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非线性时间分布阶双曲波动方程的广义 BDF2-θ 有限元方法*

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摘要： 构造了一种基于带有位移参数 θ 的广义向后差分公式(广义 BDF2-θ)的有限元(FE)方法,用于求解非线性时间分布阶双曲波动方程.时间方向由广义 BDF2-θ 近似进一步得到 FE 全离散格式.将具有高阶时间导数的模型转化为包括两个低阶方程的耦合系统.证明了格式的稳定性以及两个函数 u 和 p 的最优误差估计结果.最后,通过数值算例验证了格式的可行性和有效性.

关　键　词： 非线性时间分布阶双曲波动方程；　有限元方法；　广义 BDF2-θ；　稳定性；　误差估计；　数值模拟

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A Generalized BDF2-θ Finite Element Method for Nonlinear Distributed-Order Time-Fractional Hyperbolic Wave Equations

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Abstract: A finite element (FE) method based on the generalized backward differentiation θ formula (generalized BDF2-θ) was presented to solve nonlinear distributed-order time-fractional hyperbolic wave equations. The temporal direction was approximated with the generalized BDF2-θ to get the FE fully discrete scheme. The proposed model with high-order temporal derivatives was transformed into a coupled system including 2 lower-order equations. The stability of the proposed FE scheme and the optimal error estimates for 2 functions u and p were discussed. Several numerical examples indicate the feasibility and efficiency of the schemes.

Key words: nonlinear distributed-order time-fractional hyperbolic wave equation; FE method; generalized BDF2-θ; stability; error estimate; numerical simulation

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0 引　　言

与传统的整数阶微分方程相比,分数阶微分方程可以更好地描述具有记忆和遗传特性的材料和过程.其中时间分数阶波动方程在许多实际应用中发挥着重要作用.因此,越来越多的学者开始研究这类问题的数值方法和解析方法.数值方法主要包括有限元(FE)法[1-3]、有限差分法[4-12]、谱方法、无网格法[13]、小波法[14]、Euler-Maruyama 方法[15]以及其他方法[16].

然而,分数阶的参数是固定的,具有固定的记忆特性和非局部特征.因此,单一分数阶导数无法充分描述扩散指数随时间变化的异常扩散问题.为了克服这一缺陷,多项分数阶偏微分方程逐渐被发展成为一种有效的替代工具.作为单一分数阶微分方程的进一步改进,多项分数阶偏微分方程仍然具有有限的记忆特性和非局部性的特点.因此,学者们考虑在一定范围内对导数的阶数进行积分,由此引入了分布阶导数.分布阶模型比分数阶模型更适合描述复杂的动力系统.分布阶偏微分方程可视为单项和多项分数阶偏微分方程的推广,可用于描述单项和多项分数阶微分方程无法描绘的过程,如缓速亚扩散和加速超扩散等过程[17-26].目前,分布阶偏微分方程已在许多领域发挥着重要作用,并成为国际学术界的热门研究课题.

分布阶波动方程可以很好地描绘物理学、工程学等多个领域的问题,因此学者们研究了多种求解分布阶波动方程的数值解法,并取得了一些成效.2011 年,Atanackovic 等[27]利用 Laplace 变换方法讨论了分布阶时间分数阶波动方程.2013 年,Gorenflo 等[28]讨论了分布阶时间分数阶波动方程的基本解.2015 年,Ye 等[29]分析了分布阶时间分数阶扩散波动方程的紧差分方法.2016 年,Gao 等[30]建立了两种交替方向隐式差分方法数值求解一类二维时间分布阶波动方程.2018 年,Tomovski 等[31]利用 Fourier-Laplace 变换方法,求解了广义分布阶波动方程;Hendy 等[32]用线性化紧差分方法讨论了一类非线性分布阶扩散波方程;Dehghan 等[33]提出了一种基于谱元法的新数值格式,用于模拟分布阶阻尼扩散波动方程;Li 等[34]介绍并分析了具有 Neumann 边界条件的分布阶时间分数阶扩散波动方程的块中心有限差分法.2020 年,Janno 等[35]考虑了包含 Caputo 时间分数阶导数的扩散和波动方程的两个逆问题.2023 年,Engström 等[36]利用 Laplace 变换讨论了分布阶时间分数阶扩散波方程的数值解法.

广义 BDF2-θ 近似公式是由 Yin 等在文献[37]中提出的,其引入了一个可以变化的参数 θ,是传统 BDF2 公式的扩展.根据上述文献可以看出,关于分布阶时间分数阶波动方程的研究还相对较少.特别地,我们还未看到利用基于广义 BDF2-θ 近似公式的 FE 方法数值求解分布阶时间分数阶双曲波动方程的研究.因此,本文提出了一种利用广义 BDF2-θ 近似公式求解非线性时间分布阶双曲波动方程的 FE 方法,并证明了其稳定性,然后推导了先验误差估计.最后,我们通过数值模拟验证了算法的有效性和计算效率.

本文中,我们考虑如下非线性时间分布阶双曲波动方程:

$$\begin{cases} u_{tt}(x,y,t)+\mathrm{D}_t^{\omega,\beta}u(x,y,t)-\Delta u(x,y,t)+f(u)=g(x,y,t), & (x,y)\in\Omega,\ t\in J,\\ u(x,y,t)=0, & (x,y)\in\partial\Omega,\ t\in J,\\ u(x,y,0)=\phi(x,y), & (x,y)\in\Omega\cup\partial\Omega,\\ u_t(x,y,0)=\varphi(x,y), & (x,y)\in\Omega\cup\partial\Omega, \end{cases} \tag{1}$$

其中 $\Omega=(a,b)\times(c,d)$,边界 $\partial\Omega$ 是 Lipschitz 连续的;$J=(0,T]$ 是时间区间;非线性项 $f(u)$ 满足 $|f(u)-f(v)|\leqslant\mathscr{L}|u-v|$,其中 $\mathscr{L}>0$ 是 Lipschitz 常量;$\phi(x,y)$ 和 $\varphi(x,y)$ 是两个给定的函数.

定义

$$\mathrm{D}_t^{\omega,\beta}u(x,y,t)=\int_{3/2}^{2}\omega(\beta)\,{}_0^{\mathrm{C}}\mathrm{D}_t^{\beta}u(x,y,t)\,\mathrm{d}\beta, \tag{2}$$

其中

$${}_0^{\mathrm{C}}\mathrm{D}_t^{\beta}u(x,y,t)=\frac{1}{\Gamma(2-\beta)}\int_0^t(t-s)^{-\beta+1}\frac{\partial^2u(s)}{\partial s^2}\,\mathrm{d}s,\qquad \frac{3}{2}<\beta<2, \tag{3}$$

同时 $\omega(\beta)\geqslant0,\int_{3/2}^{2}\omega(\beta)\,\mathrm{d}\beta=C_0>0$.

本文的主要结构如下:在第 1 节中,我们给出了预备知识;第 2 节中,我们推导了基于广义 BDF2-θ 公式

的 FE 数值格式；在第 3 节中，我们分析了格式的稳定性；在第 4 节中，我们讨论了两个函数的最优误差估计；在第 5 节中，我们给出了数值算例来验证结果的正确性；第 6 节给出了本文的结论和未来的工作方向。

1　预备知识

引入记号 $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$，其中 N 是正整数，$\tau = T/N$ 是时间步长，且 $t_n = n\tau\,(n = 0,1,2,\cdots,N)$。定义 $\beta_r = r\Delta\beta + \beta_0\,(r = 0,1,2,\cdots,K)$ 属于区间 $[3/2,2]$，其中 $\Delta\beta = 1/(2K)$ 且 $3/2 \leqslant \beta_0 < \beta_1 < \beta_2 < \cdots < \beta_K \leqslant 2$。对于 $[0,T]$ 中的光滑函数 ψ，记 $\psi^n = \psi(t_n)$。

定义 $H^m(\Omega)$，$L^\infty(\Omega)$，$L^2(\Omega)$ 和 $\|\cdot\|_m$，$\|\cdot\|_\infty$，$\|\cdot\|$ 分别是传统的 Sobolev 空间和相应的范数。$L^2(\Omega)$ 下的内积记为 $(\cdot,\cdot)$。对于 $h > 0$，定义 $V_h \subset H_0^1(\Omega)$ 是有限维子空间。为了进一步研究，我们给出如下引理。

引理 1[38]　当 $w(t) \in C^3[0,T]$ 时，$\forall\theta \in [0,1/2]$，在 $t_{n-\theta}$ 处，有

$$\frac{\partial v}{\partial t}(t_{n-\theta}) = \begin{cases} \partial_t[v^{n-\theta}] + E_1^{n-\theta}, & n \geqslant 2, \\ \dfrac{v^1 - v^0}{\tau} + E_1^{1-\theta}, & n = 1, \end{cases} \tag{4}$$

其中

$$\partial_t[v^{n-\theta}] \triangleq \frac{(3-2\theta)v^n - (4-4\theta)\,v^{n-1} + (1-2\theta)v^{n-2}}{2\tau},$$

同时 $E_1^{n-\theta} = O(\tau^2)$，$E_1^{1-\theta} = O(\tau)$。

引理 2[38]　当 $v(t) \in C^2[0,T]$ 时，$\forall\theta \in [0,1/2]$，在 $t_{n-\theta}$ 处，有

$$v(t_{n-\theta}) = (1-\theta)v^n + \theta v^{n-1} + E_2^{n-\theta} \triangleq v^{n-\theta} + E_2^{n-\theta}, \tag{5}$$

且

$$f(v(t_{n-\theta})) = (2-\theta)\,f(v^{n-1}) - (1-\theta)f(v^{n-2}) + E_3^{n-\theta}, \tag{6}$$

其中 $E_2^{n-\theta} = O(\tau^2)$，$E_3^{n-\theta} = O(\tau^2)$。

引理 3[39]　令 $s(\gamma) \in C^2[3/2,2]$，$\gamma_k = (\beta_k + \beta_{k-1})/2$，$k = 1,2,\cdots,K$ 且 $\Delta\gamma = 1/(2K)$，可得到

$$\int_{3/2}^2 s(\gamma)\,\mathrm{d}\gamma = \Delta\gamma\sum_{k=1}^K s(\gamma_k) - \frac{\Delta\gamma^2}{24}s^{(2)}(\zeta), \qquad \zeta \in \left(\frac{3}{2},2\right). \tag{7}$$

引理 4[37]　对于 $1/2 < \alpha < 1$，在 $t_{n-\theta}$ 处，基于广义 BDF2-θ 的 Caputo 型微分算子的近似公式为

$${}_0^{\mathrm{C}}\mathrm{D}_t^\alpha v^{n-\theta} = \tau^{-\alpha}\sum_{i=0}^n \psi_i^{(\alpha)} v^{n-i} + E_4^{n-\theta}, \tag{8}$$

其中 $|E_4^{n-\theta}| \leqslant C\tau^2$。

引理 5[37]　记

$$\Psi_\tau^{\alpha,n} v := \tau^{-\alpha}\sum_{i=0}^n \psi_i^{(\alpha)} v^{n-i}, \tag{9}$$

其中卷积权重 $\{\psi_i^{(\alpha)}\}_{i=0}^\infty$ 是下列生成函数的系数，其关系式为 $\psi^{(\alpha)}(\xi) = \sum_{i=0}^\infty \psi_i^{(\alpha)}\xi^i$，

$$\psi^{(\alpha)}(\xi) = \left(\frac{3\alpha - 2\theta}{2\alpha} - \frac{2\alpha - 2\theta}{\alpha}\xi + \frac{\alpha - 2\theta}{2\alpha}\xi^2\right)^\alpha, \qquad 0 \leqslant \theta \leqslant \min\left\{\alpha,\frac{1}{2}\right\}. \tag{10}$$

引理 6[37]　广义 BDF2-θ 的卷积权重 $\psi_i^{(\alpha)}$ 可以通过以下递推公式得出：

$$\begin{cases} \psi_0^{(\alpha)} = \left(\dfrac{3\alpha - 2\theta}{2\alpha}\right)^\alpha, \\ \psi_1^{(\alpha)} = 2(\theta - \alpha)\left(\dfrac{2\alpha}{3\alpha - 2\theta}\right)^{1-\alpha}, \\ \psi_i^{(\alpha)} = \dfrac{2\alpha}{i(3\alpha - 2\theta)}\left[2(\alpha - \theta)\left(\dfrac{i-1}{\alpha} - 1\right)\psi_{i-1}^{(\alpha)} + (\alpha - 2\theta)\left(1 - \dfrac{i-2}{2\alpha}\right)\psi_{i-2}^{(\alpha)}\right], & i \geqslant 2. \end{cases} \tag{11}$$

2 数值格式

为了得到数值格式,我们引入 $\alpha=\beta-1$ 和 $p=\partial u/\partial t$.因此,原方程(1)可以重新写为如下耦合系统:

$$\begin{cases}\dfrac{\partial u}{\partial t}=p, \\ p_t(x,y,t)+\mathrm{D}_t^{\omega,\beta}u(x,y,t)-\Delta u(x,y,t)+f(u)=g(x,y,t), & (x,y)\in\Omega,\ t\in J, \\ u(x,y,t)=p(x,y,t)=0, & (x,y)\in\partial\Omega,\ t\in J, \\ u(x,y,0)=\phi(x,y), & (x,y)\in\Omega\cup\partial\Omega, \\ u_t(x,y,0)=\varphi(x,y), & (x,y)\in\Omega\cup\partial\Omega,\end{cases}\tag{12}$$

其中 $p=\partial u/\partial t$ 满足边界条件 $p(x,y,t)=0,(x,y)\in\partial\Omega,t\in J$.文献[1,40]也采用了类似的方法.

根据引理 3,我们可以离散分布阶方程中的积分项.假设 $s(\beta)\in C^2[3/2,2]$, 令 $s(\beta)=\omega(\beta)\,{}_0^{\mathrm{C}}\mathrm{D}_t^{\beta}u$, 可以得到

$$\mathrm{D}_t^{\omega,\beta}u=\Delta\beta\sum_{k=1}^{K}\omega(\gamma_k)\,{}_0^{\mathrm{C}}\mathrm{D}_t^{\gamma_k}u+E_5,$$

其中 $E_5=O(\Delta\beta^2)$ 且 $3/2<\beta_k<2$, $\gamma_k=(\beta_k+\beta_{k-1})/2$.

由于

$$\begin{aligned}\mathrm{D}_t^{\omega,\beta}u&=\int_{3/2}^{2}\omega(\beta)\,{}_0^{\mathrm{C}}\mathrm{D}_t^{\beta}u(x,y,t)\mathrm{d}\beta= \\ &\int_{3/2}^{2}\omega(\beta)\frac{1}{\Gamma(2-\beta)}\int_0^t(t-s)^{-\beta+1}\frac{\partial^2u(s)}{\partial s^2}\mathrm{d}s\mathrm{d}\beta= \\ &\int_{3/2}^{2}\omega(\beta)\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}\frac{\partial p(s)}{\partial s}\mathrm{d}s\mathrm{d}\beta= \\ &\int_{3/2}^{2}\omega(\beta)\,{}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha}p(x,y,t)\mathrm{d}\beta,\end{aligned}\tag{13}$$

因此可得

$$\mathrm{D}_t^{\omega,\beta}u=\Delta\beta\sum_{k=1}^{K}\omega(\gamma_k)\,{}_0^{\mathrm{C}}\mathrm{D}_t^{\gamma_k-1}p+E_5,\tag{14}$$

其中 $1/2<\alpha_k<1$.

由引理 1—6,我们可以得到在 $t_{n-\theta}$ 处式(12)的弱格式:

当 $n=1$ 时,

$$\begin{cases}\left(\dfrac{u^1-u^0}{\tau},v\right)=(p^{1-\theta},v)+(E_1^{1-\theta},v), \\ \left(\dfrac{p^1-p^0}{\tau},w\right)+\left(\Delta\beta\sum\limits_{k=1}^{K}\omega(\gamma_k)\Psi_\tau^{\gamma_k-1,1}p,w\right)+(\nabla u^{1-\theta},\nabla w)+(f(u^0),w)= \\ \qquad (g^{1-\theta},w)+\left(\sum\limits_{i=1}^{5}E_i^{1-\theta},w\right).\end{cases}\tag{15}$$

当 $n\geqslant 2$ 时,

$$\begin{cases}(\partial_t[u^{n-\theta}],v)=(p^{n-\theta},v)+(E_1^{n-\theta},v), \\ (\partial_t[p^{n-\theta}],w)+\left(\Delta\beta\sum\limits_{k=1}^{K}\omega(\gamma_k)\Psi_\tau^{\gamma_k-1,n}p,w\right)+(\nabla u^{n-\theta},\nabla w)+ \\ \qquad ((2-\theta)f(u^{n-1})-(1-\theta)f(u^{n-2}),w)=(g^{n-\theta},w)+\left(\sum\limits_{i=1}^{5}E_i^{n-\theta},w\right).\end{cases}\tag{16}$$

寻求 $u_h^n\in V_h^r,p_h^n\in V_h^r$, 可以得到式(15)和(16)的 FE 格式如下:

当 $n=1$ 时,

$$\begin{cases}\left(\dfrac{u_h^1-u_h^0}{\tau},v_h\right)=(p_h^{1-\theta},v_h),\\\left(\dfrac{p_h^1-p_h^0}{\tau},w_h\right)+\left(\Delta\beta\sum_{k=1}^{K}\omega(\gamma_k)\varPsi_\tau^{\gamma_k-1,1}p_h,w_h\right)+\\\qquad(\nabla u_h^{1-\theta},\nabla w_h)+(f(u_h^0),w_h)=(g^{1-\theta},w_h).\end{cases}\tag{17}$$

当 $n\geqslant 2$ 时,

$$\begin{cases}(\partial_t[u_h^{n-\theta}],v_h)=(p_h^{n-\theta},v_h),\\(\partial_t[p_h^{n-\theta}],w_h)+\left(\Delta\beta\sum_{k=1}^{K}\omega(\gamma_k)\varPsi_\tau^{\gamma_k-1,n}p_h,w_h\right)+(\nabla u_h^{n-\theta},\nabla w_h)+\\\qquad((2-\theta)f(u_h^{n-1})-(1-\theta)f(u_h^{n-2}),w_h)=(g^{n-\theta},w_h).\end{cases}\tag{18}$$

下面,我们将分析 FE 系统(17)、(18)的稳定性.

3 稳定性分析

为了进行稳定性分析,我们引入如下两个引理.

引理 7[37] 假设 $\{\psi_i^\alpha\}$, $\alpha\in(0,1)$ 是由式(10)定义的 $\psi^{(\alpha)}(\xi)$ 的系数, θ 满足 $0\leqslant\theta\leqslant\min\{\alpha,1/2\}$, 则有

$$\sum_{m=0}^{n-1}v^m\sum_{k=0}^{m}\psi_{m-k}^{(\alpha)}v^k\geqslant 0,\qquad\forall(v^0,v^1,\cdots,v^{n-1})\in\mathbb{R}^n,\ \forall n\geqslant 1.\tag{19}$$

引理 8[37] 假设 $\theta\leqslant 1/2$ 且 $v^0=0$, 有

$$\sum_{m=1}^{n}v^{m-\theta}v^m\geqslant 0,\qquad\forall(v^1,v^2,\cdots,v^n)\in\mathbb{R}^n,\ \forall n\geqslant 1,\tag{20}$$

其中 $v^{m-\theta}:=(1-\theta)v^m+\theta v^{m-1}$.

引理 9[38] 序列 $\{v^n\}$ $(n\geqslant 2)$ 满足如下不等式:

$$(\partial_t[v^{n-\theta}],v^{n-\theta})\geqslant\frac{1}{4\tau}(\mathscr{G}[v^n]-\mathscr{G}[v^{n-1}]),\tag{21}$$

其中

$$\mathscr{G}[v^n]=(3-2\theta)\|v^n\|^2-(1-2\theta)\|v^{n-1}\|^2+(2-\theta)(1-2\theta)\|v^n-v^{n-1}\|^2,\tag{22}$$

$$\mathscr{G}[v^n]\geqslant\|v^n\|^2.\tag{23}$$

定理 1 对于系统(17)、(18), 有如下稳定性结论:

$$\|p_h^n\|^2+\|\nabla u_h^n\|^2+\|u_h^n\|^2\leqslant C(\|u_h^0\|^2+\|p_h^0\|^2+\|\nabla u_h^0\|^2+\max_{0\leqslant l\leqslant n}\|g^l\|^2).\tag{24}$$

证明 在式(18)中将 n 替换为 m,并且取 $v_h=u_h^{m-\theta}$, $w_h=p_h^{m-\theta}$,可得当 $n\geqslant 2$ 时,

$$(\partial_t[u_h^{m-\theta}],u_h^{m-\theta})=(p_h^{m-\theta},u_h^{m-\theta}),\tag{25a}$$

$$(\partial_t[p_h^{m-\theta}],p_h^{m-\theta})+\left(\Delta\beta\sum_{k=1}^{K}\omega(\gamma_k)\varPsi_\tau^{\gamma_k-1,m}p_h,p_h^{m-\theta}\right)+(\nabla u_h^{m-\theta},\nabla p_h^{m-\theta})+$$
$$((2-\theta)f(u_h^{m-1})-(1-\theta)f(u_h^{m-2}),p_h^{m-\theta})=((1-\theta)g^m+\theta g^{m-1},p_h^{m-\theta}).\tag{25b}$$

应用引理 9,在式(25)两端同时乘以 4τ,并对 m 从 2 到 n 求和,可以推出

$$\|u_h^n\|^2\leqslant\mathscr{G}[u_h^1]+4\tau\sum_{m=2}^{n}(p_h^{m-\theta},u_h^{m-\theta}),\tag{26a}$$

$$\|p_h^n\|^2+\|\nabla u_h^n\|^2+$$
$$4\tau\sum_{m=2}^{n}(1-\theta)\Delta\beta\sum_{k=1}^{K}\omega(\gamma_k)\tau^{-(\gamma_k-1)}\sum_{i=0}^{m}\psi_{m-i}^{\gamma_k-1}(p_h^i,p_h^m)+$$
$$4\tau\sum_{m=2}^{n}\theta\Delta\beta\sum_{k=1}^{K}\omega(\gamma_k)\tau^{-(\gamma_k-1)}\sum_{i=0}^{m}\psi_{m-i}^{\gamma_k-1}(p_h^i,p_h^{m-1})\leqslant$$

$$\mathscr{G}[p_h^1]+\mathscr{G}[\nabla u_h^1]-4\tau\sum_{m=2}^{n}((2-\theta)f(u_h^{m-1})-(1-\theta)f(u_h^{m-2}),p_h^{m-\theta})+$$
$$4\tau\sum_{m=2}^{n}((1-\theta)g^m+\theta g^{m-1},p_h^{m-\theta}). \tag{26b}$$

将式(26a)、(26b)两式相加,并应用 Cauchy-Schwarz 不等式和 Young 不等式,同时根据引理 7 移除式(26b)不等号左端的两个非负项,可以得到

$$\|u_h^n\|^2+\|p_h^n\|^2+\|\nabla u_h^n\|^2\leqslant$$
$$\mathscr{G}[u_h^1]+\mathscr{G}[p_h^1]+\mathscr{G}[\nabla u_h^1]+C\tau\sum_{m=0}^{n}(\|u_h^m\|^2+\|p_h^m\|^2)+C\tau\sum_{m=1}^{n}\|g^m\|^2. \tag{27}$$

下面我们估计 $\mathscr{G}[u_h^1]$,$\mathscr{G}[p_h^1]$,$\mathscr{G}[\nabla u_h^1]$ 这三项.

在式(17)中,令 $v_h=u_h^{1-\theta}$,$w_h=p_h^{1-\theta}$,可得当 $n=1$ 时,

$$\left(\frac{u_h^1-u_h^0}{\tau},u_h^{1-\theta}\right)=(p_h^{1-\theta},u_h^{1-\theta}), \tag{28a}$$

$$\left(\frac{p_h^1-p_h^0}{\tau},p_h^{1-\theta}\right)+\left(\Delta\beta\sum_{k=1}^{K}\omega(\gamma_k)\Psi_\tau^{\gamma_k-1,1}p_h,p_h^{1-\theta}\right)+(\nabla u_h^{1-\theta},\nabla p_h^{1-\theta})+$$
$$(f(u_h^0),p_h^{1-\theta})=((1-\theta)g^1+\theta g^0,p_h^{1-\theta}). \tag{28b}$$

由于

$$\begin{cases}\left(\dfrac{u_h^1-u_h^0}{\tau},(1-\theta)u_h^1+\theta u_h^0\right)=\dfrac{\|u_h^1\|^2-\|u_h^0\|^2}{2\tau}+\dfrac{1-2\theta}{2\tau}\|u_h^1-u_h^0\|^2,\\ \left(\dfrac{p_h^1-p_h^0}{\tau},(1-\theta)p_h^1+\theta p_h^0\right)=\dfrac{\|p_h^1\|^2-\|p_h^0\|^2}{2\tau}+\dfrac{1-2\theta}{2\tau}\|p_h^1-p_h^0\|^2,\end{cases} \tag{29}$$

因此有

$$\frac{\|u_h^1\|^2-\|u_h^0\|^2}{2\tau}+\frac{1-2\theta}{2\tau}\|u_h^1-u_h^0\|^2=(p_h^{1-\theta},u_h^{1-\theta}), \tag{30a}$$

$$\frac{\|p_h^1\|^2-\|p_h^0\|^2}{2\tau}+\frac{1-2\theta}{2\tau}\|p_h^1-p_h^0\|^2+$$
$$\left(\Delta\beta\sum_{k=1}^{K}\omega(\gamma_k)\tau^{-(\gamma_k-1)}\sum_{i=0}^{1}\psi_{1-i}^{\gamma_k-1}p_h^i,(1-\theta)p_h^1+(\theta)p_h^0\right)+$$
$$\frac{\|\nabla u_h^1\|^2-\|\nabla u_h^0\|^2}{2\tau}+\frac{1-2\theta}{2\tau}\|\nabla u_h^1-\nabla u_h^0\|^2=$$
$$-(f(u_h^0),p_h^{1-\theta})+((1-\theta)g^1+\theta g^0,p_h^{1-\theta}). \tag{30b}$$

由于 $1-2\theta\geqslant0$, 在式(30a)、(30b)两端同时乘以 2τ, 并应用 Cauchy-Schwarz 不等式和 Young 不等式,可得

$$\|u_h^1\|^2\leqslant C\|u_h^0\|^2+C\tau(\|p_h^1\|^2+\|p_h^0\|^2)+C\tau\|u_h^1\|^2, \tag{31a}$$

$$\|p_h^1\|^2+\|\nabla u_h^1\|^2\leqslant$$
$$C\tau\|p_h^1\|^2+C\tau\|u_h^0\|^2+C\|p_h^0\|^2+C\|\nabla u_h^0\|^2+C\tau(\|g^1\|^2+\|g^0\|^2). \tag{31b}$$

将式(31a)、(31b)相加,可得

$$\|u_h^1\|^2+\|p_h^1\|^2+\|\nabla u_h^1\|^2\leqslant$$
$$C\|p_h^0\|^2+C\|u_h^0\|^2+C\|\nabla u_h^0\|^2+$$
$$C\tau(\|p_h^1\|^2+\|u_h^1\|^2)+C\tau(\|g^1\|^2+\|g^0\|^2). \tag{32}$$

根据引理 9 和式(32),可得

$$\mathscr{G}[p_h^1]+\mathscr{G}[\nabla u_h^1]+\mathscr{G}[u_h^1]\leqslant$$
$$C(\|u_h^1\|^2+\|p_h^1\|^2+\|\nabla u_h^1\|^2+\|u_h^0\|^2+\|p_h^0\|^2+\|\nabla u_h^0\|^2)\leqslant$$

$$C(\|u_h^0\|^2+\|p_h^0\|^2+\|\nabla u_h^0\|^2)+$$
$$C\tau(\|p_h^1\|^2+\|u_h^1\|^2)+C\tau(\|g^1\|^2+\|g^0\|^2). \tag{33}$$

联立式(27)、(33)并根据 Gronwall 不等式,可以推出

$$\|u_h^n\|^2+\|p_h^n\|^2+\|\nabla u_h^n\|^2\leqslant C\left(\|u_h^0\|^2+\|p_h^0\|^2+\|\nabla u_h^0\|^2+\tau\sum_{m=0}^{n}\|g^m\|^2\right). \tag{34}$$

至此,完成了关于稳定性的证明. □

4 误差估计

引理 10[38] 定义 Ritz 投影算子 $\mathfrak{R}_h: H_0^1(\Omega)\to V_h$, 满足

$$(\nabla(u-\mathfrak{R}_h u),\nabla v_h)=0,\qquad \forall v_h\in V_h,$$

且有如下不等式成立:

$$\|u-\mathfrak{R}_h u\|+h\|u-\mathfrak{R}_h u\|_1\leqslant Ch^{r+1}\|u\|_{r+1},\qquad \forall u\in H_0^1(\Omega)\cap H^{r+1}(\Omega), \tag{35}$$

其中范数定义为 $\|u\|_l=\sqrt{\sum_{0\leqslant|k|\leqslant l}\int_\Omega|\mathrm{D}^k u|^2\mathrm{d}u}$.

为了简化记号,我们引入

$$u(t_n)-u_h^n=(u(t_n)-\mathfrak{R}_h u^n)+(\mathfrak{R}_h u^n-u_h^n)=\xi^n+\eta^n,$$
$$p(t_n)-p_h^n=(p(t_n)-\mathfrak{R}_h p^n)+(\mathfrak{R}_h p^n-p_h^n)=\lambda^n+\varsigma^n.$$

定理 2 设 $u(t_n)$ 和 u_h^n 分别是式(15)、(16)和式(17)、(18)的解.对足够光滑的解 $u\in C^3[0,T]$, $p\in C^3[0,T]$, 可得如下误差估计结果:

$$\|u(t_n)-u_h^n\|^2+\|p(t_n)-p_h^n\|^2\leqslant C(h^{2r+2}+\tau^4+\Delta\beta^4), \tag{36}$$

其中 C 是不依赖于空间步长 h 和时间步长 τ 的正常量.

证明 由式(16)—(18),并将 n 替换为 m,取 $v_h=\eta^{m-\theta}$,$w_h=\varsigma^{m-\theta}$, 可以得到如下误差方程:

$$(\partial_t[\eta^{m-\theta}],\eta^{m-\theta})=-(\partial_t[\xi^{m-\theta}],\eta^{m-\theta})+(\lambda^{m-\theta}+\varsigma^{m-\theta},\eta^{m-\theta})+(E_1^{m-\theta},\eta^{m-\theta}), \tag{37a}$$

$$(\partial_t[\varsigma^{m-\theta}],\varsigma^{m-\theta})+\Delta\beta\sum_{k=1}^{K}\omega(\gamma_k)\Psi_\tau^{\gamma_k-1,m}(\varsigma,\varsigma^{m-\theta})+(\nabla\eta^{m-\theta},\nabla\varsigma^{m-\theta})=$$
$$((2-\theta)(f(u^{m-1})-f(u_h^{m-1}))-(1-\theta)(f(u^{m-2})-f(u_h^{m-2})),\varsigma^{m-\theta})-$$
$$(\partial_t[\lambda^{m-\theta}],\varsigma^{m-\theta})-\Delta\beta\sum_{k=1}^{K}\omega(\gamma_k)\Psi_\tau^{\gamma_k-1,m}(\lambda,\varsigma^{m-\theta})-$$
$$(\nabla\xi^{m-\theta},\nabla\varsigma^{m-\theta})+\left(\sum_{i=1}^{5}E_i^{m-\theta},\varsigma^{m-\theta}\right). \tag{37b}$$

应用引理 9 并在式(37)两端同时乘以 4τ,同时对 m 从 2 到 n 求和,可以推出

$$\|\eta^n\|^2\leqslant\mathscr{G}[\eta^1]+4\tau\sum_{m=2}^{n}(-(\partial_t[\xi^{m-\theta}],\eta^{m-\theta})+(\lambda^{m-\theta}+\varsigma^{m-\theta},\eta^{m-\theta})+(E_1^{m-\theta},\eta^{m-\theta})), \tag{38a}$$

$$\|\varsigma^n\|^2+\|\nabla\eta^n\|^2+4\tau\sum_{m=2}^{n}(1-\theta)\Delta\beta\sum_{k=1}^{K}\omega(\gamma_k)\tau^{-(\gamma_k-1)}\sum_{i=0}^{m}\psi_{m-i}^{\gamma_k-1}(\varsigma^i,\varsigma^m)+$$
$$4\tau\sum_{m=2}^{n}\theta\Delta\beta\sum_{k=1}^{K}\omega(\gamma_k)\tau^{-(\gamma_k-1)}\sum_{i=0}^{m}\psi_{m-i}^{\gamma_k-1}(\varsigma^i,\varsigma^{m-1})\leqslant\mathscr{G}[\varsigma^1]+\mathscr{G}[\nabla\eta^1]+$$
$$4\tau\sum_{m=2}^{n}((2-\theta)(f(u^{m-1})-f(u_h^{m-1}))-(1-\theta)(f(u^{m-2})-f(u_h^{m-2})),\varsigma^{m-\theta})-$$
$$4\tau\sum_{m=2}^{n}(\partial_t[\lambda^{m-\theta}],\varsigma^{m-\theta})-\sum_{m=2}^{n}4\tau\Delta\beta\sum_{k=1}^{K}\omega(\gamma_k)\Psi_\tau^{\gamma_k-1,m}(\lambda,\varsigma^{m-\theta})-$$
$$4\tau\sum_{m=2}^{n}(\nabla\xi^{m-\theta},\nabla\varsigma^{m-\theta})+4\tau\sum_{m=2}^{n}\left(\sum_{i=1}^{5}E_i^{m-\theta},\varsigma^{m-\theta}\right). \tag{38b}$$

将式(38a)、(38b)两式相加,并根据引理 7 移除式(38b)不等号左端的两个非负项,可以得到

$$
\begin{aligned}
&\|\boldsymbol{\eta}^n\|^2+\|\varsigma^n\|^2+\|\nabla\boldsymbol{\eta}^n\|^2\leqslant \\
&\quad \mathscr{G}[\boldsymbol{\eta}^1]+\mathscr{G}[\varsigma^1]+\mathscr{G}[\nabla\boldsymbol{\eta}^1]+ \\
&\quad 4\tau\sum_{m=2}^{n}\left((2-\theta)(f(u^{m-1})-f(u_h^{m-1}))-(1-\theta)(f(u^{m-2})-f(u_h^{m-2})),\varsigma^{m-\theta}\right)+ \\
&\quad 4\tau\sum_{m=2}^{n}\left(\sum_{i=1}^{5}E_i^{m-\theta},\varsigma^{m-\theta}\right)-4\tau\sum_{m=2}^{n}(\partial_t[\lambda^{m-\theta}],\varsigma^{m-\theta})- \\
&\quad 4\tau\sum_{m=2}^{n}\Delta\beta\sum_{k=1}^{K}\omega(\gamma_k)\Psi_\tau^{\gamma_k-1,m}(\lambda,\varsigma^{m-\theta})-4\tau\sum_{m=2}^{n}(\nabla\xi^{m-\theta},\nabla\varsigma^{m-\theta})- \\
&\quad 4\tau\sum_{m=2}^{n}(\partial_t[\xi^{m-\theta}],\boldsymbol{\eta}^{m-\theta})+4\tau\sum_{m=2}^{n}(\lambda^{m-\theta}+\varsigma^{m-\theta},\boldsymbol{\eta}^{m-\theta})+4\tau\sum_{m=2}^{n}(E_1^{m-\theta},\boldsymbol{\eta}^{m-\theta})= \\
&\quad \mathscr{G}[\boldsymbol{\eta}^1]+\mathscr{G}[\varsigma^1]+\mathscr{G}[\nabla\boldsymbol{\eta}^1]+R_1+R_2+R_3+R_4+R_5+R_6+R_7+R_8.
\end{aligned}
\tag{39}
$$

下面我们估计 $R_1+R_2+R_3+R_4+R_5+R_6+R_7+R_8$.

根据三角不等式、Cauchy-Schwarz 不等式和 Young 不等式,可以推出

$$
\begin{aligned}
R_1&=4\tau\sum_{m=2}^{n}\left((2-\theta)(f(u^{m-1})-f(u_h^{m-1}))-(1-\theta)(f(u^{m-2})-f(u_h^{m-2})),\varsigma^{m-\theta}\right)\leqslant \\
&\quad C\tau\sum_{m=0}^{n}(\|\xi^m\|^2+\|\boldsymbol{\eta}^m\|^2+\|\varsigma^m\|^2).
\end{aligned}
\tag{40}
$$

同时,可得

$$
R_2=4\tau\sum_{m=2}^{n}\left(\sum_{i=1}^{5}E_i^{m-\theta},\varsigma^{m-\theta}\right)\leqslant C\tau\sum_{m=1}^{n}(\tau^4+\Delta\beta^4+\|\varsigma^m\|^2).
\tag{41}
$$

根据 Cauchy-Schwarz 不等式和 Young 不等式,可得

$$
\begin{aligned}
R_3&=-4\tau\sum_{m=2}^{n}(\partial_t[\lambda^{m-\theta}],\varsigma^{m-\theta})\leqslant \\
&\quad C\int_{t_0}^{t_n}\|\lambda_t\|^2\mathrm{d}s+C\tau\sum_{m=1}^{n}\|\varsigma^m\|^2.
\end{aligned}
\tag{42}
$$

根据 Cauchy-Schwarz 不等式、Young 不等式以及引理 10,可得

$$
\begin{aligned}
R_4&=-4\tau\sum_{m=2}^{n}\Delta\beta\sum_{k=1}^{K}\omega(\gamma_k)\Psi_\tau^{\gamma_k-1,m}(\lambda,\varsigma^{m-\theta})= \\
&\quad -4\tau\sum_{m=2}^{n}\Delta\beta\sum_{k=1}^{K}\omega(\gamma_k)({}_0^{\mathrm{C}}\mathrm{D}_t^{\gamma_k-1}\lambda^{m-\theta},\varsigma^{m-\theta})+4\tau\sum_{m=2}^{n}\Delta\beta\sum_{k=1}^{K}\omega(\gamma_k)(E_4^{m-\theta},\varsigma^{m-\theta})\leqslant \\
&\quad C\tau\sum_{m=1}^{n}(h^{2r+2}+\tau^4)+C\tau\sum_{m=1}^{n}\|\varsigma^m\|^2.
\end{aligned}
\tag{43}
$$

根据投影的性质,有

$$
R_5=-\sum_{m=2}^{n}4\tau(\nabla\xi^{m-\theta},\nabla\varsigma^{m-\theta})=0.
\tag{44}
$$

采取与计算 R_3 类似的过程,可得

$$
\begin{aligned}
R_6&=-4\tau\sum_{m=2}^{n}(\partial_t[\xi^{m-\theta}],\boldsymbol{\eta}^{m-\theta})\leqslant \\
&\quad C\int_{t_0}^{t_n}\|\xi_t\|^2\mathrm{d}s+C\tau\sum_{m=1}^{n}\|\boldsymbol{\eta}^m\|^2.
\end{aligned}
\tag{45}
$$

根据 Cauchy-Schwarz 不等式和 Young 不等式, 有

$$
\begin{aligned}
R_7&=4\tau\sum_{m=2}^{n}(\lambda^{m-\theta}+\varsigma^{m-\theta},\boldsymbol{\eta}^{m-\theta})\leqslant \\
&\quad C\tau\sum_{m=1}^{n}(\|\lambda^m\|^2+\|\varsigma^m\|^2+\|\boldsymbol{\eta}^m\|^2).
\end{aligned}
\tag{46}
$$

采取与计算 R_2 类似的过程,可得

$$R_8 = 4\tau \sum_{m=2}^{n} (E_1^{m-\theta}, \eta^{m-\theta}) \leqslant$$
$$C\tau \sum_{m=1}^{n} (\tau^4 + \| \eta^m \|^2). \tag{47}$$

将式(40)、(47)代入到式(39)中,可以推出

$$\| \eta^n \|^2 + \| \varsigma^n \|^2 + \| \nabla \eta^n \|^2 \leqslant$$
$$\mathscr{G}[\eta^1] + \mathscr{G}[\varsigma^1] + \mathscr{G}[\nabla \eta^1] + C\tau \sum_{m=0}^{n} (\| \xi^m \|^2 + \| \eta^m \|^2 + \| \lambda^m \|^2 + \| \varsigma^m \|^2) +$$
$$C\tau \sum_{m=1}^{n} (h^{2r+2} + \tau^4 + \Delta\beta^4) + C\int_{t_0}^{t_n} \| \lambda_t \|^2 \mathrm{d}s + C\int_{t_0}^{t_n} \| \xi_t \|^2 \mathrm{d}s. \tag{48}$$

下面,给出 $\mathscr{G}[\eta^1]$,$\mathscr{G}[\varsigma^1]$ 和 $\mathscr{G}[\nabla \eta^1]$ 的估计.

用式(15)减去式(17),并取 $v_h = \eta^{1-\theta}, w_h = \varsigma^{1-\theta}$, 可得

$$\left(\frac{\eta^1 - \eta^0}{\tau}, \eta^{1-\theta}\right) = -\left(\frac{\xi^1 - \xi^0}{\tau}, \eta^{1-\theta}\right) + (\lambda^{1-\theta} + \varsigma^{1-\theta}, \eta^{1-\theta}) + (E_1^{1-\theta}, \eta^{1-\theta}), \tag{49a}$$

$$\left(\frac{\varsigma^1 - \varsigma^0}{\tau}, \varsigma^{1-\theta}\right) + \Delta\beta \sum_{k=1}^{K} \omega(\gamma_k) \Psi_\tau^{\gamma_k - 1, 1}(\varsigma, \varsigma^{1-\theta}) + (\nabla \eta^{1-\theta}, \nabla \varsigma^{1-\theta}) =$$
$$(f(u^0) - f(u_h^0), \varsigma^{1-\theta}) + \left(\sum_{i=1}^{5} E_i^{1-\theta}, \varsigma^{1-\theta}\right) - \left(\frac{\lambda^1 - \lambda^0}{\tau}, \varsigma^{1-\theta}\right) -$$
$$\Delta\beta \sum_{k=1}^{K} \omega(\gamma_k) \Psi_\tau^{\gamma_k - 1, 1}(\lambda, \varsigma^{1-\theta}) - (\nabla \xi^{1-\theta}, \nabla \varsigma^{1-\theta}). \tag{49b}$$

由于

$$\begin{cases} \left(\dfrac{\eta^1 - \eta^0}{\tau}, (1-\theta)\eta^1 + \theta\eta^0\right) = \dfrac{\| \eta^1 \|^2 - \| \eta^0 \|^2}{2\tau} + \dfrac{1-2\theta}{2\tau} \| \eta^1 - \eta^0 \|^2, \\ \left(\dfrac{\varsigma^1 - \varsigma^0}{\tau}, (1-\theta)\varsigma^1 + \theta\varsigma^0\right) = \dfrac{\| \varsigma^1 \|^2 - \| \varsigma^0 \|^2}{2\tau} + \dfrac{1-2\theta}{2\tau} \| \varsigma^1 - \varsigma^0 \|^2, \\ (\nabla \eta^{1-\theta}, \nabla \varsigma^{1-\theta}) = \dfrac{\| \nabla \eta^1 \|^2 - \| \nabla \eta^0 \|^2}{2\tau} + \dfrac{1-2\theta}{2\tau} \| \nabla \eta^1 - \nabla \eta^0 \|^2. \end{cases} \tag{50}$$

因此,可以得到

$$\frac{\| \eta^1 \|^2 - \| \eta^0 \|^2}{2\tau} + \frac{1-2\theta}{2\tau} \| \eta^1 - \eta^0 \|^2 =$$
$$-\left(\frac{\xi^1 - \xi^0}{\tau}, \eta^{1-\theta}\right) + (\lambda^{1-\theta} + \varsigma^{1-\theta}, \eta^{1-\theta}) + (E_1^{1-\theta}, \eta^{1-\theta}), \tag{51a}$$

$$\frac{\| \varsigma^1 \|^2 - \| \varsigma^0 \|^2}{2\tau} + \frac{1-2\theta}{2\tau} \| \varsigma^1 - \varsigma^0 \|^2 + \Delta\beta \sum_{k=1}^{K} \omega(\gamma_k) \Psi_\tau^{\gamma_k - 1, 1}(\varsigma, \varsigma^{1-\theta}) +$$
$$\frac{\| \nabla \eta^1 \|^2 - \| \nabla \eta^0 \|^2}{2\tau} + \frac{1-2\theta}{2\tau} \| \nabla \eta^1 - \nabla \eta^0 \|^2 =$$
$$(f(u^0) - f(u_h^0), \varsigma^{1-\theta}) + \left(\sum_{i=1}^{5} E_i^{1-\theta}, \varsigma^{1-\theta}\right) - \left(\frac{\lambda^1 - \lambda^0}{\tau}, \varsigma^{1-\theta}\right) -$$
$$\Delta\beta \sum_{k=1}^{K} \omega(\gamma_k) \Psi_\tau^{\gamma_k - 1, 1}(\lambda, \varsigma^{1-\theta}) - (\nabla \xi^{1-\theta}, \nabla \varsigma^{1-\theta}). \tag{51b}$$

由于 $1 - 2\theta \geqslant 0$, 在式(51)两端同时乘以 2τ, 并应用 Cauchy-Schwarz 不等式和 Young 不等式,可以推出

$$\| \eta^1 \|^2 \leqslant \| \eta^0 \|^2 - 2\tau \left(\frac{\xi^1 - \xi^0}{\tau}, \eta^{1-\theta}\right) + 2\tau(\lambda^{1-\theta} + \varsigma^{1-\theta}, \eta^{1-\theta}) + 2\tau(E_1^{1-\theta}, \eta^{1-\theta}), \tag{52a}$$

$$\| \varsigma^1 \|^2 + \| \nabla \eta^1 \|^2 \leqslant$$

$$2\tau(f(u^0)-f(u_h^0),\varsigma^{1-\theta})+2\tau\left(\sum_{i=1}^{5}E_i^{1-\theta},\varsigma^{1-\theta}\right)-2\tau\left(\frac{\lambda^1-\lambda^0}{\tau},\varsigma^{1-\theta}\right)-$$
$$2\tau\left(\Delta\beta\sum_{k=1}^{K}\omega(\gamma_k)\tau^{-(\gamma_k-1)}\sum_{i=0}^{1}\psi_{1-i}^{\gamma_k-1}\lambda^i,\varsigma^{1-\theta}\right)-$$
$$2\tau(\nabla\xi^{1-\theta},\nabla\varsigma^{1-\theta})+\|\varsigma^0\|^2+\|\nabla\eta^0\|^2. \tag{52b}$$

将式(52a)、(52b)相加,可得

$$\|\eta^1\|^2+\|\varsigma^1\|^2+\|\nabla\eta^1\|^2\leqslant$$
$$\|\varsigma^0\|^2+\|\eta^0\|^2+\|\nabla\eta^0\|^2+2\tau(f(u^0)-f(u_h^0),\varsigma^{1-\theta})+2\tau\left(\sum_{i=1}^{5}E_i^{1-\theta},\varsigma^{1-\theta}\right)-$$
$$2\tau\left(\frac{\lambda^1-\lambda^0}{\tau},\varsigma^{1-\theta}\right)-2\tau\left(\Delta\beta\sum_{k=1}^{K}\omega(\gamma_k)\tau^{-(\gamma_k-1)}\sum_{i=0}^{1}\psi_{1-i}^{\gamma_k-1}\lambda^i,\varsigma^{1-\theta}\right)-$$
$$2\tau(\nabla\xi^{1-\theta},\nabla\varsigma^{1-\theta})-2\tau\left(\frac{\xi^1-\xi^0}{\tau},\eta^{1-\theta}\right)+2\tau(\lambda^{1-\theta}+\varsigma^{1-\theta},\eta^{1-\theta})+2\tau(E_1^{1-\theta},\eta^{1-\theta})=$$
$$\|\varsigma^0\|^2+\|\eta^0\|^2+\|\nabla\eta^0\|^2+R_{11}+R_{12}+R_{13}+R_{14}+R_{15}+R_{16}+R_{17}+R_{18}. \tag{53}$$

采用计算 $R_1\sim R_8$ 类似的过程,可以得到

$$\|\eta^1\|^2+\|\varsigma^1\|^2+\|\nabla\eta^1\|^2\leqslant$$
$$\|\varsigma^0\|^2+\|\eta^0\|^2+\|\nabla\eta^0\|^2+$$
$$C\tau(\|\xi^0\|^2+\|\lambda^0\|^2+\|\lambda^1\|^2+\|\eta^0\|^2+\|\eta^1\|^2+\|\varsigma^0\|^2+\|\varsigma^1\|^2)+$$
$$C(h^{2r+2}+\tau^4+\Delta\beta^4)+C\|\lambda^1\|^2+C\|\xi^1\|^2. \tag{54}$$

因此,根据引理 9 和式(54),可以推出

$$\mathscr{G}[\eta^1]+\mathscr{G}[\varsigma^1]+\mathscr{G}[\nabla\eta^1]\leqslant$$
$$C(\|\eta^1\|^2+\|\varsigma^1\|^2+\|\nabla\eta^1\|^2+\|\eta^0\|^2+\|\varsigma^0\|^2+\|\nabla\eta^0\|^2)\leqslant$$
$$C(\|\varsigma^0\|^2+\|\eta^0\|^2+\|\nabla\eta^0\|^2)+$$
$$C\tau(\|\xi^0\|^2+\|\lambda^0\|^2+\|\lambda^1\|^2+\|\eta^0\|^2+\|\eta^1\|^2+\|\varsigma^0\|^2+\|\varsigma^1\|^2)+$$
$$C(h^{2r+2}+\tau^4+\Delta\beta^4)+C\|\lambda^1\|^2+C\|\xi^1\|^2. \tag{55}$$

联立式(48)、(55),并应用 Cauchy-Schwarz 不等式、Young 不等式、Gronwall 引理以及引理 10,可以得到

$$\|\eta^n\|^2+\|\varsigma^n\|^2+\|\nabla\eta^n\|^2\leqslant C(h^{2r+2}+\tau^4+\Delta\beta^4). \tag{56}$$

联立式(56)、(35),并使用三角不等式就可以得到定理 2 的结果.至此,我们完成了定理的证明. □

5　数 值 算 例

本节将通过一个数值例子验证理论结果的正确性.

例 1　在时空区域$[0,1]^2\times[0,1/2]$中, 取 $\omega(\beta)=\Gamma(4-\beta)$, 非线性项 $f(u)=\sin(u)$, 源项为

$$g(x,y,t)=\left(6t+\frac{6(t\sqrt{t}-t)}{\ln t}+2\pi^2t^3\right)\sin(\pi x)\sin(\pi y)+\sin(t^3\sin(\pi x)\sin(\pi y)), \tag{57}$$

精确解为

$$u=t^3\sin(\pi x)\sin(\pi y). \tag{58}$$

在表 1 中, 取 $\theta=0.2$, $\Delta\alpha=1/400$, $\Delta t=h_x=h_y=1/8,1/16,1/32,1/64$,并给出了 u 的误差估计结果,收敛阶以及计算时间.在表 2 中, 取 $\theta=0.5$, $\Delta\alpha=1/400$, $\Delta t=h_x=h_y=1/8,1/16,1/32,1/64$,并给出了 u 的误差估计结果,收敛阶以及计算时间.数据结果表明,本文所构造的算法时空收敛阶均接近于二阶,与理论分析结果一致,本算法可以高效地求解非线性时间分布阶双曲波动方程.

在图 1—4 中, 给出了当 $\theta=0.2$, $\Delta\alpha=1/400$, $\Delta t=h_x=h_y=1/8,1/16,1/32,1/64$ 时, 在 $t=0.5$ 处的数值解 u_h 的表面.从图像上可以直观地看出,剖分越细, 数值解的图像越接近于精确解的图像.最后, 为了展示不同网格剖分情况下数值解与精确解之间的误差行为表现, 在图 5—8 中, 给出了当 $\theta=0.2$, $\Delta\alpha=1/400$,

$\Delta t = h_x = h_y = 1/8, 1/16, 1/32, 1/64$ 时，在 $t = 0.5$ 处的误差 $u - u_h$ 的图像.根据图像展示可以看到,该数值方法在求解非线性时间分布阶双曲波动方程时是可行且有效的.

表 1 当 $\theta = 0.2$, $\Delta\alpha = 1/400$, $\Delta t = h_x = h_y = 1/8, 1/16, 1/32, 1/64$ 时的误差和收敛阶

Table 1 The space-time errors and convergence orders with $\theta = 0.2$, $\Delta\alpha = 1/400$, $\Delta t = h_x = h_y = 1/8, 1/16, 1/32, 1/64$

Δt	$h_x = h_y$	$\Vert u - u_h \Vert$	rate	T/s
1/8	1/8	5.3364×10^{-3}	–	0.61
1/16	1/16	1.4775×10^{-3}	1.852 7	2.54
1/32	1/32	3.7845×10^{-4}	1.965 0	15.57
1/64	1/64	9.5188×10^{-5}	1.991 3	134.66

表 2 当 $\theta = 0.5$, $\Delta\alpha = 1/400$, $\Delta t = h_x = h_y = 1/8, 1/16, 1/32, 1/64$ 时的误差和收敛阶

Table 2 The space-time errors and convergence orders with $\theta = 0.5$, $\Delta\alpha = 1/400$, $\Delta t = h_x = h_y = 1/8, 1/16, 1/32, 1/64$

Δt	$h_x = h_y$	$\Vert u - u_h \Vert$	rate	T/s
1/8	1/8	1.2946×10^{-3}	–	1.92
1/16	1/16	3.3269×10^{-4}	1.960 3	2.57
1/32	1/32	8.4615×10^{-5}	1.975 2	16.45
1/64	1/64	2.1377×10^{-5}	1.984 9	130.07

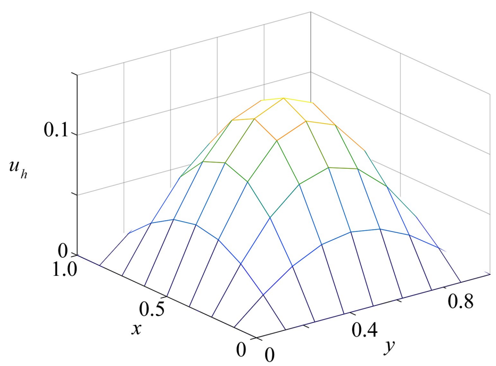

图 1 当 $\theta = 0.2$, $\Delta\alpha = 1/400$, $\Delta t = h_x = h_y = 1/8$ 时，在 $t = 0.5$ 处的数值解 u_h

Fig. 1 The numerical solution u_h at $t = 0.5$ with $\theta = 0.2$, $\Delta\alpha = 1/400$, $\Delta t = h_x = h_y = 1/8$

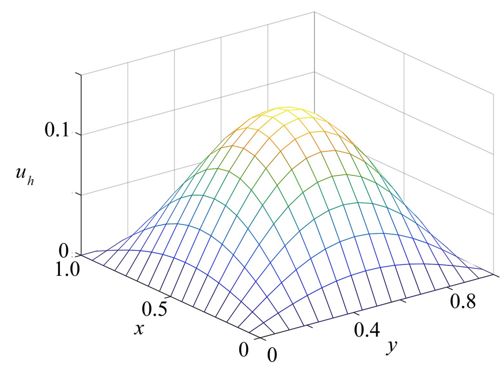

图 2 当 $\theta = 0.2$, $\Delta\alpha = 1/400$, $\Delta t = h_x = h_y = 1/16$ 时，在 $t = 0.5$ 处的数值解 u_h

Fig. 2 The numerical solution u_h at $t = 0.5$ with $\theta = 0.2$, $\Delta\alpha = 1/400$, $\Delta t = h_x = h_y = 1/16$

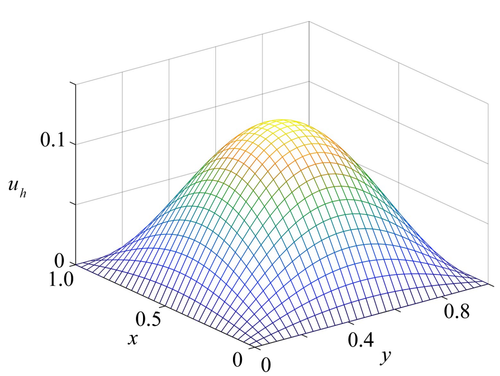

图 3 当 $\theta = 0.2$, $\Delta\alpha = 1/400$, $\Delta t = h_x = h_y = 1/32$ 时，在 $t = 0.5$ 处的数值解 u_h

Fig. 3 The numerical solution u_h at $t = 0.5$ with $\theta = 0.2$, $\Delta\alpha = 1/400$, $\Delta t = h_x = h_y = 1/32$

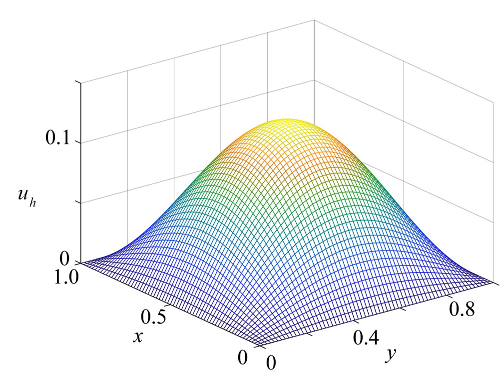

图 4 当 $\theta = 0.2$, $\Delta\alpha = 1/400$, $\Delta t = h_x = h_y = 1/64$ 时，在 $t = 0.5$ 处的数值解 u_h

Fig. 4 The numerical solution u_h at $t = 0.5$ with $\theta = 0.2$, $\Delta\alpha = 1/400$, $\Delta t = h_x = h_y = 1/64$

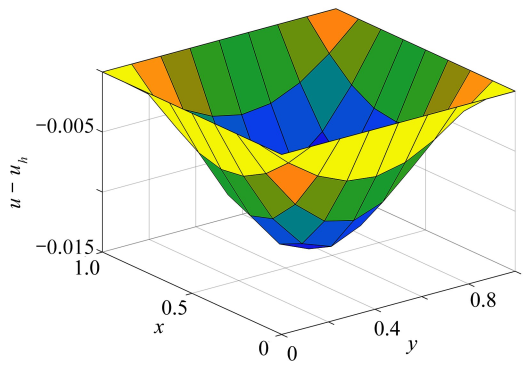

图 5　当 $\theta=0.2$，$\Delta\alpha=1/400$，$\Delta t=h_x=h_y=1/8$ 时，在 $t=0.5$ 处的误差 $u-u_h$

Fig. 5　The error $u-u_h$ at $t=0.5$ with $\theta=0.2$, $\Delta\alpha=1/400$, $\Delta t=h_x=h_y=1/8$

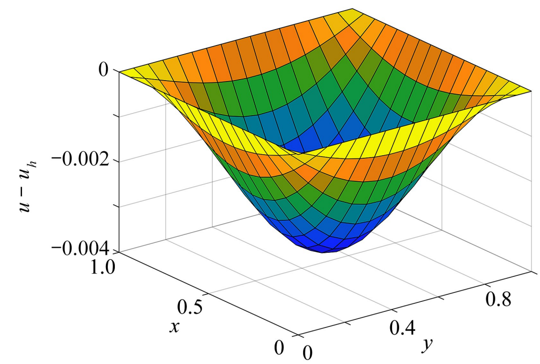

图 6　当 $\theta=0.2$，$\Delta\alpha=1/400$，$\Delta t=h_x=h_y=1/16$ 时，在 $t=0.5$ 处的误差 $u-u_h$

Fig. 6　The error $u-u_h$ at $t=0.5$ with $\theta=0.2$, $\Delta\alpha=1/400$, $\Delta t=h_x=h_y=1/16$

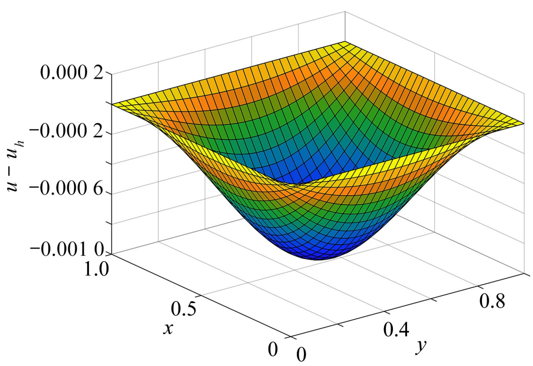

图 7　当 $\theta=0.2$，$\Delta\alpha=1/400$，$\Delta t=h_x=h_y=1/32$ 时，在 $t=0.5$ 处的误差 $u-u_h$

Fig. 7　The error $u-u_h$ at $t=0.5$ with $\theta=0.2$, $\Delta\alpha=1/400$, $\Delta t=h_x=h_y=1/32$

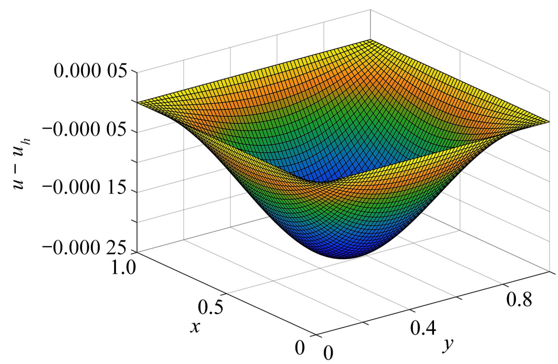

图 8　当 $\theta=0.2$，$\Delta\alpha=1/400$，$\Delta t=h_x=h_y=1/64$ 时，在 $t=0.5$ 处的误差 $u-u_h$

Fig. 8　The error $u-u_h$ at $t=0.5$ with $\theta=0.2$, $\Delta\alpha=1/400$, $\Delta t=h_x=h_y=1/64$

注 1　在初始时刻，分数阶导数具有弱奇异性，当解具有较低的正则性时，通常会导致收敛阶掉阶．为了处理这个问题，一些学者提出了分层网格方法[41-43]、校正方法[44-45]等技术．在未来的研究中，我们可采用这些相关方法来处理当前研究的问题．

6　结　　论

本文构造了一种基于广义 BDF2-θ 的 FE 方法，该方法可以很好地数值求解非线性时间分布阶双曲波动方程．本文详细证明了稳定性，并给出了两个函数 u 和 p 的最优误差估计结果．通过实际的算例也可以看到，所构造的算法在时间和空间方向均达到了二阶收敛精度．

未来，我们将把所讨论的方法应用于其他分布阶偏微分方程，例如非线性空间分布阶偏微分方程、非线性时空分布阶偏微分方程等．此外我们还可以结合其他数值方法（如 ADI FE 方法、谱方法等）去构造一些新的二阶方法．

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