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细长杆的 Cosserat 动力学模型和变分原理*

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摘要: 利用 Cosserat 理论建立了细长杆的三维非线性动力学模型. 借助伪刚体法和变分原理得到了 Cosserat 杆的包括各种形变的三维空间运动方程.

关 键 词: 细长杆; 变分原理; Cosserat 理论
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引 言

三维空间的一根弹性的细长杆可以看成由一条空间曲线以及一些小的截面组成. 细长杆被广泛研究是因为它可以用于模拟很多实际问题, 譬如微电子系统的部件、活动机械臂甚至是 DNA 链等. 这方面早期的工作可以追溯到 Kirchhoff 和 Euler 关于弹性细长杆的形变在平衡状态下的分析[1]. Faulkner 和 Steigmann[2-3] 得到了弹性细长杆在三维空间的形变的传统的 Kirchhoff-Clebsch 理论的改进形式. 传统的有限元方法提供了分析弹性细长杆的一般方法. 均匀形变的细长杆可以看作伪刚体, 关于细长杆的变分原理由 Cohen 和 Muncaster 做了综述[4]. Papadopoulos[5] 建立了高维伪刚体模型, 它可以用于分析形变梯度是线性变化的情形.

在 Cosserat 理论中, 细长杆结构在三维空间的运动可由 1 条基准曲线和 3 条相互垂直的单位向量的行为来描述. 这条基准曲线叫做 Cosserat 曲线, 3 条相互垂直的单位向量称为方向. Cosserat 连续体和传统连续体的主要区别是传统连续体的每一个物质点只有位移向量场而在 Cosserat 连续体中每一个物质点同时还有转动向量场. 因此 Cosserat 细长杆的控制方程包括 3 个转动自由度和 3 个通常的位移自由度. 除了有转动自由度, Cosserat 细长杆还考虑了传统应力的耦合以及在本构模型中引进了内在长度变量.

许多学者对 Cosserat 理论中细长杆的位形分析进行了研究. Green 等[6-7] 发展了一套包括许多位形形变的细长杆的一般理论. Naghdi 和 Rubin[8] 讨论了对这些位形附加了限制条件的情形. Rubin[9-10] 研究了 Cosserat 理论下的一般的 Bernoulli-Euler 杆. 最近 Zhang 等[11-12] 借助参数变分原理发展了一种对 Cosserat 连续体的弹塑性分析的算法. Neff[13], Sansour 和 Skatul-

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la[14-15]在这方面也做了一些很好的工作. Cao 等[16]和本文作者[17]建立了 Cosserat 杆元法和改进的 Cosserat 杆元法, 可以有效地模拟一些动力系统中的细长结构.

依赖于伪刚体法和 Cosserat 理论我们将建立一种杆的模拟的技巧方法. 我们假设细长杆的截面只改变位置和方向而不改变形状, 因此我们容易写出它的 Lagrange 方程. 细长杆的动力学方程由熟知的作用原理而得到. 这样得到的细长杆的动力学方程是可靠的, 因为作用原理保证了基本的守恒定律.

本文的目的是应用 Cosserat 理论得出细长杆的动力学方程. 这个模型的最大优势是它能够有效的模拟弹性细长杆在各种形变下的线性和非线性行为, 其形变可以是弯曲、伸长、扭曲、剪切等. 在对系统的多个形变了的部件合成时有简单的数学表示形式并且不需要讨论其收敛性. 在模拟含有细长结构的部件的复杂的多体系统时, 我们只需将细长结构的部件分成少数几个 Cosserat 杆元, 有时甚至一个 Cosserat 杆元即可. 利用 Cosserat 杆单元及其合成在模拟有较大形变的非线性动力系统时更为有效且只需较少的自由度. 应用举例可见文献[17-18].

本文中向量$\left\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\right\}$组成固定的正交基. 求和符号有时被省略.

1 预备知识

本节将对建立 Cosserat 杆元模型的方法技巧做一个概述. 在初始坐标系$(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$中取直角坐标$(x, y, z)$和Newton时间$t$. 一小段杆可以用Cosserat杆元来模拟, 它的运动状况由中心轴线$\boldsymbol{r}(s, t)$(Cosserat曲线)和3条相互垂直的向量$\boldsymbol{d}_i(s, t)$, $(i = 1, 2, 3)$ (Cosserat方向)来刻画. 这里s和t分别表示长度参数和时间. 我们假定在任何时间$\partial_s \boldsymbol{r} \cdot \boldsymbol{d}_3 > 0$, ∂_s表示对长度参数s求微分. 由假设条件可知: (ⅰ) 因为$|\partial_s \boldsymbol{r}| > 0$, 中心轴线局部形变后的长度与原长度的比率不会为0; (ⅱ) 杆的截面$(s = s_0)$的形变不会退化为完全剪切, 详细说明可见文献[19].

在初始坐标系$(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$中记

$$\boldsymbol{r}(s, t) = r_i(s, t)\boldsymbol{e}_i = x(s, t)\boldsymbol{e}_1 + y(s, t)\boldsymbol{e}_2 + z(s, t)\boldsymbol{e}_3. \tag{1}$$

设$\boldsymbol{d}_i(s, t) = d_{ij}(s, t)\boldsymbol{e}_j$满足正交条件. Cosserat 杆的运动涉及到Cosserat 曲线的速度$\partial_t \boldsymbol{r}$和截面的角速度$\boldsymbol{w} = w_i \boldsymbol{d}_i$满足$\partial_t \boldsymbol{d}_i = \boldsymbol{w} \times \boldsymbol{d}_i$. 同样地, Cosserat 杆的应变可以分成"线性应变"向量$\boldsymbol{v} = \partial_s \boldsymbol{r} = v_i \boldsymbol{d}_i$和"角形应变"向量$\boldsymbol{u} = u_i \boldsymbol{d}_i$满足$\partial_s \boldsymbol{d}_i = \boldsymbol{u} \times \boldsymbol{d}_i$.

类似于刚体动力系统, 杆的单位长度的动能可以写成:

$$\mathcal{T} = \frac{1}{2}\left\{\mu \partial_t \boldsymbol{r} \cdot \partial_t \boldsymbol{r} + \boldsymbol{I}(\boldsymbol{w}, \boldsymbol{w})\right\}, \tag{2}$$

这里, $\mu = \rho A$, ρ是杆的密度, A是截面的面积. $\boldsymbol{I} = I_{ij}\boldsymbol{d}_i \otimes \boldsymbol{d}_j$表示杆的单位长度的惯性矩张量.

当应变较小时, 杆的单位长度的应变能可以通过应变向量$\boldsymbol{u}$和$\boldsymbol{v}$来表示:

$$\mathcal{U} = \frac{1}{2}\left\{\boldsymbol{J}(\boldsymbol{u} - \hat{\boldsymbol{u}}, \boldsymbol{u} - \hat{\boldsymbol{u}}) + \boldsymbol{K}(\boldsymbol{v} - \hat{\boldsymbol{v}}, \boldsymbol{v} - \hat{\boldsymbol{v}})\right\}, \tag{3}$$

这里, $\hat{\boldsymbol{v}}$和$\hat{\boldsymbol{u}}$是杆在未发生形变时的应变向量, $\boldsymbol{J} = J_{ij}\boldsymbol{d}_i \otimes \boldsymbol{d}_j$和$\boldsymbol{K} = K_{ij}\boldsymbol{d}_i \otimes \boldsymbol{d}_j$是对称张量, 它们分别决定杆的位移和转动形变的刚度.

上述动能和应变能的表示可用于定义 Lagrange 密度:

$$\mathcal{L} = \mathcal{T} - \mathcal{U}. \tag{4}$$

从而通过变分原理得到 Cosserat 杆的动力学模型, 它的作用泛函是

$$\mathcal{S} = \int \mathcal{L} \, \mathrm{d}s \, \mathrm{d}t . \tag{5}$$

2 角变分微商

设

$$\boldsymbol{d}_i(s, t) = d_{ij}(s, t) \boldsymbol{e}_j . \tag{6}$$

由方向向量的正交性得到

$$d_{ik} d_{jk} = \delta_{ij}, \quad d_{ki} d_{kj} = \delta_{ij}, \tag{7}$$

这里 δ_{ij} 是 Kronecker 数.

记角速度 $\boldsymbol{w}$, 应变向量 $\boldsymbol{u}$ 和 $\boldsymbol{v}$ 在参考坐标系 $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ 和运动坐标系 $\{\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3\}$ 中有如下的形式:

$$\boldsymbol{w} = w_i \boldsymbol{e}_i = w_i^{\mathrm{d}} \boldsymbol{d}_i, \tag{8}$$

$$\boldsymbol{u} = u_i \boldsymbol{e}_i = u_i^{\mathrm{d}} \boldsymbol{d}_i, \tag{9}$$

$$\boldsymbol{v} = v_i \boldsymbol{e}_i = v_i^{\mathrm{d}} \boldsymbol{d}_i . \tag{10}$$

由方向向量的正交性及上述记号我们得到

$$w_i = \frac{1}{2} \epsilon_{ijk} d_{lj} \partial_t d_{lk}, \tag{11}$$

$$w_i^{\mathrm{d}} = -\frac{1}{2} \epsilon_{ijk} d_{jl} \partial_t d_{kl}, \tag{12}$$

$$u_i = \frac{1}{2} \epsilon_{ijk} d_{lj} \partial_s d_{lk}, \tag{13}$$

$$u_i^{\mathrm{d}} = -\frac{1}{2} \epsilon_{ijk} d_{jl} \partial_s d_{kl}, \tag{14}$$

$$v_i = \partial_s r_i, \tag{15}$$

$$v_i^{\mathrm{d}} = \partial_s r_j d_{ij}, \tag{16}$$

这里 ϵ_{ijk} 满足 $\epsilon_{ijk} = (\boldsymbol{e}_i \times \boldsymbol{e}_j) \cdot \boldsymbol{e}_k$.

我们用“•”表示变分微分. 由方向向量的正交性(7) 式, 有

$$\dot{d}_{ia} = -d_{ja} d_{ib} \dot{d}_{jb}, \tag{17}$$

令

$$\boldsymbol{D} = (\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3)^{\mathrm{T}} . \tag{18}$$

显然 $\boldsymbol{D} \in \mathrm{SO}(3)$. SO(3) 是三维欧氏空间的旋转李群[20]. 并且

$$\dot{\boldsymbol{D}} = (\dot{\boldsymbol{d}}_1, \dot{\boldsymbol{d}}_2, \dot{\boldsymbol{d}}_3)^{\mathrm{T}} . \tag{19}$$

为简单起见, 我们引进“角变分微商” Ω:

$$\dot{\boldsymbol{D}} = \dot{\boldsymbol{D}} \times \boldsymbol{D}^{\mathrm{T}} \times \boldsymbol{D} := \Omega \times \boldsymbol{D} \tag{20}$$

其中 $\Omega = \dot{\boldsymbol{D}} \times \boldsymbol{D}^{\mathrm{T}}$. 易见 $\Omega \in \mathrm{so}(3)$, so(3) 是和李群 SO(3) 相对应的李代数.

由(19) 式和(20) 式, 我们有

$$\dot{\boldsymbol{d}}_i = \Omega \times \boldsymbol{d}_i, \qquad i = 1, 2, 3, \tag{21}$$

其分量为

$$\Omega_i^{\mathrm{d}} = -\frac{1}{2} \epsilon_{ijk} d_{jl} \dot{d}_{kl} . \tag{22}$$

借助于(12) 式、(14) 式和(16) 式得到

$$\dot{w}_i^d = -\frac{1}{2}\epsilon_{ijk}(\dot{d}_{jl}\partial_t d_{kl} + d_{jl}\partial_t \dot{d}_{kl}) = \partial_t \Omega_i^d + \epsilon_{ijk} w_j^d \Omega_k^d, \tag{23}$$

$$\dot{u}_i^d = -\frac{1}{2}\epsilon_{ijk}(\dot{d}_{jl}\partial_s d_{kl} + d_{jl}\partial_s \dot{d}_{kl}) = \partial_s \Omega_i^d + \epsilon_{ijk} u_j^d \Omega_k^d, \tag{24}$$

$$\dot{v}_i^d = \partial_s \dot{\boldsymbol{r}} \cdot \boldsymbol{d}_i - \Omega \cdot (\partial_s \boldsymbol{r} \times \boldsymbol{d}_i). \tag{25}$$

3 有剪切形变的 Cosserat 杆

我们知道 $\boldsymbol{r}(s, t)$ 是一个机械系统的运动形式, 充要条件是它的作用泛函的变分为 0, 即 $\dot{\mathcal{S}} = 0$. 下面我们将直接计算 Cosserat 杆的作用泛函 $\mathcal{S}$ 的变分.

每一单位薄片的动能可以写成下面的形式:

$$\mathcal{T} = \frac{1}{2}\left\{\mu\partial_t \boldsymbol{r}\cdot\partial_t \boldsymbol{r} + \boldsymbol{I}(\boldsymbol{w}, \boldsymbol{w})\right\} = \frac{1}{2}\left\{\mu\partial_t r_i \partial_t r_i + I_{ij}^d w_i^d w_j^d\right\}, \tag{26}$$

其中 $\mu = \rho A$, A 是截面面积.

每一单位薄片的应变能可以写成应变向量 $\boldsymbol{u}$ 和 $\boldsymbol{v}$ 的二次型:

$$\mathcal{U} = \frac{1}{2}\left\{\boldsymbol{J}(\boldsymbol{u} - \hat{\boldsymbol{u}}, \boldsymbol{u} - \hat{\boldsymbol{u}}) + \boldsymbol{K}(\boldsymbol{v} - \hat{\boldsymbol{v}}, \boldsymbol{v} - \hat{\boldsymbol{v}})\right\} = \\ \frac{1}{2}\left\{J_{ij}^d(u_i^d - \hat{u}_i^d)(u_j^d - \hat{u}_j^d) + K_{ij}^d(v_i^d - \hat{v}_i^d)(v_j^d - \hat{v}_j^d)\right\} \tag{27}$$

和

$$\mathcal{U} := \mathcal{U}_1 + \mathcal{U}_2. \tag{28}$$

为方便起见, 我们假设初始杆没有变形. 因此 I_{ij}, J_{ij} 和 K_{ij} 均为常数. 由(12) 式和(23) 式得到同余全微分:

$$\dot{\mathcal{T}} = \mu\partial_t \dot{r}_i \cdot \partial_t r_i + (\partial_t \Omega_i^d + \epsilon_{ikl} w_k^d \Omega_l^d) I_{ij}^d w_j^d = \\ -\dot{r}_i \cdot \mu\partial_{tt} r_i - \Omega_1^d[\partial_t(I_{1j}^d w_j^d) - w_3^d I_{2j}^d w_j^d + w_2^d I_{3j}^d w_j^d] - \\ \Omega_2^d[\partial_t(I_{2j}^d w_j^d) + w_3^d I_{1j}^d w_j^d - w_1^d I_{3j}^d w_j^d] - \\ \Omega_3^d[\partial_t(I_{3j}^d w_j^d) - w_2^d I_{1j}^d w_j^d + w_1^d I_{2j}^d w_j^d]. \tag{29}$$

进一步地, 由(24) 式和(25) 式得到同余全微分:

$$\dot{\mathcal{U}}_1 = \dot{u}_i^d J_{ij}^d(u_j^d - \hat{u}_j^d) = (\partial_s \Omega_i^d + \epsilon_{ikl} u_k^d \Omega_l^d) J_{ij}^d(u_j^d - \hat{u}_j^d) = \\ -\Omega_1^d[\partial_s(J_{1j}^d(u_j^d - \hat{u}_j^d)) - u_3^d J_{2j}^d(u_j^d - \hat{u}_j^d) + u_2^d J_{3j}^d(u_j^d - \hat{u}_j^d)] - \\ \Omega_2^d[\partial_s(J_{2j}^d(u_j^d - \hat{u}_j^d)) + u_3^d J_{1j}^d(u_j^d - \hat{u}_j^d) - u_1^d J_{3j}^d(u_j^d - \hat{u}_j^d)] - \\ \Omega_3^d[\partial_s(J_{3j}^d(u_j^d - \hat{u}_j^d)) - u_2^d J_{1j}^d(u_j^d - \hat{u}_j^d) + u_1^d J_{2j}^d(u_j^d - \hat{u}_j^d)] \tag{30}$$

和

$$\dot{\mathcal{U}}_2 = \dot{v}_i^d K_{ij}^d(v_j^d - \hat{v}_j^d) = \\ [\partial_s \dot{\boldsymbol{r}} \cdot \boldsymbol{d}_i - \Omega \cdot (\partial_s \boldsymbol{r} \times \boldsymbol{d}_i)] K_{ij}^d(v_j^d - \hat{v}_j^d) = \\ -\dot{\boldsymbol{r}} \cdot \partial_s[K_{ij}^d(v_j^d - \hat{v}_j^d)\boldsymbol{d}_i] - \Omega \cdot [\partial_s \boldsymbol{r} \times K_{ij}^d(v_j^d - \hat{v}_j^d)\boldsymbol{d}_i]. \tag{31}$$

令

$$\boldsymbol{K}(\boldsymbol{v} - \hat{\boldsymbol{v}}) = \sum_{i=1}^{3}\left(\sum_{j=1}^{3} K_{ij}^d(v_j^d - \hat{v}_j^d)\right)\boldsymbol{d}_i, \tag{32}$$

$$\boldsymbol{I}(\boldsymbol{w}) = \sum_{i=1}^{3}\left(\sum_{j=1}^{3} I_{ij}^d w_j^d\right)\boldsymbol{d}_i := \sum_{i=1}^{3} h_i \boldsymbol{d}_i, \tag{33}$$

$$\boldsymbol{J}(\boldsymbol{u}-\hat{\boldsymbol{u}}) = \sum_{i=1}^{3}\left(\sum_{j=1}^{3} J_{ij}^{d}(u_j^{d}-\hat{u}_j^{d})\right)\boldsymbol{d}_i := \sum_{i=1}^{3} m_i\boldsymbol{d}_i. \tag{34}$$

则同余全微分为

$$\begin{aligned}\dot{\mathcal{L}} = & -\dot{\boldsymbol{r}}\cdot\mu\partial_{tt}\boldsymbol{r} - \Omega\cdot\left\{\partial_t(I_{ij}^{d}w_j^{d})\boldsymbol{d}_i + \boldsymbol{w}\times I_{ij}^{d}w_j^{d}\boldsymbol{d}_i\right\} + \dot{\boldsymbol{r}}\cdot\partial_s(K_{ij}^{d}(v_j^{d}-\hat{v}_j^{d})\boldsymbol{d}_i) + \\ & \Omega\cdot\left\{\partial_s(J_{ij}^{d}(u_j^{d}-\hat{u}_j^{d}))\boldsymbol{d}_i + \boldsymbol{u}\times J_{ij}^{d}(u_j^{d}-\hat{u}_j^{d})\boldsymbol{d}_i + \partial_s\boldsymbol{r}\times K_{ij}^{d}(v_j^{d}-\hat{v}_j^{d})\boldsymbol{d}_i\right\} = \\ & -\dot{\boldsymbol{r}}\cdot[\mu\partial_{tt}\boldsymbol{r} - \partial_s(K_{ij}^{d}(v_j^{d}-\hat{v}_j^{d})\boldsymbol{d}_i))] - \\ & \Omega\cdot\left\{\partial_t[(I_{ij}^{d}w_j^{d})\boldsymbol{d}_i] - [\partial_s((J_{ij}^{d}(u_j^{d}-\hat{u}_j^{d}))\boldsymbol{d}_i) + \partial_s\boldsymbol{r}\times K_{ij}^{d}(v_j^{d}-\hat{v}_j^{d})\boldsymbol{d}_i]\right\} = \\ & -\dot{\boldsymbol{r}}\cdot[\mu\partial_{tt}\boldsymbol{r} - \partial_s\boldsymbol{K}(\boldsymbol{v}-\hat{\boldsymbol{v}})] - \\ & \Omega\cdot[\partial_t\boldsymbol{I}(\boldsymbol{w}) - \partial_s\boldsymbol{J}(\boldsymbol{u}-\hat{\boldsymbol{u}}) - \partial_s\boldsymbol{r}\times\boldsymbol{K}(\boldsymbol{v}-\hat{\boldsymbol{v}})].\end{aligned} \tag{35}$$

由于 $\dot{\boldsymbol{r}}$和 Ω 相互独立, $\dot{\mathcal{S}} = 0$ 蕴含着下面的式子:

$$\mu\partial_{tt}\boldsymbol{r} = \partial_s\boldsymbol{n}, \tag{36}$$

$$\partial_t(\boldsymbol{h}) = \partial_s\boldsymbol{m} + \partial_s\boldsymbol{r}\times\boldsymbol{n}, \tag{37}$$

其中

$$\boldsymbol{h} = \boldsymbol{I}(\boldsymbol{w}). \tag{38}$$

而

$$\boldsymbol{n} = \boldsymbol{K}(\boldsymbol{v}-\hat{\boldsymbol{v}}), \tag{39}$$

$$\boldsymbol{m} = \boldsymbol{J}(\boldsymbol{u}-\hat{\boldsymbol{u}}) \tag{40}$$

则分别是接触力和力矩.

若设外力和外力矩分别是$\boldsymbol{f}(s,t)$ 和 $\boldsymbol{l}(s,t)$. 则 Cosserat 杆的动力学方程可以写成下面的形式.

$$\mu\partial_{tt}\boldsymbol{r} = \partial_s\boldsymbol{n} + \boldsymbol{f}, \tag{41}$$

$$\partial_t(\boldsymbol{h}) = \partial_s\boldsymbol{m} + \partial_s\boldsymbol{r}\times\boldsymbol{n} + \boldsymbol{l}. \tag{42}$$

这和 Newton 法则得到的结果一致.

4 没有剪切形变的 Cosserat 杆

剪切形变可以忽略的细长结构, 可以用抑制了剪切形变的 Cosserat 杆来模拟. 这相当于附加限制条件

$$\boldsymbol{d}_3 = \frac{\boldsymbol{v}}{v}, \tag{43}$$

这里 $v = |\boldsymbol{v}|$.

这时应变能密度表达式(3) 可以简化成

$$\mathcal{U} = \frac{1}{2}\left\{\boldsymbol{J}(\boldsymbol{u}-\hat{\boldsymbol{u}}, \boldsymbol{u}-\hat{\boldsymbol{u}}) + K_{33}(v-1)^2\right\}. \tag{44}$$

然而由于此时 $\dot{\boldsymbol{r}}$和 Ω 不再相互独立, 我们不能直接得到像(36) 式和(37) 式那样的动力学方程.

由(43) 式, 我们有

$$\dot{\boldsymbol{d}}_3 = \frac{\dot{\boldsymbol{v}}}{|\boldsymbol{v}|} - \frac{\dot{\boldsymbol{v}}\cdot\boldsymbol{v}}{|\boldsymbol{v}|^3}\boldsymbol{v}. \tag{45}$$

记

$$\Omega = \Omega_{\perp} + \Omega_3, \tag{46}$$

这里

$$\Omega_3 = (\Omega \cdot \boldsymbol{d}_3)\boldsymbol{d}_3 = \Omega_3^{d}\boldsymbol{d}_3, \tag{47}$$

$$\Omega_\perp = \Omega - \Omega_3 = \boldsymbol{d}_3 \times \boldsymbol{d}_3' = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} \times \frac{\boldsymbol{v}'}{|\boldsymbol{v}|} = \frac{\boldsymbol{v} \times \boldsymbol{v}'}{|\boldsymbol{v}|^2} =$$
$$\left[\frac{\boldsymbol{v} \times \boldsymbol{v}'}{|\boldsymbol{v}|^2} \cdot \boldsymbol{d}_\alpha\right]\boldsymbol{d}_\alpha = \Omega_1^{d}\boldsymbol{d}_1 + \Omega_2^{d}\boldsymbol{d}_2. \tag{48}$$

此时

$$\mathcal{U} = \frac{1}{2}\left\{\boldsymbol{J}(\boldsymbol{u} - \hat{\boldsymbol{u}}, \boldsymbol{u} - \hat{\boldsymbol{u}}) + K_{33}(|\boldsymbol{v}| - 1)^2\right\}, \tag{49}$$

$$\mathcal{U} = \frac{1}{2}\left\{J_{ij}^{d}(u_i^{d} - \hat{u}_i^{d})(u_j^{d} - \hat{u}_j^{d}) + K_{33}(|\boldsymbol{v}| - 1)^2\right\} \tag{50}$$

和

$$\mathcal{U} := \mathcal{U}_1 + \mathcal{U}_2. \tag{51}$$

利用(46)～(48)式,我们得到同余全微分:

$$\begin{aligned}
\dot{\mathcal{T}} = {} & \mu\partial_t r_i' \cdot \partial_t r_i + (\partial_t \Omega_i^{d} + \epsilon_{ikl} w_k^{d} \Omega_l^{d}) I_{ij}^{d} w_j^{d} = \\
& - r_i' \cdot \mu\partial_{tt} r_i - \Omega_1^{d}[\partial_t(I_{1j}^{d} w_j^{d}) - w_3^{d} I_{2j}^{d} w_j^{d} + w_2^{d} I_{3j}^{d} w_j^{d}] - \\
& \Omega_2^{d}[\partial_t(I_{2j}^{d} w_j^{d}) + w_3^{d} I_{1j}^{d} w_j^{d} - w_1^{d} I_{3j}^{d} w_j^{d}] - \\
& \Omega_3^{d}[\partial_t(I_{3j}^{d} w_j^{d}) - w_2^{d} I_{1j}^{d} w_j^{d} + w_1^{d} I_{2j}^{d} w_j^{d}];
\end{aligned}$$

$$\begin{aligned}
\dot{\mathcal{T}} = {} & - r_i' \cdot \mu\partial_{tt} r_i - \left\{\frac{\boldsymbol{v} \times \boldsymbol{v}'}{|\boldsymbol{v}|^2} \cdot \boldsymbol{d}_1\right\}[\partial_t(I_{1j}^{d} w_j^{d}) - w_3^{d} I_{2j}^{d} w_j^{d} + w_2^{d} I_{3j}^{d} w_j^{d}] - \\
& \left\{\frac{\boldsymbol{v} \times \boldsymbol{v}'}{|\boldsymbol{v}|^2} \cdot \boldsymbol{d}_2\right\}[\partial_t(I_{2j}^{d} w_j^{d}) + w_3^{d} I_{1j}^{d} w_j^{d} - w_1^{d} I_{3j}^{d} w_j^{d}] - \\
& \Omega_3^{d}[\partial_t(I_{3j}^{d} w_j^{d}) - w_2^{d} I_{1j}^{d} w_j^{d} + w_1^{d} I_{2j}^{d} w_j^{d}];
\end{aligned}$$

$$\begin{aligned}
\dot{\mathcal{T}} = {} & - r_i' \cdot \mu\partial_{tt} r_i + \boldsymbol{v}' \cdot [\partial_t(I_{1j}^{d} w_j^{d}) - w_3^{d} I_{2j}^{d} w_j^{d} + w_2^{d} I_{3j}^{d} w_j^{d}] \frac{\boldsymbol{d}_2}{|\boldsymbol{v}|} - \\
& \boldsymbol{v}' \cdot [\partial_t(I_{2j}^{d} w_j^{d}) + w_3^{d} I_{1j}^{d} w_j^{d} - w_1^{d} I_{3j}^{d} w_j^{d}] \frac{\boldsymbol{d}_1}{|\boldsymbol{v}|} - \\
& \Omega_3^{d}[\partial_t(I_{3j}^{d} w_j^{d}) - w_2^{d} I_{1j}^{d} w_j^{d} + w_1^{d} I_{2j}^{d} w_j^{d}];
\end{aligned}$$

$$\begin{aligned}
\dot{\mathcal{T}} = {} & - \boldsymbol{r}' \cdot \mu\partial_{tt} \boldsymbol{r} - \boldsymbol{r}' \cdot \partial_s\left\{[\partial_t(I_{1j}^{d} w_j^{d}) - w_3^{d} I_{2j}^{d} w_j^{d} + w_2^{d} I_{3j}^{d} w_j^{d}] \frac{\boldsymbol{d}_2}{|\boldsymbol{v}|}\right\} + \\
& \boldsymbol{r}' \cdot \partial_s\left\{[\partial_t(I_{2j}^{d} w_j^{d}) + w_3^{d} I_{1j}^{d} w_j^{d} - w_1^{d} I_{3j}^{d} w_j^{d}] \frac{\boldsymbol{d}_1}{|\boldsymbol{v}|}\right\} - \\
& \Omega_3^{d}[\partial_t(I_{3j}^{d} w_j^{d}) - w_2^{d} I_{1j}^{d} w_j^{d} + w_1^{d} I_{2j}^{d} w_j^{d}].
\end{aligned}$$

进一步地从(23)～(25)式和(46)～(48)式得到同余全微分:

$$\begin{aligned}
\dot{\mathcal{U}}_1 = {} & - \boldsymbol{r}' \cdot \partial_s\left\{[\partial_s(J_{1j}^{d}(u_j^{d} - \hat{u}_j^{d})) - u_3^{d} J_{2j}^{d}(u_j^{d} - \hat{u}_j^{d}) + u_2^{d} J_{3j}^{d}(u_j^{d} - \hat{u}_j^{d})] \frac{\boldsymbol{d}_2}{|\boldsymbol{v}|}\right\} + \\
& \boldsymbol{r}' \cdot \partial_s\left\{[\partial_s(J_{2j}^{d}(u_j^{d} - \hat{u}_j^{d})) + u_3^{d} J_{1j}^{d}(u_j^{d} - \hat{u}_j^{d}) - u_1^{d} J_{3j}^{d}(u_j^{d} - \hat{u}_j^{d})] \frac{\boldsymbol{d}_1}{|\boldsymbol{v}|}\right\} - \\
& \Omega_3^{d}[\partial_s(J_{3j}^{d}(u_j^{d} - \hat{u}_j^{d})) - u_2^{d} J_{1j}^{d}(u_j^{d} - \hat{u}_j^{d}) + u_1^{d} J_{2j}^{d}(u_j^{d} - \hat{u}_j^{d})]
\end{aligned} \tag{52}$$

和

$$\begin{aligned}
\dot{\mathcal{U}}_2 = {} & K_{33}(|\boldsymbol{v}| - 1) \frac{\boldsymbol{v}}{|\boldsymbol{v}|} \cdot \boldsymbol{v}' = - \boldsymbol{r}' \cdot \partial_s\left(\frac{K_{33}(|\boldsymbol{v}| - 1)}{|\boldsymbol{v}|}\boldsymbol{v}\right) = \\
& - \boldsymbol{r}' \cdot \partial_s(K_{33}(|\partial_s \boldsymbol{r}| - 1)\boldsymbol{d}_3).
\end{aligned} \tag{53}$$

注意到 $\delta\boldsymbol{r}$ 和 Ω_3 是独立的, 由 $\delta S = 0$ 得出动力学方程如下:

$$
\begin{aligned}
& -\delta\boldsymbol{r}\cdot\mu\partial_{tt}\boldsymbol{r}-\delta\boldsymbol{r}\cdot\partial_s\Big\{\left[\partial_t(I_{1j}^{d}w_j^{d})-w_3^{d}I_{2j}^{d}w_j^{d}+w_2^{d}I_{3j}^{d}w_j^{d}\right]\frac{\boldsymbol{d}_2}{|\boldsymbol{v}|}- \\
& \quad\left[\partial_t(I_{2j}^{d}w_j^{d})+w_3^{d}I_{1j}^{d}w_j^{d}-w_1^{d}I_{3j}^{d}w_j^{d}\right]\frac{\boldsymbol{d}_1}{|\boldsymbol{v}|}\Big\}= \\
& \quad -\delta\boldsymbol{r}\cdot\partial_s\Big\{\left[\partial_s(J_{1j}^{d}(u_j^{d}-\hat{u}_j^{d}))-u_3^{d}J_{2j}^{d}(u_j^{d}-\hat{u}_j^{d})+u_2^{d}J_{3j}^{d}(u_j^{d}-\hat{u}_j^{d})\right]\frac{\boldsymbol{d}_2}{|\boldsymbol{v}|}- \\
& \quad\left[\partial_s(J_{2j}^{d}(u_j^{d}-\hat{u}_j^{d}))+u_3^{d}J_{1j}^{d}(u_j^{d}-\hat{u}_j^{d})-\right. \\
& \quad\left.u_1^{d}J_{3j}^{d}(u_j^{d}-\hat{u}_j^{d})\right]\frac{\boldsymbol{d}_1}{|\boldsymbol{v}|}+\left[K_{33}(|\partial_s\boldsymbol{r}|-1)\right]\boldsymbol{d}_3\Big\},
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
& \partial_t(I_{3j}^{d}w_j^{d})-w_2^{d}I_{1j}^{d}w_j^{d}+w_1^{d}I_{2j}^{d}w_j^{d}= \\
& \quad \partial_s(J_{3j}^{d}(u_j^{d}-\hat{u}_j^{d}))-u_2^{d}J_{1j}^{d}(u_j^{d}-\hat{u}_j^{d})+u_1^{d}J_{2j}^{d}(u_j^{d}-\hat{u}_j^{d}).
\end{aligned}
\tag{55}
$$

(54) 式可以写成下面的形式:

$$
\mu\partial_{tt}\boldsymbol{r}=\partial_s\boldsymbol{n},
\tag{56}
$$

这里

$$
\begin{aligned}
\boldsymbol{n}= & -\left[\partial_t(I_{1j}^{d}w_j^{d})-w_3^{d}I_{2j}^{d}w_j^{d}+w_2^{d}I_{3j}^{d}w_j^{d}\right]\frac{\boldsymbol{d}_2}{|\boldsymbol{v}|}+ \\
& \left[\partial_t(I_{2j}^{d}w_j^{d})+w_3^{d}I_{1j}^{d}w_j^{d}-w_1^{d}I_{3j}^{d}w_j^{d}\right]\frac{\boldsymbol{d}_1}{|\boldsymbol{v}|}+ \\
& \left[\partial_s(J_{1j}^{d}(u_j^{d}-\hat{u}_j^{d}))-u_3^{d}J_{2j}^{d}(u_j^{d}-\hat{u}_j^{d})+u_2^{d}J_{3j}^{d}(u_j^{d}-\hat{u}_j^{d})\right]\frac{\boldsymbol{d}_2}{|\boldsymbol{v}|}- \\
& \left[\partial_s(J_{2j}^{d}(u_j^{d}-\hat{u}_j^{d}))+u_3^{d}J_{1j}^{d}(u_j^{d}-\hat{u}_j^{d})-u_1^{d}J_{3j}^{d}(u_j^{d}-\hat{u}_j^{d})\right]\frac{\boldsymbol{d}_1}{|\boldsymbol{v}|}+ \\
& \left[K_{33}(|\partial_s\boldsymbol{r}|-1)\right]\boldsymbol{d}_3
\end{aligned}
\tag{57}
$$

是接触力.

如果记

$$
\boldsymbol{n}=n_1\boldsymbol{d}_1+n_2\boldsymbol{d}_2+n_3\boldsymbol{d}_3,
\tag{58}
$$

则有

$$
\begin{aligned}
n_1= & \Big\{\partial_t(I_{2j}^{d}w_j^{d})+w_3^{d}I_{1j}^{d}w_j^{d}-w_1^{d}I_{3j}^{d}w_j^{d}- \\
& \partial_s(J_{2j}^{d}(u_j^{d}-\hat{u}_j^{d}))-u_3^{d}J_{1j}^{d}(u_j^{d}-\hat{u}_j^{d})+u_1^{d}J_{3j}^{d}(u_j^{d}-\hat{u}_j^{d}))\Big\}/|\boldsymbol{v}|;
\end{aligned}
\tag{59}
$$

$$
n_1=\Big\{\partial_t(h_2)+w_3^{d}h_1-w_1^{d}h_3-\partial_s(m_2)-u_3^{d}m_1+u_1^{d}m_3\Big\}/v_3^{d}
\tag{60}
$$

和

$$
\begin{aligned}
n_2= & \Big\{-\partial_t(I_{1j}^{d}w_j^{d})+w_3^{d}I_{2j}^{d}w_j^{d}-w_2^{d}I_{3j}^{d}w_j^{d}+ \\
& \partial_s(J_{1j}^{d}(u_j^{d})-\hat{u}_j^{d}))-u_3^{d}J_{2j}^{d}(u_j^{d}-\hat{u}_j^{d})+u_2^{d}J_{3j}^{d}(u_j^{d}-\hat{u}_j^{d})\Big\}/|\boldsymbol{v}|;
\end{aligned}
\tag{61}
$$

$$
n_2=\Big\{-\partial_t(h_1)+w_3^{d}h_2-w_2^{d}h_3+\partial_s(m_1)-u_3^{d}m_2+u_2^{d}m_3\Big\}/v_3^{d}
\tag{62}
$$

和

$$
n_3=K_{33}(|\partial_s\boldsymbol{r}|-1)=K_{33}(v_3^{d}-1).
\tag{63}
$$

同样(55) 式可以写成下面的形式:

$$
\begin{aligned}
\partial_t(I_{3j}^{d}w_j^{d})= & w_2^{d}I_{1j}^{d}w_j^{d}-w_1^{d}I_{2j}^{d}w_j^{d}+\partial_s(J_{3j}(u_j^{d}-\hat{u}_j^{d}))- \\
& u_2^{d}J_{1j}^{d}(u_j^{d}-\hat{u}_j^{d})+u_1^{d}J_{2j}^{d}(u_j^{d}-\hat{u}_j^{d})
\end{aligned}
\tag{64}
$$

或者

$$\partial_t(h_3) = w_2^d h_1 - w_1^d h_2 + \partial_s(m_3) - u_2^d m_1 + u_1^d m_2. \tag{65}$$

因此在没有外力和外力矩时, 剪切形变可以忽略的细长结构的动力学方程能够写成下面的形式:

$$\mu\partial_{tt}\boldsymbol{r} = \partial_s\boldsymbol{n} = n_1\boldsymbol{d}_1 + n_2\boldsymbol{d}_2 + n_3\boldsymbol{d}_3, \tag{66}$$

$$\partial_t(h_3) = w_2^d h_1 - w_1^d h_2 + \partial_s(m_3) - u_2^d m_1 + u_1^d m_2, \tag{67}$$

这里 n_i 由(60)式、(62)式和(63)式定义.

5 结　论

本文利用 Cosserat 理论建立了细长结构的三维非线性动力学模型. 借助伪刚体法和变分原理得到了 Cosserat 杆的包括各种形变的三维空间运动方程. 得到的结果与由 Newton 法则得到的结果一致. 该方法已经被用于模拟 Tang 微型谐振器, 参见文献[17-18].

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Variational Principle for a Special Cosserat Rod

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Abstract: Based on the Cosserat theory, the nonlinear models of a rod in 3-dimensional space was described. Using pseudo-rigid body method and variational principle the equations of motion of Cosserat rod including shear deformation were obtained.

Key words: rod; variational principle; Cosserat theory