumann 边值问题,给出了在转点附近发生稳定性交替的若干判别准则.对于右端不连续问题及空间对照结构的奇摄动问题,有不少学者进行了相关研究[16-18].而对于右端不连续中的转点问题,目前尚未见相关研究.在物理过程建模过程中,数据的不连续通常会导致求解问题中出现内层问题,因此带有转点的右端不连续的奇摄动问题的研究具有一定的理论意义和应用价值.

本文研究了一类存在转点的右端不连续的二阶半线性奇摄动 Dirichlet 边值问题,在间断点处将问题分为左右问题,利用 Nagumo 定理证明了左问题解的存在性,并对左右问题进行光滑缝接,得到了原问题在整个区间的形式渐近解.

1　问题提出

本文讨论了如下的奇摄动边值问题:

$$\begin{cases}\mu^2 \dfrac{\mathrm{d}^2 u}{\mathrm{d}x^2} = f(u,x,\mu), & 0 < x < 1, \\ u(0,\mu) = u^0,\ u(1,\mu) = u^1, \end{cases} \tag{1}$$

其中 $\mu > 0$ 是小参数.方程右端函数 f 在区间某点 x^0 间断,即

$$f(u,x,\mu) = \begin{cases} f_1(u,x,\mu), & 0 \leqslant x < x^0, \\ f_2(u,x,\mu), & x^0 \leqslant x \leqslant 1, \end{cases}$$

其中

$$f_1(u,x,\mu) = h_1(x)(u - \varphi_1(x))(u - \varphi_2(x)) - \mu g_1(u,x,\mu),$$
$$f_2(u,x,\mu) = h_2(x)(u - \psi_1(x))(u - \psi_2(x))(u - \psi_3(x)) - \mu g_2(u,x,\mu).$$

如图 1 所示,式(1)的解在区间 $[0,x^0]$ 与 $[x^0,1]$ 上具有不同的性质.在 $x \in [0,x^0]$ 区间上,退化解 $\varphi_1(x)$,$\varphi_2(x)$ 在点 x_0 相交,在此区间上解离开边界层后趋于退化解 $\varphi_1(x)$,并经过点 x_0 后趋于退化解 $\varphi_2(x)$;在 $x \in [x^0,1]$ 区间上具有 3 个孤立退化解 $\psi_1(x)$,$\psi_2(x)$,$\psi_3(x)$,在此区间上解先趋于 $\psi_3(x)$,在经过未知点 x^* 后转移到退化解 $\psi_1(x)$.

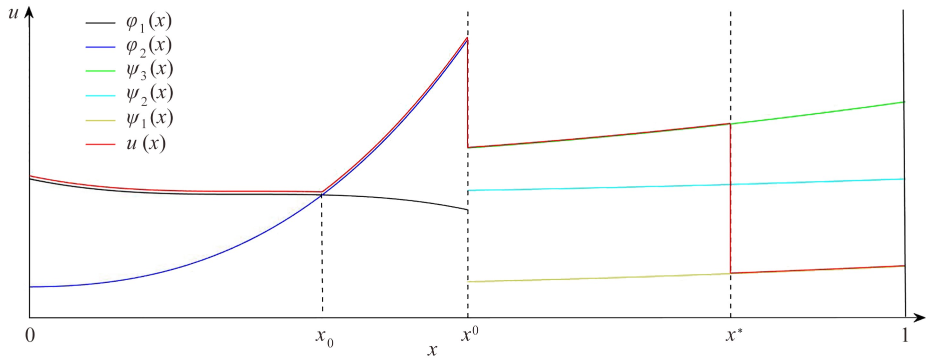

图 1　问题(1)退化解的示意图及真解 $u(x)$ 的示意图

Fig. 1　Schematic diagram of the problem (1) degenerate solution and true solution $u(x)$

注　为了解释图中的颜色,读者可以参考本文的电子网页版本,后同.

条件 1　假设 $\varphi_2(x^0) > \psi_3(x^0)$, $f_{2u}(\psi_{1,3}(x), x, 0) > 0$, $f_{2u}(\psi_2(x), x, 0) < 0$.同时要求 $h_1(x)$, $g_1(u,x,\mu)$ 和 $\varphi_1(x)$, $\varphi_2(x)$ 在 D_1 上充分光滑, $h_1(x)$, $g_2(u,x,\mu)$, $\psi_j(x)$, $j = 1,2,3$ 在 D_2 上是充分光滑的函数,且

$$h_i(x) > 0,\ g_i(u,x,\mu) > 0, \qquad i = 1,2,\ 0 \leqslant x \leqslant x^0,$$

其中

$$D_1 = \{(u,x,\mu) \mid |u| \leqslant l,\ 0 \leqslant x \leqslant x^0,\ 0 \leqslant \mu \leqslant \mu_0\},$$
$$D_2 = \{(u,x,\mu) \mid |u| \leqslant l,\ x^0 \leqslant x \leqslant 1,\ 0 \leqslant \mu \leqslant \mu_0\}.$$

2　渐近解的构造

由于式(1)的解在两个区间 $[0,x^0]$, $[x^0,1]$ 具有不同性质,为了构造原问题(1)的渐近解,我们将原问题分为下面两个边值问题,并在点 x^0 处进行光滑缝接,从而得到原问题在整个区间的渐近展开解.

左问题 $0 \leqslant x < x^0$:

$$\begin{cases} \mu^2 \dfrac{\mathrm{d}^2 u}{\mathrm{d}x^2} = f_1(u,x,\mu), \\ u(0,\mu) = u^0,\ u(x^0,\mu) = p(\mu). \end{cases} \tag{2}$$

右问题 $x^0 \leqslant x \leqslant 1$:

$$\begin{cases} \mu^2 \dfrac{\mathrm{d}^2 \bar{u}}{\mathrm{d}x^2} = f_2(\bar{u},x,\mu), \\ \bar{u}(x^0,\mu) = p(\mu),\ \bar{u}(1,\mu) = u^1. \end{cases} \tag{3}$$

它们同时满足

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}(x^0,\mu) = \frac{\mathrm{d}^2 \bar{u}}{\mathrm{d}x^2}(x^0,\mu). \tag{4}$$

对于右问题(3),其解具有空间对照结构,可将其分为以下左右两个问题 $P^{(-)}$ 和 $P^{(+)}$.

左问题 $P^{(-)}$, $x^0 \leqslant x < x^*$:

$$\begin{cases} \mu^2 \dfrac{\mathrm{d}^2 \bar{u}^{(-)}}{\mathrm{d}x^2} = f_2(\bar{u}^{(-)},x,\mu), \\ \bar{u}^{(-)}(x^*,\mu) = p(\mu),\ \bar{u}^{(-)}(x^*,\mu) = \psi_2(x^*). \end{cases} \tag{5}$$

右问题 $P^{(+)}$, $x^* \leqslant x \leqslant 1$:

$$\begin{cases} \mu^2 \dfrac{\mathrm{d}^2 \bar{u}^{(+)}}{\mathrm{d}x^2} = f_2(\bar{u}^{(+)},x,\mu), \\ \bar{u}^{(+)}(x^*,\mu) = \psi_2(x^*),\ \bar{u}^{(+)}(1,\mu) = u^1. \end{cases} \tag{6}$$

它们同时满足

$$\frac{\mathrm{d}^2 \bar{u}^{(-)}}{\mathrm{d}x^2}(x^*,\mu) = \frac{\mathrm{d}^2 \bar{u}^{(+)}}{\mathrm{d}x^2}(x^*,\mu). \tag{7}$$

后文中,我们将分别构造左问题和右问题的形式渐近解,再在点 $x = x^0$ 处进行光滑缝接.

2.1　左问题形式渐近解的构造

先讨论左问题,注意到退化解 $\varphi_1(x)$, $\varphi_2(x)$ 在 $x \in (0,x^0)$ 上的某点相交.

条件 2　假设存在 $x_0 \in (0,x^0)$, 具有

$$\begin{cases} \varphi_1(x) > \varphi_2(x), & 0 \leqslant x < x_0, \\ \varphi_1(x) = \varphi_2(x), & x = x_0, \\ \varphi_1(x) < \varphi_2(x), & x_0 < x \leqslant x^0, \end{cases}$$

并且在点 x^0 处满足

$$f_1(u,x^0,0) \neq f_2(u,x^0,0).$$

对 $f_1(u,x,0)$ 求关于 u 的导数

$$f_{1u}(u(x),x,0)=h(x)(2u(x)-\varphi_1(x)-\varphi_2(x)).\tag{8}$$

由条件 2 可得

$$\begin{cases}f_{1u}(\varphi_1(x),x,0)>0, & 0\leqslant x<x_0,\\ f_{1u}(\varphi_1(x),x,0)<0, & x_0<x\leqslant x^0,\end{cases}$$

$$\begin{cases}f_{1u}(\varphi_2(x),x,0)<0, & 0\leqslant x<x_0,\\ f_{1u}(\varphi_2(x),x,0)>0, & x_0<x\leqslant x^0.\end{cases}$$

构造退化方程 $f_1(u,x,\mu)=0$ 的复合退化解：

$$\bar{u}(x)=\begin{cases}\varphi_1(x), & 0\leqslant x\leqslant x_0,\\ \varphi_2(x), & x_0<x\leqslant x^0.\end{cases}\tag{9}$$

条件 3　假设 $g_1(\bar{u},x,0)>0$.

由于在 $x=x_0$ 处,$\bar{u}(x)$ 不可导,故将区间$[0,x^0]$分成三个部分,分别在区间$[0,x_1]$,I_δ,$[x_2,1]$上讨论问题(2),记 $I_\delta=(x_1,x_2)$.此处 $x_1=x_0-\delta,x_2=x_0+\delta$,且 δ 充分小,但当 $\mu\to 0$ 时固定.

第一步　首先在区间$[0,x_1]$上讨论问题(2),在该区间上系统满足 Vasil'eva 定理所有条件,构造形式渐近解如下：

$$U^{(-)}(x,\mu)=u^{(-)}(x,\mu)+Lu(\tau_0,\mu),\ \tau_0=\frac{x}{\mu},\tag{10}$$

其中

$$\begin{cases}u^{(-)}(x,\mu)=u_0(x)+\mu u_1(x)+\mu^2=u_2(x)+\cdots,\\ Lu(\tau_0,\mu)=L_0u(\tau_0)+\mu L_1u(\tau_0)+\mu^2L_2u(\tau_0)+\cdots,\end{cases}\tag{11}$$

这里 $u_k(x)$ 是渐近解的正则部分系数,$L_ku(\tau_0)$ 是在 $x=0$ 附近渐近解的边界层部分系数.同时要求 $L_ku(+\infty)=0$.将式(10)代入问题(2),可得

$$\mu^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}u^{(-)}(x,\mu)+\frac{\mathrm{d}^2}{\mathrm{d}\tau_0^2}Lu(\tau_0,\mu)=$$
$$f_1(u^{(-)}(x,\mu),x,\mu)+f_1(u^{(-)}(\mu\tau_0,\mu)+Lu(\tau_0,\mu),\mu\tau_0,\mu)-f_1(u^{(-)}(\mu\tau_0,\mu),\mu\tau_0,\mu).\tag{12}$$

按尺度分离,可得到正则部分满足的方程为

$$\mu^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}u^{(-)}(x,\mu)=f_1(u^{(-)}(x,\mu),x,\mu).\tag{13}$$

边界层部分满足的方程为

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau_0^2}Lu(\tau_0,\mu)=f_1(u^{(-)}(\mu\tau_0,\mu)+Lu(\tau_0,\mu),\mu\tau_0,\mu)-f_1(u^{(-)}(\mu\tau_0,\mu),\mu\tau_0,\mu).\tag{14}$$

边界条件满足

$$u_0(0)+\mu u_1(0)+\mu^2u_2(0)+\cdots+L_0u(0)+\mu L_1u(0)+\mu^2L_2u(0)+\cdots=u^0.\tag{15}$$

将式(11)代入式(12)、(14),可得确定形式渐近解各部分满足的方程为

$$\mu^2\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}u_0(x)+\mu\frac{\mathrm{d}^2}{\mathrm{d}x^2}u_1(x)+\cdots\right)=$$
$$h_1(x)(u_0(x)+\mu u_1(x)+\cdots-\varphi_1(x))(u_0(x)+\mu u_1(x)+\cdots-\varphi_2(x))-$$
$$\mu g(u_0(x)+\mu u_1(x)+\cdots,u,\mu),\tag{16}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau_0^2}(L_0u(\tau_0)+\mu L_1u(\tau_0)+\cdots)=$$
$$f_1(u_0(x)+\mu u_1(x)+\cdots+L_0u(\tau_0)+\mu L_1u(\tau_0)+\cdots,\mu\tau_0,\mu)-$$
$$f_1(u_0(\mu\tau_0)+\mu u_1(\mu\tau_0)+\cdots,\mu\tau_0,\mu).\tag{17}$$

在式(16)两端对 μ 进行展开,并比较同次幂系数,由条件 1 可得

$$u_0(x)=\varphi_1(x).$$

由条件 1、2,一次项 $u_0(x)$ 可由如下的线性代数方程唯一确定:

$$h_1(x)[u_1(\varphi_1(x)-\varphi_2(x))]-g_1(\varphi_1(x),x,0)=0.$$

当 $i\geqslant 2$ 时,

$$h_1(x)[u_i(\varphi_1(x)-\varphi_2(x))]=\bar{g}_i(x),$$

其中 $\bar{g}_i$ 是依赖于 u_j, $j<i$ 的已知函数.

在式(17)两端对 μ 进行展开,并比较同次幂系数,结合边界条件可得各项系数满足的方程.确定 $L_0u(\tau_0)$ 的方程为

$$\begin{cases}\dfrac{\mathrm{d}^2}{\mathrm{d}\tau_0^2}L_0u(\tau_0)=h_1(0)(\varphi_1(0)-\varphi_2(0))L_0u(\tau_0)+h_1(0)(L_0u(\tau_0))^2,\\ L_0u(0)=u^0-\varphi_1(0),\ L_0u(+\infty)=0.\end{cases}\tag{18}$$

求解可得

$$L_0u(\tau_0)=\frac{6k_2c\mathrm{e}^{-k_0\tau_0}}{(1-c\mathrm{e}^{-k_0\tau_0})^2},$$

其中 $k_0=(k_1k_2)^{1/2}$,$k_1=h_1(0)$,$k_2=\varphi_1(0)-\varphi_2(0)$, 且

$$6k_2c=k_3(1-c)^2,$$

其中 $k_3=u^1-\varphi_1(0)$.当 $k_3=0$ 时,$L_0u(\tau)\equiv 0$;当 $k_3\neq 0$ 时,求解上式可得两同号根

$$c_{1,2}=1+\frac{1}{k_3}[3k_2\mp(3k_2(3k_2+2k_3))^{1/2}],$$

且有

$$\frac{\mathrm{d}}{\mathrm{d}\tau}L_0u(0)=-k_0k_3-\frac{12k_0k_2c^2}{(1-c)^3}=\pm\frac{k_0k_3}{3k_2}(3k_2(3k_2+2k_3))^{1/2}.$$

即,当 $k_3>0$ 时,$c=c_2$, 显然方程(18)有解;当 $k_3<0$ 时,$c=c_2$,若 $3k_2+2k_3\geqslant 0$, 则方程(18)有解.且 $L_0u(\tau_0)$ 在 $\tau_0\to+\infty$ 时指数衰减.

当 $k\geqslant 1$ 时,有

$$\begin{cases}\dfrac{\mathrm{d}^2}{\mathrm{d}\tau_0^2}L_ku(\tau)=k_0^2L_ku(\tau_0)+2k_1L_0u(\tau_0)L_ku(\tau_0)+l_k^{(-)}(\tau_0),\\ L_ku(0)=-u_k(0),\ L_ku(+\infty)=0,\end{cases}\tag{19}$$

其中 $l_k^{(-)}(\tau_0)$ 依赖于 $L_ju(\tau)$, $j<k$,特别地,$l_0^{(-)}(\tau_0)\equiv 0$.显然方程(19)的解可以显式表达出来:

$$L_ku(\tau_0)=-u_k(0)\frac{L_0u(\tau_0)}{L_0u(0)}+L_0u(\tau_0)\int_0^{\tau_0}(L_0u(\eta))^{-2}\mathrm{d}\eta\int_{+\infty}^{\eta}L_0u(\sigma)l_k^{(-)}(\sigma)\mathrm{d}\sigma.$$

并且 $L_ku(\tau_0)$ 都有指数估计

$$|L_ku(\tau_0)|\leqslant C\exp(-\kappa\tau_0),\qquad \tau\geqslant 0,\ k\geqslant 0.$$

第二步 在区间 $[x_2,1]$ 上讨论问题(2),构造形式渐近解如下:

$$U^{(-)}(x,\mu)=u^{(-)}(x,\mu)+Ru(\tau_1,\mu),\qquad \tau_1=\frac{x-x^0}{\mu},\tag{20}$$

其中

$$\begin{cases}u^{(+)}(x,\mu)=\bar{u}_0(x)+\mu\bar{u}_1(x)+\mu^2\bar{u}_2(x)+\cdots,\\ Ru(\tau_1,\mu)=R_0u(\tau_1)+\mu R_1u(\tau_1)+\mu^2R_2u(\tau_1)+\cdots,\end{cases}\tag{21}$$

这里 $\bar{u}_k(x)$ 是渐近解的正则部分系数,$R_ku(\tau_1)$ 是在 $x=x^0$ 附近渐近解的边界层部分系数, 同时要求 $R_ku(-\infty)=0$.将式(20)代入问题(2),可得

$$\mu^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}u^{(+)}(x,\mu)+\frac{\mathrm{d}^2}{\mathrm{d}\tau_1^2}Ru(\tau_1,\mu)=$$

$$f_1(u^{(+)}(x,\mu),x,\mu)+f_1(u^{(+)}(\mu\tau_1+x^0,\mu)+Ru(\tau_1,\mu),\mu\tau_1+x^0,\mu)-$$

$$f_1(u^{(+)}(\mu\tau_1 + x^0,\mu),\mu\tau_1 + x^0,\mu). \tag{22}$$

按尺度分离，得到正则部分满足的方程为

$$\mu^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} u^{(+)}(x,\mu) = f_1(u^{(+)}(x,\mu),x,\mu), \tag{23}$$

边界层部分满足的方程为

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau_1^2} Ru(\tau_1,\mu) = f_1(u^{(+)}(\mu\tau_1 + x^0,\mu) + Ru(\tau_1,\mu),\mu\tau_1 + x^0,\mu) - f_1(u^{(+)}(\mu\tau_1 + x^0,\mu),\mu\tau_1 + x^0,\mu). \tag{24}$$

边界条件满足

$$u_0(x^0) + \mu u_1(x^0) + \mu^2 u_2(x^0) + \cdots + R_0u(0) + \mu R_1u(0) + \mu^2 R_2u(0) + \cdots = p(\mu), \tag{25}$$

此处

$$p(\mu) = p_0 + \mu p_1 + \cdots + \mu^k p_k + \cdots.$$

将式(21)代入式(23)、(24)，可得确定形式渐近解各部分满足的方程为

$$\begin{aligned}\mu^2\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\bar{u}_0(x) + \mu\frac{\mathrm{d}^2}{\mathrm{d}x^2}\bar{u}_1(x) + \cdots\right) = \\ h(x)(\bar{u}_0(x) + \mu\bar{u}_1(x) + \cdots - \varphi_1(x))(\bar{u}_0(x) + \mu\bar{u}_1(x) + \cdots - \varphi_2(x)) - \\ \mu g(\bar{u}_0(x) + \mu\bar{u}_1(x) + \cdots,x,\mu),\end{aligned} \tag{26}$$

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}\tau_1^2}(R_0u(\tau_1) + \mu R_1u(\tau_1) + \cdots g) = \\ f_1(\bar{u}_0(x) + \mu\bar{u}_1(x) + \cdots + R_0u(\tau_1) + \mu R_1u(\tau_1) + \cdots,\mu\tau_1 + x^0,\mu) - \\ f_1(u_0(\mu\tau_1 + x^0) + \mu u_1(\mu\tau_1 + x^0) + \cdots,\mu\tau_1 + x^0,\mu).\end{aligned} \tag{27}$$

在式(26)两端对 μ 进行展开，并比较同次幂系数，由条件 1 可得

$$\bar{u}_0(x) = \varphi_2(x).$$

由条件 1、2，一次项 $\bar{u}_0$ 可由如下的线性代数方程唯一确定：

$$h_1(x)[\bar{u}_1(\varphi_2(x) - \varphi_1(x))] - g(\varphi_2(x),x,0) = 0.$$

当 $i \geqslant 2$ 时，

$$h_1(x)[u_i(\varphi_1(x) - \varphi_2(x))] = \tilde{g}_i(x),$$

其中 $\tilde{g}_i$ 是依赖于 $\bar{u}_j$, $j < i$ 的已知函数.

在式(27)两端对 μ 进行展开，并比较同次幂系数，结合边界条件可得各项系数满足的方程.确定 $R_0u(\tau_1)$ 的方程为

$$\begin{cases}\dfrac{\mathrm{d}^2}{\mathrm{d}\tau_1^2}R_0u(\tau_1) = h_1(x^0)(\varphi_2(0) - \varphi_1(0)R_0u(\tau_1) + h_1(x^0)(R_0u(\tau_1))^2, \\ R_0u(0) = p_0 - \varphi_2(x^0),\ R_0u(-\infty) = 0.\end{cases} \tag{28}$$

求解可得

$$R_0u(\tau_1) = \frac{6k_2\bar{c}\mathrm{e}^{\bar{k}_0\tau_1}}{(1 - \bar{c}\mathrm{e}^{\bar{k}_0\tau_1})^2},$$

其中 $\bar{k}_0 = (\bar{k}_1\bar{k}_2)^{1/2}$, $\bar{k}_1 = h_1(x^0)$, $\bar{k}_2 = \varphi_2(x^0) - \varphi_1(x^0)$，且

$$6\bar{k}_2\bar{c} = \bar{k}_3(1 - \bar{c})^2,$$

其中 $\bar{k}_3 = p_0 - \varphi_2(x^0)$.当 $\bar{k}_3 = 0$ 时，$R_0u(\tau_1) \equiv 0$；当 $\bar{k}_3 \neq 0$ 时，求解上式可得两同号根

$$\bar{c}_{1,2} = 1 + \frac{1}{\bar{k}_3}[3\bar{k}_2 \mp (3\bar{k}_2(3\bar{k}_2 + 2\bar{k}_3))^{1/2}],$$

且有

$$\frac{\mathrm{d}}{\mathrm{d}\tau_1}R_0u(0)=\bar{k}_0\bar{k}_3+\frac{12\bar{k}_0\bar{k}_2\bar{c}^2}{(1-\bar{c})^3}=\pm\frac{\bar{k}_0\bar{k}_3}{3\bar{k}_2}(3\bar{k}_2(3\bar{k}_2+2\bar{k}_3))^{1/2}.$$

由于 $p\in[\psi_3(x^0),\varphi_2(x^0)]$,故 $\bar{k}_3<0,\bar{c}=\bar{c}_1$,当 $3\bar{k}_2+2\bar{k}_3\geqslant 0$ 时,方程(28)有解,且 $R_0u(\tau_0)$ 在 $\tau_1\to-\infty$ 时指数衰减.

当 $k\geqslant 1$ 时,有

$$\begin{cases}\dfrac{\mathrm{d}^2}{\mathrm{d}\tau_1^2}R_ku(\tau_1)=\bar{k}_0^2R_ku(\tau_1)+l_k^{(+)}(\tau_1),\\ R_ku(0)=p_k-\bar{u}_k(x^0),\ R_iu(-\infty)=0,\end{cases}\tag{29}$$

其中 $l_k^{(+)}(\tau_1)$ 依赖于 $R_ju(\tau_1)$, $j<k$,特别地,$l_0^{(+)}(\tau)=0$.显然方程(29)的解可以显式表达出来:

$$R_ku(\tau_1)=(p_k-\bar{u}_k(0))\frac{R_0u(\tau_1)}{R_0u(0)}+R_0u(\tau_1)\int_0^{\tau_1}(R_0u(\eta))^{-2}\mathrm{d}\eta\int_{-\infty}^{\eta}R_0u(\sigma)l_k^{(+)}(\sigma)\mathrm{d}\sigma.\tag{30}$$

并且 $R_ku(\tau_1)$ 都有指数估计

$$|R_ku(\tau_1)|\leqslant C\exp(\kappa\tau_1),\qquad \tau_1\leqslant 0,\ k\geqslant 1.$$

2.2 左问题退化方程的正则化

由条件 3 和式(9)可知, $\bar{u}(x)$ 是区间$[0,x^0]$上的连续函数,但在 $x=x_0$ 处不可导,且$f_{1u}(\bar{u}(x_0),x_0,0)=0$.文献[14]将退化方程$f_1(u,x,0)=0$ 修正为如下形式的正则化方程:

$$h_1(x)(u-\varphi_1(x))(u-\varphi_2(x))-\mu g_1(u,x,0)=0.\tag{31}$$

由于条件 1 在 x_0 的充分小 δ 邻域内(记为 I_δ) 有 $g_1(\bar{u},x,0)>0$, 所以在该邻域内方程(31)有两个不同光滑实根, 设其中一个大的为

$$\varphi(x,\mu)=\frac{1}{2}\{\varphi_1(x)+\varphi_2(x)+[(\varphi_1(x)-\varphi_2(x))^2+4\mu H(\varphi(x,\mu),x,0)]^{1/2}\},\tag{32}$$

式中 $H(\varphi(x,\mu),x,0)=h_1^{-1}(\varphi,x)g_1(\varphi,x,0)$.显然 $\varphi(u,x)$ 是区域 $\Omega_1=\{(x,\mu)\mid x\in[0,x^0],0\leqslant\mu\leqslant\mu_0\}$ 上的充分光滑函数.

如图 2 所示, $\bar{u}(x)$ 与正则化后求得的 $\varphi(x,\mu)$ 在区域 Ω_1 上非常接近.往下将在区域 Ω_1 上证明 $\bar{u}(x)$ 与 $\varphi(x,\mu)$ 的关系.

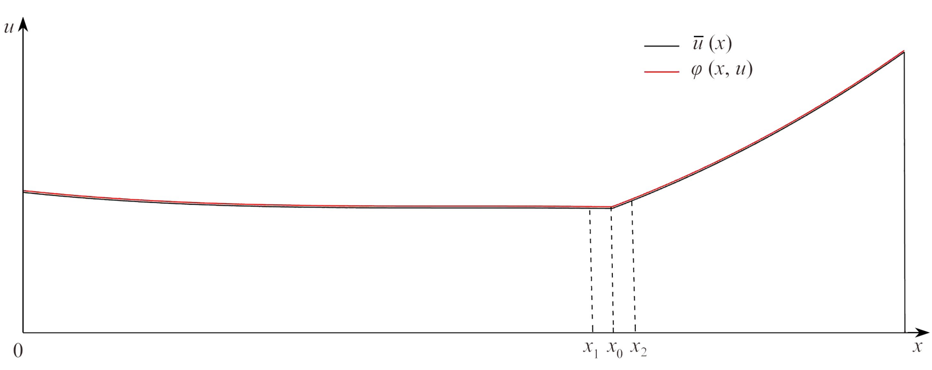

图 2 $\bar{u}(x)$ 与 $\varphi(x,\mu)$ 的示意图

Fig. 2 Schematic diagram of $\bar{u}(x)$ and $\varphi(x,\mu)$

根据式(9)和(32),当 $x\in I_\delta$ 时,$H_1(x,\varphi(x,\mu))\geqslant c_1>0$($c_1$ 是常数),此时

$$0\leqslant\varphi(x,\mu)-\bar{u}(x)\leqslant c_2\sqrt{\mu}.$$

当 $x\in[0,x^0]\backslash I_\delta$ 时,总有 $|\varphi_1(x)-\varphi_2(x)|\geqslant c_\delta$,其中 c_δ 仅依赖于 δ, 此时

$$|\varphi(x,\mu)-\bar{u}(x)|=2\mu H[((\varphi_1(x)-\varphi_2(x))^2+4\mu H)^{1/2}+|\varphi_1(x)-\varphi_2(x)|]^{-1}=O(\mu).$$

则可用光滑函数 $\varphi(u,x)$ 替换 $\bar{u}$, 且满足如下的零次近似表达式:

$$\varphi(x,\mu)-\bar{u}(x)=\begin{cases}O(\sqrt{\mu}), & x\in I_\delta,\\ O(\mu), & x\in[0,x^0]\backslash[I_\delta].\end{cases}\tag{33}$$

2.3 左问题解的存在性及渐近表达式

虽然 $L_0u(\tau_0)$，$R_0u(\tau_1)$ 分别在 $x\in[0,x_1]$，$x\in[x_2,x^0]$ 上求得，但在 $x\in(0,x^0)$ 处，$L_0u(\tau)$，$R_0u(\tau_1)$ 和其导数都是指数小。可认为 $L_0u(\tau)$，$R_0u(\tau_1)$ 在 $x\in[0,x^0]$ 都成立。即 $L_0u(\tau)$，$R_0u(\tau_1)$ 在 $x\in[0,x^0]$ 上连续。并且下述引理成立。

引理 1　若条件 1、2 满足，则对于充分小的 μ，奇摄动问题(2)存在解 $u(x,\mu)$，并且有渐近表达式：

$$u(x,\mu)=\varphi(x,\mu)+L_0u\left(\frac{x}{\mu}\right)+R_0u\left(\frac{x-x^0}{\mu}\right)+O(\mu),\qquad 0\leqslant x\leqslant x^0. \tag{34}$$

证明　记

$$U_0(x,\mu)=\varphi(x,\mu)+L_0u\left(\frac{x}{\mu}\right)+R_0u\left(\frac{x-x^0}{\mu}\right).$$

构造上下解

$$\underline{U}(x,\mu)=U_0(x,\mu)-A\mu, \tag{35}$$

$$\bar{U}(x,\mu)=U_0(x,\mu)+A\mu, \tag{36}$$

其中 A 为正常数，

① 显然，当 $A>0$ 时，

$$\underline{U}\leqslant\bar{U},$$

即当 $x\in[0,x^0]$ 时，$\underline{U}\leqslant\bar{U}$ 恒成立。

② 由式(35)和(36)，对于充分大的 A 可得

$$\begin{cases}\underline{U}(0,\mu)=\varphi(0,\mu)+u^0-\varphi_0-A\mu\leqslant 0,\\ \bar{U}(0,\mu)=\varphi(0,\mu)+u^0-\varphi_0+A\mu\geqslant 0,\\ \underline{U}(x^0,\mu)=\varphi(x^0,\mu)+p_0-\varphi_0-A\mu\leqslant 0,\\ \bar{U}(x^0,\mu)=\varphi(x^0,\mu)+p_0-\varphi_0+A\mu\geqslant 0.\end{cases} \tag{37}$$

③ 对充分小的 μ，对 $\varphi''(x,\mu)$ 进行估计，存在某正常数 C_δ，使得

$$\begin{cases}\varphi''(x,\mu)=O(1), & x\in I_\delta,\\ |\varphi''(x,\mu)|\leqslant C_\delta, & x\in[0,x^0]\backslash I_\delta.\end{cases}$$

由于 $f_{1u}(\varphi(x,\mu),x,0)=h_1(x)(2\varphi(x,\mu)-\varphi_1(x)-\varphi_2(x))$，根据条件 1，设

$$h_1(x)\geqslant m_1>0,\qquad 0\leqslant x\leqslant x^0,$$

m_1 为常数，经计算，存在 $c_1>0$，使得

$$h_1(2\varphi-\varphi_1-\varphi_2)=h[(\varphi_1(x)-\varphi_2(x))^2+4\mu H]^{1/2}\geqslant m_1c_1\sqrt{\mu}.$$

则对充分小 μ 和 δ，有

$$\begin{cases}f_{1u}(\varphi,x,0)\geqslant m_1c_1\sqrt{\mu}:=M\sqrt{\mu}, & x\in I_\delta,\\ f_{1u}(\varphi,x,0)\geqslant m_1c_\delta+O(\mu), & x\in[0,x^0]\backslash I_\delta.\end{cases} \tag{38}$$

由上式可得

$$f_{1u}(\varphi,x,0)>0. \tag{39}$$

为了验证 Nagumo 条件(2)，需要在 $x\in[0,x_1]$，$x\in I_\delta$，$x\in[x_2,x^0]$ 上进行验证，往下只在 $x\in[0,x_1]$，$x\in I_\delta$ 上进行验证。在 $x\in[x_2,x^0]$ 上的讨论完全类似于在 $x\in[0,x_1]$ 上的讨论。

在区间 $x\in[0,x_1]$ 上，

$$L_\mu\underline{U}=\mu^2\underline{U}''-f_1(\underline{U},x,\mu)=[\mu^2U_0''-f_1(U_0,x,\mu)]-[f_1(\underline{U},x,\mu)-f_1(U_0,x,\mu)]=$$

$$\mu^2\varphi''(x,\mu)-f_1(\varphi(x,\mu),x,\mu)+\frac{\mathrm{d}^2}{\mathrm{d}\tau_0^2}L_0u(\tau_0)+N_{\mathrm{NST}}+$$

$$[f_1((\varphi(\mu\tau_0,\mu),\mu)+L_0u(\tau_0),\mu\tau_0,\mu)-f_1(\varphi(\mu\tau_0,\mu),\mu\tau_0,\mu)]-$$

$$[f_1(\underline{U},x,\mu)-f_1(U_0,x,\mu)]=$$
$$O(\mu^2)+O(\mu)+A\mu f_{1u}(\varphi(x,\mu)+L_0u(\tau)+N_{\mathrm{NST}},x,0), \tag{40}$$

其中 N_{NST} 为指数小.

条件 4 假设

$$u^0\in\left[\frac{\varphi_1(0)-\varphi_2(0)}{2},l\right],\ p(\mu)\in\left[\frac{\varphi_2(0)-\varphi_1(0)}{2},\varphi_2(0)\right].$$

由条件 4,可得

$$f_{1u}(\varphi(x,\mu)+L_0u(\tau)+N_{\mathrm{NST}},x,0)\geqslant m_1(2\varphi(x,\mu)-\varphi_1(x)-\varphi_2(x)+2L_0u(\tau)+N_{\mathrm{NST}})>0.$$

对于充分大的 A 和充分小的 μ, 有

$$L_\mu\underline{U}>0,\qquad 0\leqslant x\leqslant x_1,$$

类似可得

$$L_\mu\bar{U}<0,\qquad x_2\leqslant x\leqslant x^0.$$

在区间 $x\in I_\delta$ 上,

$$L_\mu\underline{U}=\mu^2\underline{U}''-f_1(\underline{U},x,\mu)=$$
$$\mu^2\varphi''(x,\mu)-f_1(\varphi(x,\mu)+N_{\mathrm{NST}}-A\mu,x,\mu)+N_{\mathrm{NST}}=$$
$$O(\mu^{3/2})-[f_1(\varphi(x,\mu),x,0)-\mu g_1(\varphi(x,\mu),x,0)+O(\mu^2)]+A\mu f_{1u}(\varphi(x,\mu),x,\mu)+N_{\mathrm{NST}}=$$
$$O(\mu^{3/2})+A\mu f_{1u}(\varphi(x,\mu),x,0). \tag{41}$$

由式(41),对于充分大的 A 和充分小的 μ, 有

$$L_\mu\underline{U}>0,\qquad x\in I_\delta,$$

类似可得

$$L_\mu\bar{U}<0,\qquad x\in I_\delta.$$

则对于任意 $x\in[0,x^0]$, $L_\mu\underline{U}\coloneqq\mu^2\underline{U}''-f_1(\underline{U},x,\mu)\geqslant 0\geqslant L_\mu\bar{U}$ 恒成立.

综上所诉,式(35)和(36)分别为问题(2)的下解和上解.

根据 Nagumo 定理,原问题(2)的解存在,且满足

$$\underline{U}(x,\mu)\leqslant u(x,\mu)\leqslant\bar{U}(x,\mu).$$

因此式(34)成立.引理 1 得证.

由引理 1 和式(9)可得如下结果.

定理 1 若条件 1—4 成立,那么对于充分小的 $\mu>0$, 问题(2)的解 $u(x,\mu)$ 存在,且有如下表达式:

$$u(x,\mu)=\bar{u}(x)+L_0u\left(\frac{x}{\mu}\right)+R_0u\left(\frac{x-x^0}{\mu}\right)+O(\mu),\qquad 0\leqslant x\leqslant x^0.$$

定理 2 若条件 1—4 成立,那么对于充分小的 $\mu>0$, 问题(2)的解 $u(x,\mu)$ 存在, 且有如下渐近展开式:

$$u(x,\mu)=\begin{cases}\displaystyle\sum_{k=0}^{n}\mu^k u_k(x)+\sum_{k=0}^{n}\mu^k L_k u(\tau_0)+O(\mu^n), & 0\leqslant x\leqslant x_1,\\ \bar{u}(x)+O(\mu), & x_1<x<x_2,\\ \displaystyle\sum_{k=0}^{n}\mu^k\bar{u}_k(x)+\sum_{k=0}^{n}\mu^k R_k u(\tau_1)+O(\mu^n), & x_2\leqslant x\leqslant x^0.\end{cases}$$

3 右问题形式渐近解的构造

接下来讨论右问题,为讨论方便,不妨设 $\psi_1(1)<u^1<\psi_2(1)$,可认为问题(3)是由下述两个问题的解在某点 $x^*\in(x^0,1)$ 光滑缝接而成.

左问题 P^-, $x^0\leqslant x<x^*$:

$$\begin{cases}\mu^2\dfrac{\mathrm{d}^2\bar{u}^{(-)}}{\mathrm{d}x^2}=f_2(\bar{u}^{(-)},x,\mu),\\ \bar{u}^{(-)}(x^0,\mu)=p(\mu),\ \bar{u}^{(-)}(x^*,\mu)=\psi_2(x^*).\end{cases}\tag{42}$$

右问题 P^+, $x^*\leqslant x\leqslant 1$：

$$\begin{cases}\mu^2\dfrac{\mathrm{d}^2\bar{u}^{(+)}}{\mathrm{d}x^2}=f_2(\bar{u}^{(+)},x,\mu),\\ \bar{u}^{(+)}(x^*,\mu)=\psi_2(x^*),\ \bar{u}^{(+)}(1,\mu)=u^1.\end{cases}\tag{43}$$

用边界层函数法构造左右问题解的渐近表达式：

$$\bar{u}^{(-)}(x,\mu)=\bar{u}^{(-)}(x,\mu)+\bar{R}u(\tau_1,\mu)+Q^{(-)}u(\tau_2,\mu),\qquad x^0\leqslant x<x^*,\tag{44}$$

$$\bar{u}^{(+)}(x,\mu)=\bar{u}^{(+)}(x,\mu)+\tilde{R}u(\tau_3,\mu)+Q^{(+)}u(\tau_2,\mu),\qquad x^*\leqslant x\leqslant 1,\tag{45}$$

其中 $\tau_2=(x-x^*)\mu^{-1}$, $\tau_3=(x-1)\mu^{-1}$, $\bar{u}^{(\mp)}(x,\mu)$ 为渐近解的正则部分,而 $Q^{(\mp)}u(\tau_0,\mu)$ 为内部层部分, $\bar{R}u(\tau_1,\mu)$, $\tilde{R}u(\tau_3,\mu)$ 分别为 $x=x^0$, $x=1$ 的边界层部分,它们可表示成

$$\bar{u}^{(\mp)}(x,\mu)=\bar{u}_0^{(\mp)}(x)+\mu\bar{u}_1^{(\mp)}(x)+\mu^2\bar{u}_2^{(\mp)}(x)+\cdots.\tag{46}$$

式(46)为正则级数形式.

$$\begin{cases}\bar{R}u(\tau_1,\mu)=\bar{R}_0u(\tau_1)+\mu\bar{R}_1u(\tau_1)+\mu^2\bar{R}_2u(\tau_1)+\cdots,\\ \tilde{R}u(\tau_3,\mu)=\tilde{R}_0u(\tau_3)+\mu\tilde{R}_1u(\tau_3)+\mu^2\tilde{R}_2u(\tau_3)+\cdots\end{cases}\tag{47}$$

为在 $x=x^0$, $x=1$ 边界层级数形式.

$$Q^{(\mp)}u(\tau_0,\mu)=Q_0^{(\mp)}u(\tau)+\mu Q_1^{(\mp)}u(\tau)+\mu^2Q_2^{(\mp)}u(\tau)+\cdots\tag{48}$$

为在 x^* 邻域的内部层级数形式,这里

$$x^*=\bar{x}_0+\mu\bar{x}_1+\mu^2\bar{x}_2+\cdots.\tag{49}$$

往下,我们将确定未知函数 $\bar{u}_i^{(\mp)}(x)$, $Q_i^{(\mp)}u(\tau)$, $\bar{R}u(\tau_1,\mu)$, $\tilde{R}u(\tau_3,\mu)$ 和未知数 $\bar{x}_i$, $i\geqslant 0$.

首先, $\bar{u}^{(\mp)}(x,\mu)$ 满足如下方程：

$$\mu^2\bar{u}^{(\mp)\prime\prime}=f_2(\bar{u}^{(\mp)},x,\mu).\tag{50}$$

把式(46)代入式(50)可得确定 $\bar{u}_i^{(\mp)}$ 的方程

$$0=f_2(\bar{u}_0^{(\mp)},x,0),\ 0=\bar{f}_u(x)\bar{u}_1^{(\mp)}+\bar{f}_\mu(x),\ \bar{u}_0^{(\mp)\prime\prime}=\bar{f}_u(x)\bar{u}_2^{(\mp)}+f_3(x),\ \cdots,$$

其中 $f_3(x)=\dfrac{1}{2}[\bar{f}_{u^2}(x)\bar{u}_1^{(\mp)2}+2\bar{f}_{u\mu}(x)\bar{u}_1^{(\mp)}+\bar{f}_{\mu^2}(x)]$,而 $\bar{f}_{(\cdot)}(x)=f_{2,(\cdot)}(\bar{u}_0^{(\mp)}(x),x,0)$.

根据条件 1,可求得

$$\bar{u}_0^{(\mp)}(x)=\begin{cases}\psi_3(x),\qquad x^0\leqslant x<x^*,\\ \psi_1(x),\qquad x^*<x\leqslant 1,\end{cases}$$

$$\bar{u}_1^{(\mp)}(x)=-[\bar{f}_u(x)]^{-1}\bar{f}_\mu(x),\ \bar{u}_2^{(\mp)}(x)=[\bar{f}_u(x)]^{-1}(\bar{u}_0^{(\mp)\prime\prime}(x)-f_3(x)),\ \cdots.$$

将式(44)、(45)代入式(42)、(43), $\bar{R}u(\tau_1,\mu)$, $\tilde{R}u(\tau_3,\mu)$, $Q^{(\mp)}u(\tau_2,\mu)$ 满足的方程和边值为

$$\begin{cases}\dfrac{\mathrm{d}^2}{\mathrm{d}\tau_1^2}\bar{R}u=f_2(\bar{u}^{(-)}(x^0+\mu\tau_1,\mu)+\bar{R}u,x^0+\mu\tau_1,\mu)-\\ \qquad f_2(\bar{u}^{(-)}(x^0+\mu\tau_1,\mu),x^0+\mu\tau_1,\mu),\\ \bar{R}u(0,\mu)=p(\mu)-\bar{u}^{(-)}(x^0,\mu),\ \bar{R}u(+\infty,\mu)=0,\end{cases}\tag{51}$$

$$\begin{cases}\dfrac{\mathrm{d}^2}{\mathrm{d}\tau_3^2}\tilde{R}u=f_2(\bar{u}^{(+)}(1+\mu\tau_3,\mu)+\tilde{R}u,1+\mu\tau_3,\mu)-\\ \qquad f_2(\bar{u}^{(+)}(1+\mu\tau_3,\mu),1+\mu\tau_3,\mu),\\ \tilde{R}u(0,\mu)=u^1-\bar{u}^{(+)}(1,\mu),\ \tilde{R}u(-\infty,\mu)=0,\end{cases}\tag{52}$$

$$\begin{cases}\dfrac{\mathrm{d}^2}{\mathrm{d}\tau_2{}^2}Q^{(\mp)}u=f_2(\bar{u}^{(\mp)}(x^*+\mu\tau_2,\mu)+Q^{(\mp)}u,x^*+\mu\tau_2,\mu)-\\ \qquad f_2(\bar{u}^{(\mp)}(x^*+\mu\tau_2,\mu),x^*+\mu\tau_2,\mu),\\ \bar{u}^{(-)}(x^*,\mu)+Q^{(-)}u(0,\mu)=\bar{u}^{(+)}(x^*,\mu)+Q^{(+)}u(0,\mu)=\psi_2(x^*),\\ Qu^{(\mp)}(\mp\infty,\mu)=0.\end{cases}\tag{53}$$

把式(47)代入式(51),比较 μ 的同次项,可确定 $\bar{R}_0u(\tau_1,\mu)$ 的问题

$$\begin{cases}\dfrac{\mathrm{d}^2}{\mathrm{d}\tau_1^2}\bar{R}_0u=f_2(\bar{u}_0^{(-)}(x^0)+\bar{R}_0u,x^0,0)-f_2(\bar{u}_0^{(-)}(x^0),x^0,0),\\ \bar{R}_0u(0,\mu)=p(\mu)-\bar{u}_0^{(-)}(x^0),\ \bar{R}_0u(+\infty,\mu)=0,\end{cases}\tag{54}$$

对式(51)做变量替换

$$v=\bar{u}_0^{(-)}(x^0)+\bar{R}_0u(\tau_1),\ z=\frac{\mathrm{d}v}{\mathrm{d}\tau_1},$$

可得方程和边值

$$\begin{cases}\dfrac{\mathrm{d}v}{\mathrm{d}\tau_1}=z,\\ \dfrac{\mathrm{d}z}{\mathrm{d}\tau_1}=f(v,x^0,0),\\ \tilde{u}^{(\mp)}(x^0)=p_0,\ \tilde{u}^{(\mp)}(\mp\infty)=\psi_3(x^*).\end{cases}\tag{55}$$

引入辅助系统

$$\frac{\mathrm{d}\tilde{z}}{\mathrm{d}\tau}=f(\tilde{u},\bar{x},0),\ \frac{\mathrm{d}\tilde{u}}{\mathrm{d}\tau}=\tilde{z}.$$

这里 $\bar{x}\in[x^0,1]$ 作为参数, 根据条件 1 在相平面 $(\tilde{u},\tilde{z})$ 对固定的 $\bar{x},\bar{x}\in[x^0,1]$, 平衡点$(\psi_{1,3}(\bar{x}),0)$ 是鞍点.在求解边界层函数主项时,下面的边值问题在 $\tilde{x}=x^0,\tilde{x}=\bar{x}_0,\tilde{x}=1$ 的可解性起着重要的作用

$$\begin{cases}\dfrac{\mathrm{d}\tilde{z}}{\mathrm{d}\tau_i}=f_2(\tilde{u},\tilde{x},0),\ \dfrac{\mathrm{d}\tilde{u}}{\mathrm{d}\tau_i}=\tilde{z},\\ \tilde{u}(0)=p^i,\ \tilde{u}(\mp\infty)=\psi(x^i),\end{cases}\tag{56}$$

这里 $i=1,2,3$.当 $i=1$ 时,$\tau_1=(x-x^0)\mu^{-1}$,$\tilde{x}=x^0$,$p^0=p_0$,$\psi(x^1)=\psi_3(x^0)$;当 $i=2$ 时,$\tau_2=(x-\bar{x}_0)\mu^{-1}$,$\tilde{x}=\bar{x}_0$,$p^1=\psi_2(\bar{x}_0)$,$\psi(x^2)=\psi_2(\bar{x}_0)$;当 $i=3$ 时,$\tau_3=(x-1)\mu^{-1}$,$\tilde{x}=1$,$p^1=u^1$,$\psi(x^3)=\psi_1(1)$.因为方程(56)是可积系统,所以

$$W^s(\psi_3):=\sqrt{2}\left(\int_{\psi_3(x^0)}^{\tilde{u}}f_2(s,x^*,0)\,\mathrm{d}s\right)^{1/2},\qquad \tilde{u}>\psi_3(x^0)$$

是过平衡点 $(\psi_3(x^0),0)$ 的稳定流形.而

$$\bar{W}^u(\psi_3):=\sqrt{2}\left(\int_{\psi_3(\bar{x}_0)}^{\tilde{u}}f_2(s,\bar{x}_0,0)\,\mathrm{d}s\right)^{1/2},\qquad \tilde{u}<\psi_3(\bar{x}_0)$$

是过平衡点 $(\psi_3(\bar{x}_0),0)$ 的不稳定流形,且

$$W^s(\psi_1):=\sqrt{2}\left(\int_{\psi_1(\bar{x}_0)}^{\tilde{u}}f_2(s,x_0,0)\,\mathrm{d}s\right)^{1/2},\qquad \tilde{u}>\psi_1(\bar{x}_0)$$

是过平衡点 $(\psi_1(\bar{x}_0),0)$ 的稳定流形,

$$\bar{W}^u(\psi_1):=\sqrt{2}\left(\int_{\psi_1(1)}^{\tilde{u}}f_2(s,x_0,0)\,\mathrm{d}s\right)^{1/2},\qquad \tilde{u}>\psi_1(1)$$

是过平衡点 $(\psi_1(1),0)$ 的不稳定流形.

条件 5 假设在相平面 $(\tilde{u},\tilde{z})$ 上, 直线 $\tilde{u}=\psi_2(\bar{x}_0)$ 与 $\bar{W}^u(\psi_3(\bar{x}_0)$ 和$W^s(\psi_1(\bar{x}_0))$ 横截相交, 即, 对任意

$\tilde{u} \in (\psi_1(\bar{x}_0), \psi_3(\bar{x}_0))$, 有

$$\int_{\psi_1(\bar{x}_0)}^{\tilde{u}} f_2(s, \bar{x}_0, 0)\,\mathrm{d}s > 0,\ \int_{\psi_3(\bar{x}_0)}^{\tilde{u}} f_2(s, \bar{x}_0, 0)\,\mathrm{d}s > 0.$$

记

$$I(\bar{x}_0) := \int_{\psi_3(\bar{x}_0)}^{\psi_1(\bar{x}_0)} f_2(u, \bar{x}_0, 0)\,\mathrm{d}u. \tag{57}$$

条件 6　假设方程 $I(\bar{x}_0) = 0$ 有解, $\bar{x}_0 \in (x^0, 1)$, 并且 $I'(\bar{x}_0) \neq 0$.

由条件 5 可知,问题(51)的解 $\bar{R}_0 u(\tau_1)$ 存在且有指数估计

$$|\bar{R}_0 u(\tau_1)| \leqslant c\mathrm{e}^{-\kappa\tau_1}. \tag{58}$$

确定 $\bar{R}_k u(\tau_1)$ 的问题为

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}\tau_1^2}\bar{R}_k u = f_{2u}(\tau_1)\bar{R}_k u + h_k(\tau_1), \\ \bar{R}_k u(0,\mu) = p_k - \bar{u}_k^{(-)}(x^0),\ \bar{R}_k u(+\infty,\mu) = 0, \end{cases} \tag{59}$$

其中 $f_{2u}(\tau_1) = f_{2u}(\tau_1)(\psi_3(x^*) + \bar{R}_0 u, x^*, 0)$, 而 $h_k(\tau_1)$ 是已知函数, 它依赖于 $\bar{u}_j^{(-)}(x^*)$, $0 \leqslant j \leqslant k$ 和 $\bar{R}_j u(\tau_1)$, $0 \leqslant j \leqslant k-1$.

显然,线性方程(59)的解可以显式表示出来:

$$\bar{R}_k u = (p_k - \bar{u}_k^{(-)}(x^0))\frac{\bar{R}_0 u(\tau_1)}{\bar{R}_0 u(0)} + \bar{R}_0 u(\tau_1)\int_0^{\tau_1}(\bar{R}_0 u(\eta))^{-2}\mathrm{d}\eta\int_{+\infty}^{\eta}\bar{R}_0 u(\sigma)h_k(\sigma)\,\mathrm{d}\sigma. \tag{60}$$

把式(47)代入式(51)可确定 $\tilde{R}_0 u(\tau_3)$ 的问题为

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}\tau_3^2}\tilde{R}_0 u = f_2(\bar{u}_0^{(-)}(1) + \tilde{R}_0 u, 1, 0) - f_2(\bar{u}_0^{(-)}(1), 1, 0), \\ \tilde{R}_0 u(0,\mu) = u^1 - \bar{u}_0^{(+)}(1),\ \tilde{R}_0 u(-\infty,\mu) = 0. \end{cases} \tag{61}$$

类似于式(52)、(61)也有解,且有指数估计

$$|\tilde{R}_0 u(\tau_3)| \leqslant c\mathrm{e}^{\kappa\tau_3}. \tag{62}$$

确定 $\tilde{R}_k u(\tau_1)$ 的问题为

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}\tau_3^2}\tilde{R}_k u = f_{2u}(\tau_3)\tilde{R}_k u + \bar{h}_k(\tau_3), \\ \tilde{R}_k u(0,\mu) = -\bar{u}_k^{(+)}(1),\ \tilde{R}_k u(+\infty,\mu) = 0, \end{cases} \tag{63}$$

其中 $f_{2u}(\tau_3) = f_{2u}(\tau_3)(\psi_1(1) + \tilde{R}_0 u, 1, 0)$,而 $\bar{h}_k(\tau_3)$ 是已知函数,它依赖于 $\bar{u}_j^{(+)}(1)$, $0 \leqslant j \leqslant k$ 和 $\tilde{R}_j u(\tau_3)$, $0 \leqslant j \leqslant k-1$.

显然,线性方程(63)的解可以显式表示出来:

$$\tilde{R}_k u = (-\bar{u}_k^{(+)}(1))\frac{\tilde{R}_0 u(\tau_3)}{\tilde{R}_0 u(0)} + \tilde{R}_0 u(\tau_3)\int_0^{\tau_3}(\tilde{R}_0 u(\eta))^{-2}\mathrm{d}\eta\int_{-\infty}^{\eta}\tilde{R}_0 u(\sigma)\bar{h}_k(\sigma)\,\mathrm{d}\sigma. \tag{64}$$

把式(48)代入式(52),可确定 $Q_0^{(\mp)}u(\tau_2)$ 的问题为

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}\tau_2^2}Q_0^{(\mp)}u = f_2(\bar{u}_0^{(\mp)}(\bar{x}_0) + Q_0^{(\mp)}u, \bar{x}_0, 0) - f_2(\bar{u}_0^{(\mp)}(\bar{x}_0), \bar{x}_0, 0) = \\ \qquad f(\bar{u}_0^{(\mp)}(\bar{x}_0) + Q_0^{(\mp)}u, \bar{x}_0, 0), \\ Q_0^{(\mp)}u(0) = \psi_2(\bar{x}_0) - \bar{u}_0^{(\mp)}(\bar{x}_0),\ Q_0^{(\mp)}u(\mp\infty) = 0. \end{cases} \tag{65}$$

由条件 5 可知,问题(65)的解 $Q_0^{(\mp)}u(\tau_2)$ 存在,且有指数估计

$$|Q_0^{(\mp)}u(\tau_2)| \leqslant c\mathrm{e}^{-\kappa|\tau_2|},\qquad \tau_2 \in \mathbf{R}. \tag{66}$$

确定 $Q_1^{(\mp)}u(\tau_2)$ 的问题为

$$\begin{cases}\dfrac{\mathrm{d}^2}{\mathrm{d}\tau_2^2}Q_1^{(\mp)}u=f_{2,u}(\tau)Q_1^{(\mp)}u+h_1^{(\mp)}(\tau_2),\\ Q_1^{(\mp)}u(0)=(\psi'_2(\bar{x}_0)-\psi'_{1,3}(\bar{x}_0))\bar{x}_1-\bar{u}_1^{(\mp)}(\bar{x}_0),\ Q_1^{(\mp)}u(\mp\infty)=0,\end{cases}\tag{67}$$

其中

$$h_1^{(\mp)}(\tau_2)=f_{2,u}(\tau_2)[\psi'_{1,3}(\bar{x}_0)(\bar{x}_1+\tau_2)+\bar{u}_1^{(\mp)}(\bar{x}_0)]+f_{2,x}(\tau_2)(\bar{x}_1+\tau_2)+f_{2,\mu}(\tau_2),$$

这里 $f_{2,(\cdot)}(\tau_2)=f_{2,(\cdot)}(\tilde{u}^{(\mp)}+Q_0^{(\mp)}u,\bar{x}_0,0)$.

问题(67)的解可以显式表达出来:

$$Q_1^{(\mp)}u(\tau_2)=Q_1^{(\mp)}u(0)\frac{\tilde{z}(\tau_2)}{\tilde{z}(0)}+\tilde{z}(\tau_2)\int_0^{\tau_2}\tilde{z}^{-2}(\eta)\int_{\mp\infty}^{\eta}\tilde{z}(\sigma)h_1^{(\mp)}(\sigma)\mathrm{d}\sigma\mathrm{d}\eta.\tag{68}$$

它具有类似于式(66)的指数估计,其中 $\tilde{z}(\tau_2)=\dfrac{\mathrm{d}}{\mathrm{d}\tau_2}Q_0^{\mp}(\tau_2)$.

对 $Q_1^{(\mp)}u(\tau_2)$ 求导:

$$\frac{\mathrm{d}}{\mathrm{d}\tau_2}Q_1^{(\mp)}u(\tau_2)=Q_1^{(\mp)}u(0)\frac{\tilde{z}'(\tau_2)}{\tilde{z}(0)}+\tilde{z}^{-1}(\tau_2)\int_{\mp\infty}^{\tau_2}\tilde{z}(\sigma)h_1^{(\mp)}(\sigma)\mathrm{d}\sigma+$$
$$\tilde{z}'(\tau_2)\int_0^{\tau_2}\tilde{z}^{-2}(\eta)\int_{\mp\infty}^{\eta}\tilde{z}(\sigma)h_1^{(\mp)}(\sigma)\mathrm{d}\sigma\mathrm{d}\eta,$$

计算可得

$$\frac{\mathrm{d}}{\mathrm{d}\tau_2}Q_1^{(\mp)}u(0)=\tilde{z}^{-1}(0)\int_{\mp\infty}^{0}\tilde{z}(\sigma)h_1^{(\mp)}(\sigma)\mathrm{d}\sigma.\tag{69}$$

确定 $Q_k^{\mp}u(\tau)$ 的方程和边值为

$$\begin{cases}\dfrac{\mathrm{d}^2}{\mathrm{d}\tau_2^2}Q_k^{\mp}u=f_u(\tau)Q_k^{\mp}u+h_k^{\mp}(\tau_2),\\ Q_k^{\mp}u(0)=(\psi'_2(\bar{x}_0)-\psi'_{1,3}(\bar{x}_0))\bar{x}_k+q_k,\ Q_k^{\mp}u(\mp\infty)=0,\end{cases}\tag{70}$$

其中 $q_k=q_k(\bar{x}_0,\bar{x}_1,\cdots,\bar{x}_{k-1})$ 是已知数,而

$$h_k^{\mp}(\tau_2)=f_{2u}(\tau_2)\psi'_{1,3}(\bar{x}_0)\bar{x}_k+f_x(\tau_2)\bar{x}_k+\rho_k(\sigma),$$

这里 ρ_k 是依赖于 $\bar{u}_j^{(\mp)}$, $0\leqslant j\leqslant k$ 和 $Q_j^{(\mp)}u(\tau_2)$, $0\leqslant j\leqslant k-1$ 以及 x_j, $0\leqslant j\leqslant k-1$ 的已知函数.

问题(70)的解 $Q_k^{(-)}u(\tau_2)$ 可显式表示:

$$Q_k^{(-)}u(\tau_2)=Q_k^{(-)}u(0)\frac{\tilde{z}(\tau_2)}{\tilde{z}(0)}+\tilde{z}(\tau_2)\int_0^{\tau}\tilde{z}^{-2}(\eta)J_k(\eta)\mathrm{d}\eta,$$

其中 $J_k(\eta)=\int_{-\infty}^{\eta}\tilde{z}(\sigma)h_k^{(-)}(\sigma)\mathrm{d}\sigma$, 计算它的导数

$$\frac{\mathrm{d}}{\mathrm{d}\tau_2}Q_k^{(-)}u(\tau_2)=Q_k^{(-)}u(0)\frac{\tilde{z}'(\tau_2)}{\tilde{z}(0)}+\tilde{z}'(\tau_2)\int_0^{\tau_2}\tilde{z}^{-2}(\eta)J_k(\eta)\mathrm{d}\eta+z^{-1}(\tau_2)J_k(\tau_2).$$

因为 $\tilde{z}'(0)=\dfrac{\mathrm{d}^2}{\mathrm{d}\tau_2^2}Q_0^{(-)}u(0)=f_2(\psi_2(\bar{x}_0),\bar{x}_0,0)=0$, 所以

$$\frac{\mathrm{d}}{\mathrm{d}\tau_2}Q_k^{(-)}u(0)=\tilde{z}^{-1}(0)\int_{-\infty}^{0}\tilde{z}(\sigma)h_k^{(-)}(\sigma)\mathrm{d}\sigma.\tag{71}$$

同理可得

$$\frac{\mathrm{d}}{\mathrm{d}\tau_2}Q_k^{(+)}u(0)=\tilde{z}^{-1}(0)\int_{+\infty}^{0}\tilde{z}(\sigma)h_k^{(+)}(\sigma)\mathrm{d}\sigma.$$

引理 2　如果 $Q_1^{(\mp)}u$ 满足式(69),则有如下表达式:

$$\tilde{z}(0)\left[\frac{\mathrm{d}Q_1^{(-)}u}{\mathrm{d}\tau_2}(0)-\frac{\mathrm{d}Q_1^{(+)}u}{\mathrm{d}\tau_2}(0)\right]=$$

$$\tilde{z}(0)[-\psi'_3(\bar{x}_0)+\psi'_1(\bar{x}_0)]+\bar{x}_1\int_{\psi_3(\bar{x}_0)}^{\psi_1(\bar{x}_0)}f_{2x}(u,\bar{x}_0,0)\,\mathrm{d}u+$$
$$\int_{-\infty}^{+\infty}f_{2u}(\tau_2)\tau_2\tilde{z}(\tau_2)\,\mathrm{d}\tau+\int_{\psi_3(\bar{x}_0)}^{\psi_1(\bar{x}_0)}f_{2\mu}(u,\bar{x}_0,0)\,\mathrm{d}u. \tag{72}$$

把光滑缝接条件(7)写成

$$\mu\frac{\mathrm{d}^2\bar{u}^{(-)}}{\mathrm{d}x^2}(x^*,\mu)-\mu\frac{\mathrm{d}^2\bar{u}^{(+)}}{\mathrm{d}x^2}(x^*,\mu)=0.$$

因为 $\tilde{z}(0)\neq 0$, 所以我们记

$$H_1(x^*,\mu)=\tilde{z}(0)\left[\mu\frac{\mathrm{d}}{\mathrm{d}\tau}u^{(-)}(x^*,\mu)-\mu\frac{\mathrm{d}}{\mathrm{d}\tau}u^{(+)}(x^*,\mu)\right]=0. \tag{73}$$

把式(44)—(49)代入式(73),并展开成μ 的幂级数:

$$H_1(x^*,\mu)=I(\bar{x}_0)+\mu[I'(\bar{x}_0)\bar{x}_1+I_1(\bar{x}_0)]+\cdots+\mu^k[I'(\bar{x}_0)x_k+I_k]+\cdots=0, \tag{74}$$

其中 $I(\bar{x}_0)$ 由式(57)确定, $I'(\bar{x}_0)\bar{x}_1+I_1(\bar{x}_0)$ 就是引理 1 中的式(72).

根据条件 6,由式(74)可求得

$$\bar{x}_k=-(I'(\bar{x}_0))^{-1}I_k(\bar{x}_0,\cdots,\bar{x}_{k-1}),\qquad k\geqslant 2, \tag{75}$$

其中 I_k 是仅依赖于 $\bar{x}_j,0\leqslant j\leqslant k-1$ 的实数.

到此为止,我们可以确定式(44)—(49)中的所有未知函数,也就是说构造好了形式渐近解(44)、(45).

3.1　右问题解的存在性及渐近解的估计

记 $U_n^{(\mp)}(x,\mu)$ 为已构造的幂级数(44)、(45)的部分和

$$\bar{U}_n^{(-)}(x,\mu)=\sum_{k=0}^{n}\mu^k(\bar{u}_k^{(-)}(x)+\bar{R}_ku(\tau_1)+Q_k^{(-)}u(\tau_2)),$$
$$\bar{U}_n^{(+)}(x,\mu)=\sum_{k=0}^{n}\mu^k(\bar{u}_k^{(+)}(x)+\tilde{R}_ku(\tau_3)+Q_k^{(+)}u(\tau_2)).$$

由文献[17]的余项估计,可得如下定理.

定理 3　如果满足假设条件 1、5、6,那么对于充分小的$\mu>0$, 问题 P^-(42),问题 P^+(43)的解 $\bar{u}^{(\mp)}(x,\mu)$ 存在,且有如下的渐近表达式:

$$\begin{cases}\bar{u}^{(-)}(x,\mu)=\bar{U}_n^{(-)}(x,\mu)+O(\mu^{n+1}),\\ \bar{u}^{(+)}(x,\mu)=\bar{U}_n^{(+)}(x,\mu)+O(\mu^{n+1}).\end{cases} \tag{76}$$

对导数 $\mathrm{d}\bar{u}^{(\mp)}(x,\mu)/\mathrm{d}x$ 有如下的渐近表达式:

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}x}\bar{u}^{(-)}(x,\mu)=\dfrac{\mathrm{d}}{\mathrm{d}x}\bar{U}_n^{(-)}(x,\mu)+O(\mu^n),\qquad 0\leqslant x<x^*,\\ \dfrac{\mathrm{d}}{\mathrm{d}x}\bar{u}^{(+)}(x,\mu)=\dfrac{\mathrm{d}}{\mathrm{d}x}\bar{U}_n^{(+)}(x,\mu)+O(\mu^n),\qquad x^*\leqslant x\leqslant 1.\end{cases} \tag{77}$$

接下来为了证明右问题存在阶梯状解,重新回到左右问题 $P^{(-)},P^{(+)}$, 并假设

$$x^*=x_{\bar{\delta}}\equiv\tilde{x}_0+\mu\tilde{x}_1+\cdots+\mu^n\tilde{x}_n+\mu^{n+1}(\tilde{x}_{n+1}+\bar{\delta}),$$

这里 $\bar{x}_i,i=0,1,\cdots,n+1$ 是由条件 8 和式(75)所确定的, $\bar{\delta}$ 是任意实数,它可以依赖于μ,但是,当$\mu\to 0$时, $\bar{\delta}$ 是有界的.

根据定理 3,对充分小的$\mu>0$,左右问题 $P^{(\mp)}$ 的解 $\bar{u}^{(\mp)}(x,\mu,\bar{\delta})$ 是存在的,且有渐近表达式(76),虽然 $Q_k^{\mp}u(\tau_2)$ $(k\geqslant 0)$ 换成了 $Q_k^{\mp}u(\tau_{\bar{\delta}})$, $\tau_{\bar{\delta}}=\dfrac{x-x_{\bar{\delta}}}{\mu}$, 但是式(77)中的误差估计仍保持不变.

考虑下面导数的差,即

$$\mu\left(\frac{\mathrm{d}}{\mathrm{d}x}\bar{u}^{(-)}(x_{\bar{\delta}},\mu,\bar{\delta})-\frac{\mathrm{d}}{\mathrm{d}x}\bar{u}^{(+)}(x_{\delta},\mu,\delta)\right)=$$

$$I(\tilde{x}_0)+\sum_{k=1}^{n+1}\mu^k[I'(\bar{x}_0)\tilde{x}_k+I_k]+\mu^{n+1}I'(\tilde{x}_0)\bar{\delta}+O(\mu^{n+2})=$$
$$\mu^{n+1}(I'(\tilde{x}_0)\bar{\delta}+O(\mu)). \tag{78}$$

因为 $I'(\bar{x}_0)\neq 0$,所以当 μ 充分小时,式(78)的符号取决于 $\bar{\delta}$ 的符号.由介值定理可知,存在 $\bar{\delta}=\bar{\delta}(\mu)=O(\mu)$, 使得式(78)为零,即

$$\frac{\mathrm{d}}{\mathrm{d}x}u^{(-)}(x_{\bar{\delta}},\mu,\bar{\delta})=\frac{\mathrm{d}}{\mathrm{d}x}u^{(+)}(x_{\bar{\delta}},\mu,\bar{\delta}).$$

由此可知,函数

$$u(x,\mu)=\begin{cases}\bar{u}^{(-)}(x,\mu,\bar{\delta}), & x^*\leqslant x<x_{\bar{\delta}},\\ \bar{u}^{(+)}(x,\mu,\bar{\delta}), & x_{\bar{\delta}}\leqslant x\leqslant 1\end{cases}$$

是问题 $P^{(\mp)}$ 在 $x=x_{\bar{\delta}}$ 附近具有内部边界层的解.根据定理3,右问题解有如下表达式:

$$\bar{u}(x,\mu)=\begin{cases}\sum_{k=0}^{n}\mu^k(\bar{u}_k^{(-)}(x)+\bar{R}_k u(\tau_1)+Q_k^{(-)}u(\tilde{\tau}_2))+O(\mu^{n+1}), & x^*\leqslant x<\tilde{x},\\ \sum_{k=0}^{n}\mu^k(\bar{u}_k^{(+)}(x)+\tilde{R}_k u(\tau_3)+Q_k^{(+)}u(\tilde{\tau}_2))+O(\mu^{n+1}), & \tilde{x}\leqslant x\leqslant 1,\end{cases} \tag{79}$$

其中 $\tilde{x}=\tilde{x}_0+\mu\tilde{x}_1+\cdots+\mu^n\tilde{x}_n$, $\tilde{\tau}_2=(x-\tilde{x})/\mu$.

虽然式(79)中把 $x_{\bar{\delta}}$ 换成了 $\tilde{x}$, 但在式(79)中还保持了原来的误差估计量阶.

4 解的存在性与渐近解的余项估计

在上一节中,我们证明了左问题(2)在 $x\in[0,x^0]\backslash I_\delta$ 上存在光滑解,右问题(3)在 $x\in[x^0,1]$ 上存在光滑解,接下来在点 $x=x^*$ 进行光滑缝接.首先引入两个边值问题:

左边值问题 $x_2\leqslant x<x^0$:

$$\begin{cases}\mu^2\dfrac{\mathrm{d}^2\tilde{x}^{(-)}}{\mathrm{d}x^2}=f_1(\tilde{x}^{(-)},x,\mu),\\ \tilde{x}^{(-)}(x_2,\mu)=\varphi_2(x_2),\ \tilde{x}^{(-)}(x^0,\mu)=p(\mu).\end{cases} \tag{80}$$

右边值问题 $x^0\leqslant x\leqslant x^*$:

$$\begin{cases}\mu^2\dfrac{\mathrm{d}^2\tilde{x}^{(+)}}{\mathrm{d}x^2}=f_2(\tilde{x}^{(+)},x,\mu),\\ \tilde{x}^{(+)}(x^0,\mu)=p(\mu),\ \tilde{x}^{(+)}(x^*,\mu)=\psi_2(x^*).\end{cases} \tag{81}$$

为了确定在 $x=x^0$ 处的内部层主项,我们需要下面边值问题的可解性,由辅助系统,可得

$$\begin{cases}\dfrac{\mathrm{d}\tilde{z}^{(-)}}{\mathrm{d}\tau_1}=f_1(\tilde{u}^{(-)},x^0,0),\ \dfrac{\mathrm{d}\tilde{u}^{(-)}}{\mathrm{d}\tau_1}=\tilde{z}^{(-)},\\ \tilde{u}^{(-)}(0)=p,\ \tilde{u}^{(-)}(-\infty)=\varphi_2(x^0),\end{cases} \tag{82}$$

$$\begin{cases}\dfrac{\mathrm{d}\tilde{z}^{(+)}}{\mathrm{d}\tau_1}=f_2(\tilde{u}^{(+)},x^0,,0),\ \dfrac{\mathrm{d}\tilde{u}^{(+)}}{\mathrm{d}\tau_1}=\tilde{z}^{(+)},\\ \tilde{u}^{(+)}(0)=p,\ \tilde{u}^{(+)}(+\infty)=\psi_3(x^0).\end{cases} \tag{83}$$

因为方程(82)、(83)是可积系统,所以

$$W^u(\varphi_2):=\sqrt{2}\left(\int_{\varphi_2(x^0)}^{\tilde{u}}f_1(s,x^0,0)\mathrm{d}s\right)^{1/2},\qquad \tilde{u}<\varphi_2(x^0)$$

是过平衡点 $(\varphi_2(x^0),0)$ 的不稳定流形,而

$$W^s(\psi_3):=\sqrt{2}\left(\int_{\psi_3(x^0)}^{\tilde{u}}f_2(s,x^0,0)\mathrm{d}s\right)^{1/2},\qquad \tilde{u}>\psi_3(x^0)$$

是过平衡点 $(\psi_3(x^0),0)$ 的稳定流形.

条件 7　假设在相平面 $(\tilde{u},\tilde{z})$ 上,直线 $\tilde{u}=p$ 与 $W^u(\varphi_2(x^0))$ 和 $W^s(\psi_3(x^0))$ 横截相交,即,对任意 $\tilde{u}\in(\psi_3(x^0),\varphi_2(x^0))$,有

$$\int_{\psi_3(x^0)}^{\tilde{u}} f_2(s,x^0,0)\,\mathrm{d}s>0,\ \int_{\varphi_2(x^0)}^{\tilde{u}} f_1(s,x^0,0)\,\mathrm{d}s>0,$$

记

$$H(p)=\left(2\int_{\varphi_2(x^0)}^{p} f_1(u,x^0,0)\,\mathrm{d}u\right)^{1/2}-\left(2\int_{\psi_3(x^0)}^{p} f_2(u,x^0,0)\,\mathrm{d}u\right)^{1/2}. \tag{84}$$

条件 8　假设方程 $H(p)=0$ 有解 $p=p_0,p_0\in(\varphi_2(x^0),\psi_1(x^0))$.很容易把方程 $H(p)=0$ 化成

$$\int_{\varphi_2(x^0)}^{p} f_1(u,x^0,0)\,\mathrm{d}u=\int_{\psi_3(x^0)}^{p} f_2(u,x^0,0)\,\mathrm{d}u,$$

易得

$$\frac{\mathrm{d}H(p_0)}{\mathrm{d}p}=\frac{f_1(p_0,x^0,0)-f_2(p_0,x^0,0)}{\left(2\int_{\varphi_2(x^0)}^{p_0} f_1(u,x^0,0)\,\mathrm{d}u\right)^{1/2}}\neq 0.$$

利用光滑缝接条件(4)来确定 p_k,

$$\begin{cases}\dfrac{\mathrm{d}R_0u(0)}{\mathrm{d}\tau_1}=\dfrac{\mathrm{d}\bar{R}u(0)}{\mathrm{d}\tau_1}, \qquad \tilde{z}^{(-)}(0)=\tilde{z}^{(+)}(0),\\ \varphi_1'(x_0)+\dfrac{\mathrm{d}}{\mathrm{d}\tau}Q_1^{(-)}u(0)=\varphi_2'(x_0)+\dfrac{\mathrm{d}}{\mathrm{d}\tau}Q_1^{(+)}u(0),\\ \bar{u}_{k-1}'(x_0)+\dfrac{\mathrm{d}}{\mathrm{d}\tau_1}R_ku(0)=\bar{u}_{k-1}^{(-)\prime}(x_0)+\dfrac{\mathrm{d}}{\mathrm{d}\tau_1}\bar{R}_ku(0).\end{cases} \tag{85}$$

由式(28)、(54),可得

$$\begin{cases}(R_0u(0))^2=(\tilde{z}^{(-)}(0))^2=2\int_{\varphi_2(x_0)}^{p_0} f_1(u,x_0,0)\,\mathrm{d}u,\\ (\bar{R}_0u(0))^2=(\tilde{z}^{(+)}(0))^2=2\int_{\psi_3(x_0)}^{p_0} f_2(u,x_0,0)\,\mathrm{d}u.\end{cases}$$

根据条件 8 可求得 p_0,考虑到 $\tilde{z}^{(-)}(0)=\tilde{z}^{(+)}(0)$, 把式(30)、(60)代入式(85)可得确定 p_k 的方程

$$\frac{\mathrm{d}H(p_0)}{\mathrm{d}p}p_k=\bar{u}_{k-1}^{(-)\prime}(x_0)-\bar{u}_{k-1}'(x_0)+F_k,$$

其中

$$F_k=(\tilde{z}^{(-)}(0))^{-1}\Big(f_1(p_0,x_0,0)\bar{u}_k(x_0)-f_2(p_0,x_0,0)\bar{u}_k^{(-)}(x_0)+\int_{+\infty}^{0}\tilde{z}^{(+)}(\sigma)h_k(\sigma)\,\mathrm{d}\sigma-\int_{-\infty}^{0}\tilde{z}^{(-)}(\sigma)l_k^{(+)}(\sigma)\,\mathrm{d}\sigma\Big).$$

由条件 1、8 可求出 p_k:

$$p_k=\left(\frac{\mathrm{d}H(p_0)}{\mathrm{d}p}\right)^{-1}(\bar{u}_{k-1}^{(-)\prime}(x_0)-\bar{u}_{k-1}'(x_0)+F_k).$$

接下来证整体解的存在性,令 $p_0+\mu p_1+\cdots+\mu^{n+1}(p_{n+1}+\tilde{\delta})$,$\tilde{\delta}$ 是参数.众所周知,对给定的 $p(\mu)$ 和 x^*,左右问题的解 $\tilde{x}^{(\mp)}(x,\mu,\tilde{\delta})$ 是存在的,且有如下的渐近表达式:

$$\tilde{x}^{(-)}(x,\mu,\tilde{\delta})=\sum_{k=0}^{n}\mu^k(\bar{u}_k^{(-)}(x)+L_ku(\bar{\tau}_0)+Q_k^{(-)}u(\bar{\tau}_1))+O(\mu^{n+1}),$$

其中 $\bar{\tau}_0=(x-x_2)\mu^{-1},\bar{\tau}_1=(x-x^0)\mu^{-1}$,

$$\tilde{x}^{(+)}(x,\mu,\tilde{\delta})=\sum_{k=0}^{n}\mu^k(\bar{u}_k^{(+)}(x)+Q_k^{(+)}u(\tau_1)+R_ku(\tau_2))+O(\mu^{n+1}),$$

式中 $\bar{\tau}_2 = (x - x^*)\mu^{-1}$.

由式(80)、(81)可知, $\tilde{u}^{(\mp)}(x,\mu,\tilde{\delta})$ 在 $x = x^0$ 处是连续的.令

$$G(\delta,\mu) = \frac{\mathrm{d}}{\mathrm{d}x}\tilde{u}^{(-)}(x^0,\mu,\tilde{\delta}) - \frac{\mathrm{d}}{\mathrm{d}x}\tilde{u}^{(+)}(x^0,\mu,\tilde{\delta}).$$

根据渐近解的构造方法和光滑缝接条件 8 可得

$$G(\tilde{\delta},\mu) = \mu^n\left(\bar{u}'_n(x^0) - \bar{u}_n^{(-)\prime}(x^0) + \frac{\mathrm{d}}{\mathrm{d}\tau_1}R_{n+1}u(0) - \frac{\mathrm{d}}{\mathrm{d}\tau_1}\bar{R}_{n+1}u(0)\right) + O(\mu^{n+1}) =$$
$$\mu^{n+1}\left(\tilde{\delta}\frac{\mathrm{d}H_1(p_0)}{\mathrm{d}p} + O(1)\right).$$

只要 $\tilde{\delta}$ 充分大,μ 充分小,$G(\tilde{\delta},\mu)$ 的符号就随 $\tilde{\delta}$ 符号而变,所以由介值定理知,存在 $\tilde{\delta} = \delta^*$ 有 $G(\delta^*,\mu) = 0$,这样,就有了如下关系式成立:

$$\frac{\mathrm{d}}{\mathrm{d}x}\tilde{u}^{(-)}(x,\mu,\delta^*) = \frac{\mathrm{d}}{\mathrm{d}x}\tilde{u}^{(+)}(x,\mu,\delta^*).$$

即,我们证明了问题(2)、(3)在区间 $[0,1]\backslash I_\delta$ 上光滑解的存在性,而且得到了问题(1)解的渐近展开式.这归结为如下定理.

定理 4 如果满足假设条件 1—8,那么问题(1)存在解 $u(x,\mu)$, 它有如下的渐近表达式:

$$u(x,\mu) = \begin{cases} \sum_{k=0}^{n}\mu^k(u_k(x) + L_k u(\tau_0)) + O(\mu^{n+1}), & 0 \leqslant x \leqslant x_1, \\ \bar{u}(x) + O(\mu), & x_1 < x < x_2, \\ \sum_{k=0}^{n}\mu^k(\bar{u}_k(x) + R_k u(\tau_1)) + O(\mu^{n+1}), & x_2 \leqslant x < x^0, \\ \sum_{k=0}^{n}\mu^k(\bar{u}_k^{(-)}(x) + \bar{R}_k u(\tau_1) + Q_k^{(-)}u(\tau_2)) + O(\mu^{n+1}), & x^0 \leqslant x < x^*, \\ \sum_{k=0}^{n}\mu^k(\bar{u}_k^{(+)}(x) + \tilde{R}_k u(\tau_3) + Q_k^{(+)}u(\tau_2)) + O(\mu^{n+1}), & x^* \leqslant x \leqslant 1. \end{cases}$$

5 算 例

考虑 Dirichlet 边值问题

$$\begin{cases} \mu^2\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \begin{cases} (u + 4x - 1)(u - 4x + 1) - \mu, & 0 \leqslant x \leqslant \frac{1}{2}, \\ u\left(u + \frac{3}{4}\right)(u - x) + \mu, & \frac{1}{2} < x < 1, \end{cases} \\ u(0,\mu) = \frac{3}{2},\ u(1,\mu) = -\frac{1}{2}, \end{cases} \tag{86}$$

其中 $\mu > 0$ 是小参数.这里

$$f_1(u,x,\mu) = (u + 4x - 1)(u - 4x + 1) - \mu,\ h_1(x) = 1,\ g_1(u,x,\mu) = 1,$$
$$f_2(u,x,\mu) = u\left(u + \frac{3}{4}\right)(u - x) + \mu.$$

显然,退化方程

$$f_1(u,x,0) = 0 \tag{87}$$

有两个解 $\varphi_1(x) = -4x + 1$,$\varphi_2(x) = 4x - 1$,且 $\varphi_1(x)$ 和 $\varphi_2(x)$ 在 $x = 1/4$ 处相交,即在该点发生稳定性交替.将退化方程(87)修正为 $f_1(u,x,\mu) = (u + 4x - 1)(u - 4x + 1) - \mu$.则有

$$\varphi(x,\mu) = ((4x - 1)^2 + \mu)^{1/2}.$$

而退化方程

$$f_2(u,x,0)=u\left(u+\frac{3}{4}\right)(u-x)$$

有 3 个解

$$\psi_3(x)=x,\ \psi_2(x)=0,\ \psi_1=-\frac{1}{4}.$$

由

$$H_1(p)=\int_1^p(u^2-1)\,\mathrm{d}u-\int_{1/2}^p u\left(u+\frac{3}{4}\right)\left(u-\frac{1}{2}\right)\mathrm{d}u=0, \tag{88}$$

可得 $p_0=0.788\,3$, 容易验证条件都满足.由

$$I(\bar{x}_0)=\int_{\bar{x}_0}^{-3/4}u\left(u+\frac{3}{4}\right)(u-x)\,\mathrm{d}u=0, \tag{89}$$

可得 $\bar{x}_0=0.75$, 容易验证条件都满足.

确定 $L_0u(\tau_0),R_0u(\tau_1),\bar{R}_0u(\tau_1),Q_0^{(\mp)}u(\tau_2),\tilde{R}(\tau_3)$ 的问题为

$$\begin{cases}\dfrac{\mathrm{d}^2}{\mathrm{d}\tau_0^2}L_0u(\tau_0)=2L_0u(\tau_0)+(L_0u(\tau_0))^2,\\ L_0u(0)=\dfrac{1}{2},\ L_0u(+\infty)=0.\end{cases} \tag{90}$$

求解得

$$L_0u(\tau_0)=\frac{12(13+2\sqrt{42})\mathrm{e}^{-\sqrt{2}\tau_0}}{(1-(13+2\sqrt{42})\mathrm{e}^{-\sqrt{2}\tau_0})^2},$$

$$\begin{cases}\dfrac{\mathrm{d}^2}{\mathrm{d}\tau_1^2}R_0u(\tau_1)=2R_0u(\tau_1)+(R_0u(\tau_1))^2,\\ R_0u(0)=p_0-1,\ R_0u(-\infty)=0.\end{cases} \tag{91}$$

求解得

$$R_0u(\tau_1)=\frac{12[1+(6+\sqrt{36+12(p_0-1)})/(p_0-1)]\mathrm{e}^{\sqrt{2}\tau_1}}{[1-(1+(6+\sqrt{36+12(p_0-1)})/(p_0-1))\mathrm{e}^{\sqrt{2}\tau_1}]^2},$$

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}\tau_1}\bar{R}_0u(\tau_1)=\left(\dfrac{(\bar{R}_0u(\tau_1))^2}{2}+\dfrac{7\bar{R}_0u(\tau_1)}{6}+\dfrac{5}{8}\right)^{1/2}\bar{R}_0u(\tau_1),\\ \bar{R}_0u(0)=p_0-\dfrac{1}{2},\ \bar{R}_0u(+\infty)=0.\end{cases} \tag{92}$$

$\bar{R}_0u(\tau_1)$ 满足

$$\frac{\mathrm{d}}{\mathrm{d}\tau_1}\bar{R}_0u(\tau_1)\leqslant\left(\frac{(\bar{R}_0u(\tau_1))^2}{2}+\frac{7\bar{R}_0u(\tau_1)}{6}+\frac{49}{72}\right)^{1/2}\bar{R}_0u(\tau_1).$$

由比较原理

$$\bar{R}_0u(\tau_1)\leqslant\frac{\sqrt{2}c_1\mathrm{e}^{-c_2\tau_1}}{1-c_1\mathrm{e}^{-c_2\tau_1}},\qquad c_1=\frac{2p_0-1}{2p_0+2\sqrt{2}-1},\ c_2=\frac{7\sqrt{2}}{12}.$$

$Q_0^{(\mp)}u(\tau_2)$ 满足

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}\tau_2}Q_0^{(-)}u(\tau_2)=\left(\dfrac{\sqrt{2}Q_0^{(-)}u(\tau_2)}{2}+\dfrac{3\sqrt{2}}{4}\right)Q_0^{(-)}u(\tau_2),\\ Q_0^{(-)}u(0)=-\dfrac{3}{4},\ Q_0^{(-)}u(-\infty)=0.\end{cases} \tag{93}$$

求解得

$$Q_0^{(-)}u(\tau_2)=\frac{\sqrt{2}c_3\mathrm{e}^{c_4\tau_2}}{1-c_3\mathrm{e}^{c_4\tau_2}},\qquad c_3=\frac{-3}{4\sqrt{2}-3},\ c_4=\frac{3\sqrt{2}}{4}.$$

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}\tau_2}Q_0^{(+)}u(\tau_2)=\left(\dfrac{3\sqrt{2}}{4}-\dfrac{\sqrt{2}Q_0^{(+)}u(\tau_2)}{2}\right)Q_0^{(-)}u(\tau_2),\\ Q_0^{(+)}u(0)=\dfrac{3}{4},\ Q_0^{(+)}u(+\infty)=0.\end{cases}\tag{94}$$

求解得

$$Q_0^{(+)}u(\tau_2)=\frac{3\mathrm{e}^{-c_4\tau_2}}{2(1+\mathrm{e}^{-c_4\tau_2})},$$

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}\tau_3}\tilde{R}_0u(\tau_3)=\left(\dfrac{(\bar{R}_0u(\tau_3))^2}{2}-\dfrac{5\bar{R}_0u(\tau_3)}{3}+\dfrac{21}{16}\right)^{1/2}\tilde{R}_0u(\tau_3),\\ \tilde{R}_0u(0)=\dfrac{1}{4},\ \tilde{R}_0(-\infty)=0.\end{cases}\tag{95}$$

$\tilde{R}_0u(\tau_3)$ 满足

$$\frac{\mathrm{d}}{\mathrm{d}\tau_3}\tilde{R}_0u(\tau_3)\leqslant\left(\frac{(\bar{R}_0u(\tau_3))^2}{2}-\frac{5\bar{R}_0u(\tau_1)}{3}+\frac{25}{18}\right)^{1/2}\tilde{R}_0u(\tau_3).$$

由比较原理

$$\tilde{R}_0u(\tau_3)\leqslant-\frac{5\mathrm{e}^{c_5\tau_3}}{17+3\mathrm{e}^{c_5\tau_3}},\qquad c_5=\frac{5\sqrt{2}}{6}.$$

如图 3 所示，在整个区间我们构造了问题(1)的零次近似，表明问题(1)存在连续的且在左区间 $[0,x^0]$ 有转点，右区间 $[x^0,1]$ 有空间对照结构的解。

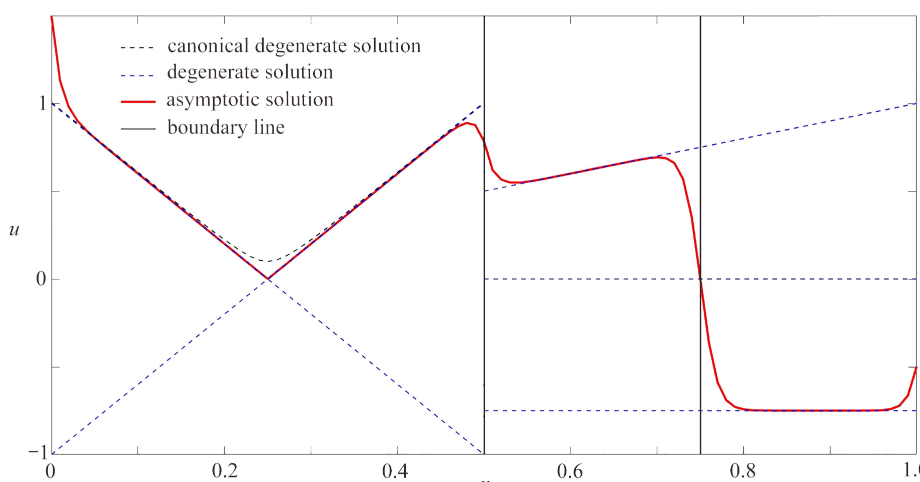

图 3 问题(1)的零阶渐近解 ($\mu=0.01$)

Fig. 3 The zero-order asymptotic solution of problem (1) ($\mu=0.01$)

6 总 结

本文针对具有转点的右端不连续二阶半线性奇摄动 Dirichlet 边值问题(1)，在一定条件下，得到了在间断处左边发生稳定性交替，右边具有空间对照结构的解。即解在边界两端产生边界层，在间断处产生内部层，并且在间断处右边还产生一内部层现象，丰富了具有转点的奇摄动问题与右端不连续奇摄动问题的研究。

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