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粘弹性流体流经作依赖时间运动伸展面的三维流动及其传质问题*

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摘要: 研究粘弹性流体流经伸展面时的三维边界层流动. 假定伸展面的运动速度依赖于时间. 进一步考虑了更高阶次化学反应对传质的影响. 应用同伦分析法(HAM)进行计算. 精确地分析了所得级数解的收敛性. 用图形讨论了各参数变化对速度和浓度的影响. 还给出了面传质数值的计算,将所得结果与前人的数值解进行了比较.

关 键 词: 不稳定流动; 传质; 粘弹性流体; HAM 解
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引 言

众所周知,非 Newton 流体经常在工业和工程中出现[1-7]. 特别是近年来,粘性非 Newton 流体边界层流经伸展平面,成为相当有意义问题,原因有二:第一,这样流动的控制方程为非线性方程,为研究者从不同的角度提供了有趣味的挑战;第二,这样的流动实际联系着,聚合物薄膜的挤出、玻璃的吹制、塑料膜的生产、晶体的生长、热轧等其他应用. 过去的研究主要集中在稳定的二维伸展流动. 但是,还很少有人关注到伸展流动依赖时间的问题[8-11]. 粘弹性流体的边界层流动也没有受到广泛关注. Beard 和 Walters[12] 对粘弹性流体的流动,给出了开创性的工作,他们研究了粘弹性流体的驻点流动,并得到了以粘弹性参数为小参数的摄动解. Ariel[13] 考虑了部分滑动,对粘弹性流体驻点流动的影响. 最近,Hayat 等[14] 在有自由流速度出现的时候,研究粘弹性流体的二维稳定流动. 应用同伦分析法(HAM),导出微分方程的解. Cortell[15-16] 分析了传质对粘弹性流体二维流动的影响. Nazar 和 Latip[17] 考虑了二级流体的三维流动,并得到了数值解. 最近,Mukhopadhyay[18] 就不稳定混合对流及其传热,流经多孔伸展平面问题,调查热辐射的影响. 本研究就流经不稳定伸展平面问题,分析不稳定三维流动及其传质特性. 应用 Liao[19] 推荐的同伦分析法(HAM),得到动量和传质的解析解. 这种方法已在多个感兴趣的问题中得到了成功地应用[20-34]. 本文详细讨论了不同参数值时的图形结果,计算了表面传质的数值.

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1　数学公式化表示

就粘弹性流体流经不稳定伸展面时,研究其不稳定的三维流动及其传质特性,表面浓度为 C_w,周围静止流体浓度为 C_∞,C_∞ 与 C_w 相比足够小(见图1).不稳定的三维边界层流动及其传质的控制方程由下列方程给出[17,30]:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}= \nu\frac{\partial^2 u}{\partial z^2}-k_0\left[\frac{\partial^3 u}{\partial z^2\partial t}+u\frac{\partial^3 u}{\partial x\partial z^2}+ w\frac{\partial^3 u}{\partial z^3}-\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial z^2}-\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2}- 2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial x\partial z}-2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}\right], \tag{2}$$

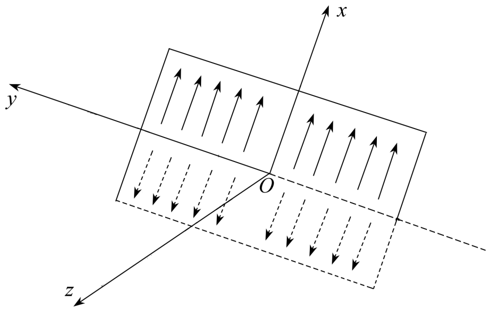


图1　问题的几何表示

Fig. 1　Geometry of the problem

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}= \nu\frac{\partial^2 v}{\partial z^2}-k_0\left[\frac{\partial^3 v}{\partial z^2\partial t}+v\frac{\partial^3 v}{\partial y\partial z^2}+w\frac{\partial^3 v}{\partial z^3}-\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial z^2}- \frac{\partial v}{\partial z}\frac{\partial^2 w}{\partial z^2}-2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial y\partial z}-2\frac{\partial w}{\partial z}\frac{\partial^2 v}{\partial z^2}\right], \tag{3}$$

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+w\frac{\partial C}{\partial z}=D\frac{\partial^2 C}{\partial z^2}-K_n(t)C^n, \tag{4}$$

其中, u, v 和 w 为速度分量, ρ 为流体密度, ν 为运动学粘性系数, μ 为动力学粘性系数,k_0 为流体的材料参数, C 为流体组分浓度, D 为流体中不同组分的扩散系数, $K_n(t)=k_n(1-\alpha t)$ 为反应率,$\alpha t<1$.

所考虑问题的边界条件为

$$\begin{cases} z=0\ 时, & u=u_w(x)=\dfrac{ax}{1-\alpha t},\ v=v_w(y)=\dfrac{by}{1-\alpha t},\ w=0,\ C=C_w, \\ z\to\infty\ 时, & u\to 0,\ v\to 0,\ w=0,\ C\to C_\infty, \end{cases} \tag{5}$$

其中, a 和 b 为正常数. 为了使问题无量纲化,定义

$$\begin{cases} \eta=\sqrt{\dfrac{a}{\nu(1-\alpha t)}}z,\ u=\dfrac{ax}{1-\alpha t}f'(\eta),\ v=\dfrac{ay}{1-\alpha t}g'(\eta), \\ w=-\sqrt{\dfrac{a\nu}{1-\alpha t}}\{f(\eta)+g(\eta)\},\ \phi=\dfrac{C-C_\infty}{C_w-C_\infty}. \end{cases} \tag{6}$$

注意到,不可压缩性条件(1)得到自动满足,方程(2)~(5)呈如下形式:

$$f'''-f'^2-\zeta\left(\frac{\eta}{2}f''+f'\right)+(f+g)f''+ K\left[-\zeta\left(\frac{\eta}{2}f^{\mathrm{iv}}+2f'''\right)+(f+g)f^{\mathrm{iv}}+(f''-g'')f''-2(f'+g')f'''\right]=0, \tag{7}$$

$$g''' - g'^2 - \zeta\left(\frac{\eta}{2}f'' + f'\right) + (f+g)g'' + K\left[-\zeta\left(\frac{\eta}{2}f^{\mathrm{iv}} + 2f'''\right) + (f+g)g^{\mathrm{iv}} + (g''-f'')g'' - 2(f'+g')g'''\right] = 0, \tag{8}$$

$$\phi'' + Sc(f+g)\phi' - Sc\gamma\phi^n - Sc\frac{\zeta}{2}\eta\phi' = 0, \tag{9}$$

$$\begin{cases}\eta = 0 \text{ 时}, \quad f(0)=0,\ g(0)=0,\ f'(0)=1,\ g'(0)=c,\ \phi=1, \\ \eta\to\infty \text{ 时}, \quad f'(\infty)=0,\ g'(\infty)=0,\ \phi=0,\end{cases} \tag{10}$$

其中,ζ 为不稳定性参数,K 为粘弹性参数,Sc 为 Schimdt 数,c 为表面的伸展率,γ 为化学反应参数,它们定义如下

$$\zeta = \frac{\alpha}{a},\ K = \frac{k_0 a}{\upsilon(1-\alpha t)},\ c = \frac{b}{a},\ Sc = \frac{\upsilon}{D},\ \gamma = \frac{k_n C_{\mathrm{w}}^{n-1}}{a}. \tag{11}$$

应该指出,当 $c=0$ 时,得到二维问题($g=0$) 的结果:

$$f''' - f'^2 - \zeta\left(\frac{\eta}{2}f'' + f'\right) + ff'' + K\left[-\zeta\left(\frac{\eta}{2}f^{\mathrm{iv}} + 2f'''\right) + ff^{\mathrm{iv}} + f''^2 - 2f'f'''\right] = 0, \tag{12}$$

$$\phi'' + Scf\phi' - Sc\gamma\phi^n - Sc\frac{\zeta}{2}\eta\phi' = 0, \tag{13}$$

$$\begin{cases}\eta = 0 \text{ 时}, \quad f(0)=0,\ f'(0)=1,\ f'(\infty)=0,\ \phi=1, \\ \eta\to\infty \text{ 时}, \quad \phi=0.\end{cases} \tag{14}$$

对于稳定状态 ($\zeta=0$), 方程(12) ~ (14)有下面形式的精确解:

$$f(\eta) = 1 - \exp(-m\eta),\ m = 1/\sqrt{1-K} \qquad (0 < K \leqslant 1), \tag{15}$$

$$\phi(\eta) = \exp[-(\alpha_1+\alpha_2)]\frac{{}_1F_1(\alpha_1+\alpha_2, 1+2\alpha_2, -(Sc/m^2)\exp(-m\eta))}{{}_1F_1(\alpha_1+\alpha_2, 1+2\alpha_2, -(Sc/m^2))}, \tag{16}$$

其中

$$\alpha_1 = \frac{Sc}{2m^2},\ \alpha_2 = \frac{\sqrt{4Sc\gamma m^2 + Sc^2}}{2m^2}, \tag{17}$$

${}_1F_1$ 为合流超几何函数.

对于 $c=1$, $f=g$, 对应于轴对称流动

$$f''' - f'^2 - \zeta\left(\frac{\eta}{2}f'' + f'\right) + 2ff'' + K\left[-\zeta\left(\frac{\eta}{2}f^{\mathrm{iv}} + 2f'''\right) + 2ff^{\mathrm{iv}} - 4f'f'''\right] = 0, \tag{18}$$

$$\phi'' + 2Scf\phi' - Sc\gamma\phi^n - Sc\frac{\zeta}{2}\eta\phi' = 0, \tag{19}$$

其边界条件为式(14).

定义表面摩擦因数公式(20)和(21),以及表面质量传递公式(22):

$$Re_x^{1/2}C_{fx} = \left[f'' + K\left\{\frac{\zeta}{2}(3f'' + \eta f''') + 3(f'+g')f'' - (f+g)f'''\right\}\right]_{\eta=0}, \tag{20}$$

$$Re_y^{1/2}C_{fy} = \left[g'' + K\left\{\frac{\zeta}{2}(3g'' + \eta g''') + 3(f'+g')g'' - (f+g)g'''\right\}\right]_{\eta=0}, \tag{21}$$

$$\phi'(0)=\left.\frac{\partial\phi}{\partial z}\right|_{z=0},\tag{22}$$

其中，$Re_x=u_wx/\nu$ 和 $Re_y=v_wy/\nu$ 为局部 Reynolds 数。

2　同伦分析法的解

2.1　0 阶变形问题

为了通过同伦分析法(HAM)求解方程(7)~(10)，采用基函数集

$$\{\eta^k\exp(-r\eta)\mid k\geqslant 0,r\geqslant 0\}\tag{23}$$

得到速度分布 $f(\eta)$，$g(\eta)$ 以及浓度场 $\phi(\eta)$ 的表达式如下：

$$\begin{cases}f(\eta)=a_{0,0}^0+\sum\limits_{r=0}^{\infty}\sum\limits_{k=0}^{\infty}a_{m,r}^k\eta^k\exp(-r\eta),\\ g(\eta)=A_{0,0}^0+\sum\limits_{r=0}^{\infty}\sum\limits_{k=0}^{\infty}A_{m,r}^k\eta^k\exp(-r\eta),\\ \phi(\eta)=\sum\limits_{r=0}^{\infty}\sum\limits_{k=0}^{\infty}b_{m,r}^k\eta^k\exp(-r\eta),\end{cases}\tag{24}$$

其中，$a_{m,r}^k$，$A_{m,r}^k$ 和 $b_{m,r}^k$ 为系数。由 $f(\eta)$，$g(\eta)$，$\phi(\eta)$ 解表达式规则以及方程(7)~(10)，得到初始猜测值 $f_0(\eta)$，$g_0(\eta)$，$\phi_0(\eta)$ 为

$$\begin{cases}f_0(\eta)=1-\exp(-\eta),\\ g_0(\eta)=c(1-\exp(-\eta)),\\ \phi_0(\eta)=\exp(-\eta),\end{cases}\tag{25}$$

以及线性算子 $\mathcal{L}_1$ 和 $\mathcal{L}_2$ 为

$$\begin{cases}\mathcal{L}_1(f)=f'''-f',\\ \mathcal{L}_2(f)=f''-f,\end{cases}\tag{26}$$

且有性质：

$$\begin{cases}\mathcal{L}_1[C_1+C_2\exp(\eta)+C_3\exp(-\eta)]=0,\\ \mathcal{L}_2[C_4\exp(\eta)+C_5\exp(-3\eta)]=0,\end{cases}\tag{27}$$

其中，C_1 ~ C_5 为常数。方程(7)~(10)更进一步提出如下非线性算子定义：

$$\begin{aligned}&\mathcal{N}_1[\bar f(\eta,p),\bar g(\eta,p)]=\\ &\quad\frac{\partial^3\bar f(\eta,p)}{\partial\eta^3}-\left(\frac{\partial\bar f(\eta,p)}{\partial\eta}\right)^2-\zeta\left(\frac{\eta}{2}\frac{\partial^2\bar f(\eta,p)}{\partial\eta^2}+\frac{\partial\bar f(\eta,p)}{\partial\eta}\right)+\\ &\quad(\bar f(\eta,p)+\bar g(\eta,p))\frac{\partial^2\bar f(\eta,p)}{\partial\eta^2}-\\ &\quad K\zeta\left(\frac{\eta}{2}\frac{\partial^4\bar f(\eta,p)}{\partial\eta^4}+2\frac{\partial^3\bar f(\eta,p)}{\partial\eta^3}\right)+\\ &\quad K\Bigg[(\bar f(\eta,p)+\bar g(\eta,p))\frac{\partial^4\bar f(\eta,p)}{\partial\eta^4}+\\ &\quad\left(\frac{\partial^2\bar f(\eta,p)}{\partial\eta^2}-\frac{\partial^2\bar g(\eta,p)}{\partial\eta^2}\right)\frac{\partial^2\bar f(\eta,p)}{\partial\eta^2}-\end{aligned}$$

$$2\left(\frac{\partial \bar{f}(\eta,p)}{\partial\eta}+\frac{\partial\bar{g}(\eta,p)}{\partial\eta}\right)\frac{\partial^3 \bar{f}(\eta,p)}{\partial\eta^3}\Bigg], \tag{28}$$

$$\begin{aligned}&\mathcal{N}_2[\bar{f}(\eta,p),\bar{g}(\eta,p)]=\\&\frac{\partial^3\bar{g}(\eta,p)}{\partial\eta^3}-\left(\frac{\partial\bar{g}(\eta,p)}{\partial\eta}\right)^2-\zeta\left(\frac{\eta}{2}\frac{\partial^2\bar{g}(\eta,p)}{\partial\eta^2}+\frac{\partial\bar{g}(\eta,p)}{\partial\eta}\right)+\\&(\bar{f}(\eta,p)+\bar{g}(\eta,p))\frac{\partial^2\bar{g}(\eta,p)}{\partial\eta^2}-\\&K\zeta\left(\frac{\eta}{2}\frac{\partial^4\bar{g}(\eta,p)}{\partial\eta^4}+\frac{\partial^3\bar{g}(\eta,p)}{\partial\eta^3}\right)+\\&K\Bigg[(\bar{f}(\eta,p)+\bar{g}(\eta,p))\frac{\partial^4\bar{g}(\eta,p)}{\partial\eta^4}-\\&\left(\frac{\partial^2\bar{f}(\eta,p)}{\partial\eta^2}-\frac{\partial^2\bar{g}(\eta,p)}{\partial\eta^2}\right)\frac{\partial^2\bar{g}(\eta,p)}{\partial\eta^2}-\\&2\left(\frac{\partial\bar{f}(\eta,p)}{\partial\eta}+\frac{\partial\bar{g}(\eta,p)}{\partial\eta}\right)\frac{\partial^3\bar{g}(\eta,p)}{\partial\eta^3}\Bigg],\end{aligned} \tag{29}$$

$$\begin{aligned}&\mathcal{N}_3[\bar{f}(\eta,p),\bar{g}(\eta,p),\bar{\phi}(\eta,p)]=\\&\frac{\partial^2\bar{\phi}}{\partial\eta^2}+Sc(\bar{f}(\eta,p)+\bar{g}(\eta,p))\frac{\partial\bar{\phi}(\eta,p)}{\partial\eta}-\\&Sc\gamma\,\bar{\phi}^n(\eta,p)-Sc\zeta\frac{\eta}{2}\frac{\partial\bar{\phi}(\eta,p)}{\partial\eta}.\end{aligned} \tag{30}$$

相应的 0 阶问题归纳为

$$(1-p)\mathcal{L}_1[\bar{f}(\eta,p)-f_0(\eta)]=p\hbar_f\mathcal{N}_1[\bar{f}(\eta,p),\bar{g}(\eta,p)], \tag{31}$$

$$(1-p)\mathcal{L}_1[\bar{g}(\eta,p)-g_0(\eta)]=p\hbar_g\mathcal{N}_2[\bar{f}(\eta,p),\bar{g}(\eta,p)], \tag{32}$$

$$(1-p)\mathcal{L}_2[\bar{\phi}(\eta,p)-\phi_0(\eta)]=p\hbar_{\bar{\phi}}\mathcal{N}_3[\bar{f}(\eta,p),\bar{g}(\eta,p),\bar{\phi}(\eta,p)], \tag{33}$$

$$\begin{cases}\bar{f}(0,p)=0,\ \bar{f}'(0,p)=1,\ \bar{g}(0,p)=0,\ \bar{g}'(0,p)=1,\ \bar{\phi}(0,p)=1,\\ \bar{f}'(\infty,p)=0,\ \bar{g}'(\infty,p)=0,\ \bar{\phi}(\infty,p)=1.\end{cases} \tag{34}$$

其中，$\hbar$ 为辅助的非零参数，$p\in[0,1]$ 为内嵌参数. $p=0$ 和 $p=1$ 时，可以定义

$$\begin{cases}\bar{f}(\eta,0)=f_0(\eta),\ \bar{f}(\eta,1)=f(\eta),\\ \bar{g}(\eta,0)=g_0(\eta),\ \bar{f}(\eta,1)=g(\eta),\\ \bar{\phi}(\eta,0)=\phi_0(\eta),\ \bar{\phi}(\eta,1)=\phi(\eta).\end{cases} \tag{35}$$

当 p 从 0 变到 1，初始猜测值 $f_0(\eta)$，$g_0(\eta)$，$\phi_0(\eta)$，分别逼近于 $f(\eta)$，$g(\eta)$，$\phi(\eta)$，由 Taylor 级数展开式，得到

$$\begin{cases}\bar{f}(\eta,p)=f_0(\eta)+\sum\limits_{m=1}^{\infty}f_m(\eta)p^m,\\ \bar{g}(\eta,p)=g_0(\eta)+\sum\limits_{m=1}^{\infty}g_m(\eta)p^m,\\ \bar{\phi}(\eta,p)=\phi_0(\eta)+\sum\limits_{m=1}^{\infty}\phi_m(\eta)p^m,\end{cases} \tag{36}$$

$$\begin{cases} f_m(\eta) = \dfrac{1}{m!} \dfrac{\partial^m \bar{f}(\eta,p)}{\partial p^m}\Bigg|_{p=0}, \\ g_m(\eta) = \dfrac{1}{m!} \dfrac{\partial^m \bar{g}(\eta,p)}{\partial p^m}\Bigg|_{p=0}, \\ \phi_m(\eta) = \dfrac{1}{m!} \dfrac{\partial^m \bar{\phi}(\eta,p)}{\partial p^m}\Bigg|_{p=0}. \end{cases} \tag{37}$$

级数(36)的收敛性,强烈地依赖于 $\hbar_f,\hbar_g$ 和 $\hbar_{\bar{\phi}}$. $\hbar_f,\hbar_g$ 和 $\hbar_{\bar{\phi}}$ 值的选取,通过 $p=1$ 时级数(36)收敛的方式得到. 因此,方程(36)变为

$$\begin{cases} f(\eta) = f_0(\eta) + \sum\limits_{m=1}^{\infty} f_m(\eta), \\ g(\eta) = g_0(\eta) + \sum\limits_{m=1}^{\infty} g_m(\eta), \\ \phi(\eta) = \phi_0(\eta) + \sum\limits_{m=1}^{\infty} \phi_m(\eta). \end{cases} \tag{38}$$

2.2　*m* 阶变形问题

m 阶次的数学问题归纳为

$$\mathcal{L}_1[f_m(\eta,p) - \chi_m f_{m-1}(\eta)] = \hbar_f \mathcal{R}_{1,m}(\eta), \tag{39}$$

$$\mathcal{L}_1[g_m(\eta,p) - \chi_m g_{m-1}(\eta)] = \hbar_g \mathcal{R}_{2,m}(\eta), \tag{40}$$

$$\mathcal{L}_2[\phi_m(\eta,p) - \chi_m \phi_{m-1}(\eta)] = \hbar_{\bar{\phi}} \mathcal{R}_{3,m}(\eta), \tag{41}$$

$$f_m(0) = f_m'(0) = f_m'(\infty) = f_m''(\infty) = g_m(0) = \\ g_m'(\infty) = g_m''(\infty) = \phi_m(0) = \phi_m(\infty) = 0, \tag{42}$$

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1, \end{cases} \tag{43}$$

$$\mathcal{R}_{1,m}(\eta) = f_{m-1}''' - \zeta\left(\frac{\eta}{2} f_{m-1}'' + f_{m-1}'\right) - K\zeta\left(\frac{\eta}{2} f_{m-1}'''' + 2f_{m-1}''\right) + \\ \sum_{k=0}^{m-1} [(f_{m-1-k} + g_{m-1-k}) f_k'' - f_{m-1-k}' f_k' + K((f_{m-1-k} + g_{m-1-k}) f_k^{\mathrm{iv}} + \\ (f_{m-1-k}'' - g_{m-1-k}'') f_k'' - 2(f_{m-1-k}' + g_{m-1-k}') f_k''')], \tag{44}$$

$$\mathcal{R}_{2,m}(\eta) = g_{m-1}''' - \zeta\left(\frac{\eta}{2} g_{m-1}'' + g_{m-1}'\right) - K\zeta\left(\frac{\eta}{2} g_{m-1}'''' + 2fg_{m-1}''\right) + \\ \sum_{k=0}^{m-1} [(f_{m-1-k} + g_{m-1-k}) g_k'' - g_{m-1-k}' g_k' + K((f_{m-1-k} + g_{m-1-k}) g_k^{\mathrm{iv}} + \\ (f_{m-1-k}'' - g_{m-1-k}'') g_k'' - 2(f_{m-1-k}' + g_{m-1-k}') g_k''')]. \tag{45}$$

当 $n=1$ 时, 可以得到

$$\mathcal{R}_{3,m}(\eta) = \phi_{m-1}'' - Sc\gamma\phi_{m-1} - Sc\zeta\,\frac{\eta}{2}\,\phi_{m-1}' + \sum_{k=0}^{m-1} Sc\phi_{m-1-k}'(f_{m-1-k} + g_{m-1-k}). \tag{46}$$

当 $n=2$ 时, 可以得到

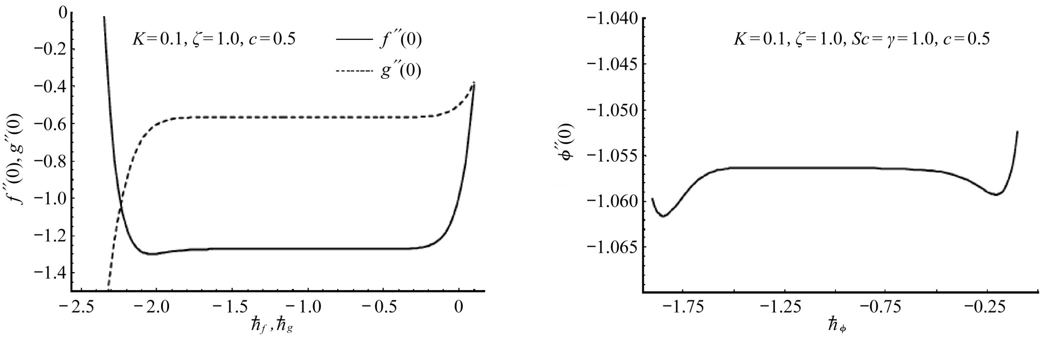
$$\mathcal{R}_{3,m}(\eta) = \phi_{m-1}'' - Sc\zeta\,\frac{\eta}{2}\,\phi_{m-1}' - Sc\gamma \sum_{k=0}^{m-1} \phi_{m-1-k}\phi_k + \\ \sum_{k=0}^{m-1} Sc\phi_{m-1-k}'(f_{m-1-k} + g_{m-1-k}). \tag{47}$$

当 $n = 3$ 时，可以得到

$$\mathcal{R}_{3,m}(\eta) = \phi''_{m-1} - Sc\zeta \frac{\eta}{2} \phi'_{m-1} - Sc\gamma \sum_{k=0}^{m-1} \phi_{m-1-k} \sum_{l=0}^{k} \phi_{k-l}\phi_l + \sum_{k=0}^{m-1} Sc\phi'_{m-1-k}(f_{m-1-k} + g_{m-1-k}) . \tag{48}$$

3 级数解的收敛性

如果能够保证由方程(38)得到的级数收敛,即得到当前所考虑流动问题的级数解. 这些解的收敛区域和逼近速度,强烈地依赖于 $\hbar_f$,$\hbar_g$ 和 $\hbar_\phi$. 辅助参数 $\hbar$的容许值,取直到15 阶的近似曲线. 图2 表示,当 $K = 0.1$,$\zeta = 0.1$, $Sc = \gamma = 1.0$ 和 $c = 0.5$ 时,$\hbar_f$,$\hbar_g$ 和 $\hbar_\phi$ 的容许值,分别是 $-1.6 \leqslant \hbar_f \leqslant -0.5$, $-1.4 \leqslant \hbar_g \leqslant -0.5$, $-1.5 \leqslant \hbar_\phi \leqslant -0.75$. 图3 给出了函数$f$和 g 的残差曲线 $\hbar$. 可以发现,在 $\hbar$ - 曲线范围内,可以得到小数点后 8 位数精度的结果. 计算表明,当 $\hbar_f = \hbar_g =- 0.9$ 和 $\hbar_\phi$ $- 1.2$ 时,级数解(38) 在整个 η 区域内收敛.

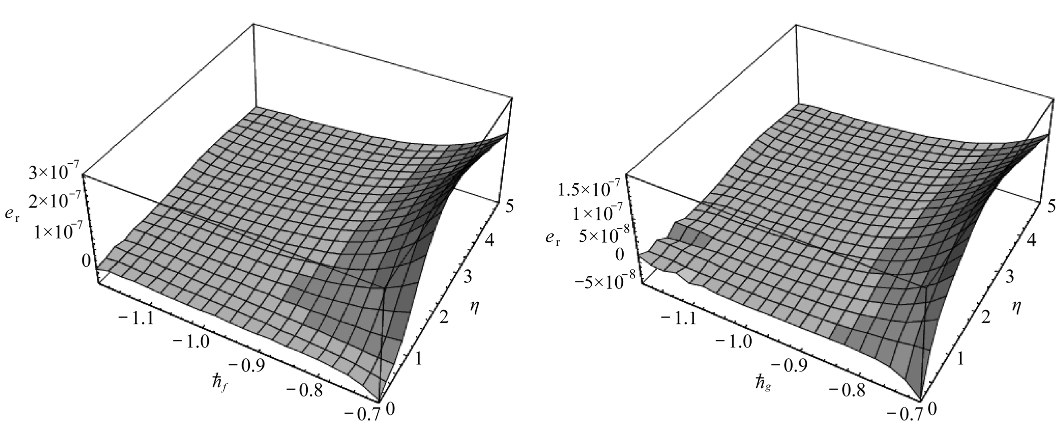


(a) f,g 函数 (b) ϕ 函数

(a) Functions f and g (b) Function ϕ

图2 函数f,g 和 ϕ 直到 15 阶近似的 $\hbar$曲线

Fig. 2 $\hbar$ - curves at 15th-order of approximations



(a) f 函数 (b) g 函数

(a) Function f (b) Function g

图3 当 $K = \zeta = 0.1$ 时的残差曲线

Fig. 3 Residual error when $K = \zeta = 0.1$

4　结果及讨论

本节检查相关参数对速度曲线 f'，g' 和浓度场 ϕ 的影响. 对于稳定流动，将当前所得的 HAM 解与之前所得到的数值解，进行比较研究.

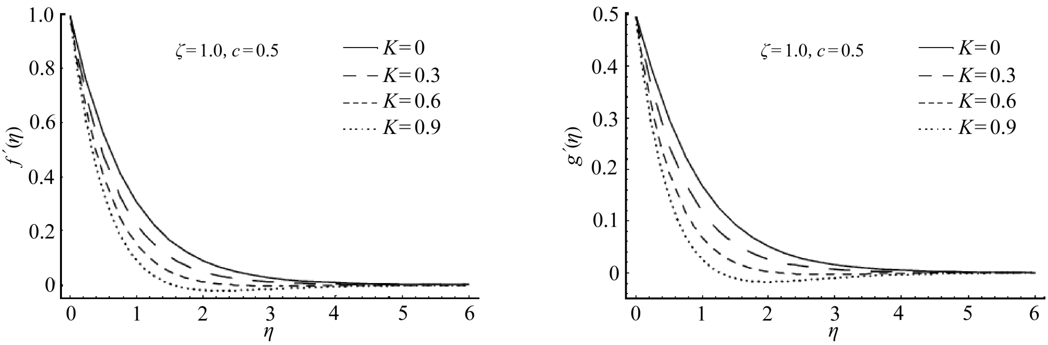


图 4　粘弹性参数 K 对 f' 和 g' 的影响

Fig. 4　Influence of K on f' and g'

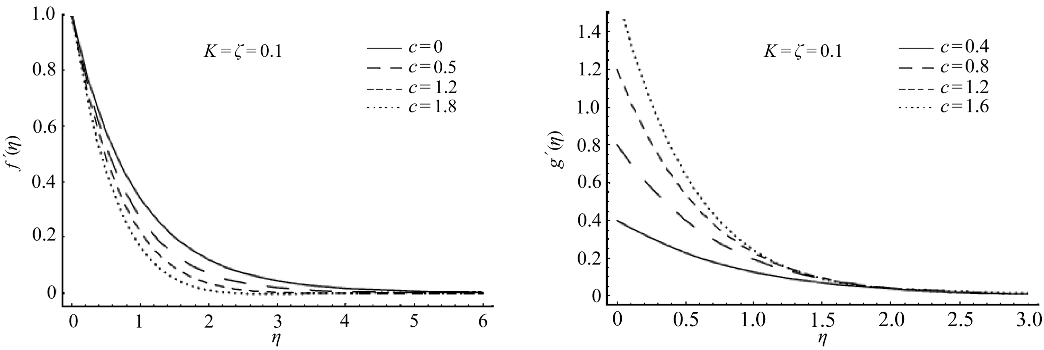


图 5　伸展率 c 对 f' 和 g' 的影响

Fig. 5　Influence of c on f' and g'

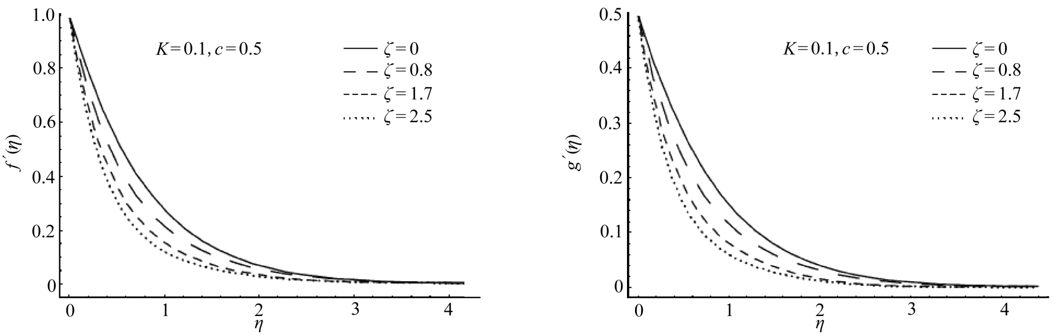


图 6　ζ 对 f' 和 g' 的影响

Fig. 6　Influence of ζ on f' and g'

图 4 表明了粘弹性参数 K 对 f' 和 g' 的影响. 显然，速度场和边界层厚度随着 K 的增大而减小. 图 5 显示了伸展率 c 对速度曲线的影响. 可以发现，x-方向的速度分量和边界层厚度随着 c 的增大而减小，而 y-方向的速度分量和边界层厚度随着 c 的增大而增大. 图 6 给出了不稳定参

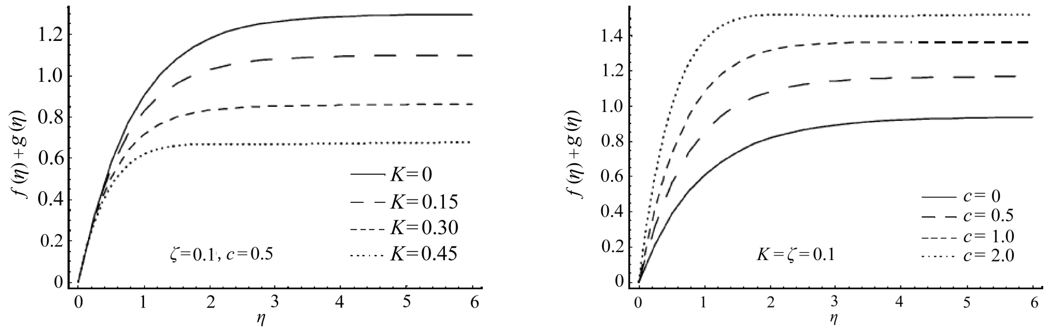


图 7　粘弹性参数 K 和伸展率 c 对 $f+g$ 的影响

Fig. 7　Influence of K and c on $f+g$

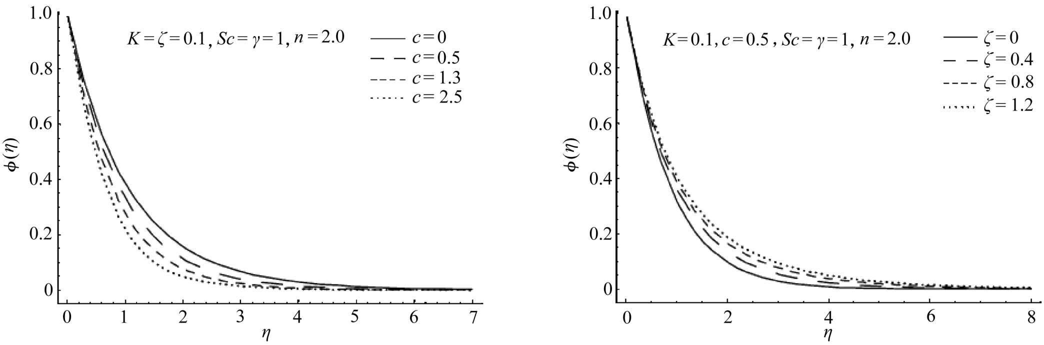


图 8　c 和 ζ 分别对 ϕ 的影响

Fig. 8　Influence of c and ζ on ϕ

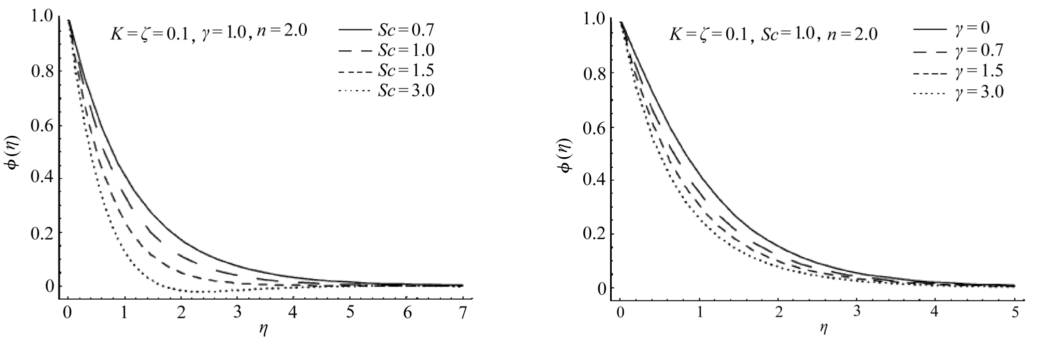
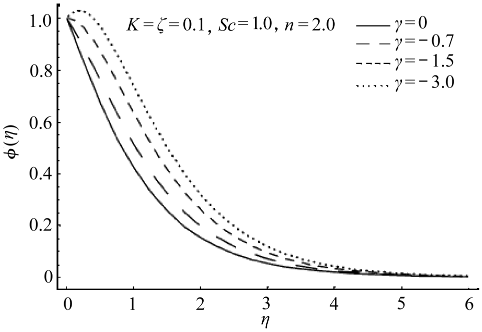


图 9　Schimdt 数 Sc 和化学反应参数 $\gamma(>0)$ 对 ϕ 的影响

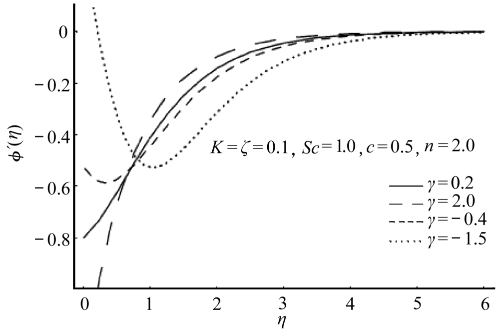
Fig. 4　Influence of Sc and $\gamma(>0)$ on ϕ

数 ζ 对速度曲线 f' 和 g' 的影响. 不稳定参数 ζ 增大时, f' 和 g' 有一个显著的减小. 图 7 给出了粘弹性参数 K 和伸展率 c 对 z-方向速度分量 w 的影响,其中 w 可由 $f+g$ 代表. z-方向速度分量 w 随着 K 的增大而减小,而随着 c 的增大而增大. 图 8 ~ 图 10(a)描述了不稳定参数 ζ、伸展率 c、Schmidt 数 Sc 和化学反应参数 γ 对浓度场 ϕ 的影响. c 和 ζ 对 ϕ 的影响如图 8 所示. 可以发现, ϕ 随着 c 的增大而减小,但随着 ζ 的增大而增大. Schmidt 数 Sc 对浓度场 ϕ 的影响如图 9(a)所示. 热扩散相同时,随着 Sc 的逐渐增大,相应的分子扩散减弱和边界层厚度变薄. 图 9(b)给出了放热化学反应参数 γ 为正(>0) 时, ϕ 随着 γ 的增大而减小. 但是, ϕ 随着吸热化学反应参数 $\gamma(<0)$ 绝对值的增大而增大,如图 10(a)所示. 图 10(b)表示化学反应参数 γ 对传质梯度



（a）化学反应参数 $\gamma(<0)$ 对 ϕ 的影响

（a）Influence of $\gamma(<0)$ on ϕ



（b）化学反应参数 $\gamma(<0)$ 对 $\phi'(\eta)$ 的影响

（b）Influence of $\gamma(<0)$ on $\phi'(\eta)$

图 10　化学反应参数 γ 对 ϕ 和 $\phi'(\eta)$ 的影响

Fig. 10　Influence of $\gamma(<0)$ on $\phi(\eta)$ and influence of γ on $\phi'(\eta)$

$\phi'(\eta)$ 的影响.

表 1 对级数解进行了分析. 表中清楚地显示，直到 15 阶近似时，f 和 g 达到收敛，而 ϕ 的收敛更晚一些，直到 25 阶近似时，开始收敛. 表 2 给出了达到稳定状态时，表面摩擦因数值的对比情况. 表 3 给出了不同内嵌参数时的面传质数值. 可以发现，面传质数值是 K 和 ζ 的一个减函数. 但是对大数值的 Sc 和 γ，$\phi'(0)$ 值大得很多.

表 1　$K=0.1$, $\zeta=1.0$, $Sc=\gamma=1$ 和 $c=0.5$ 时，HAM 解在不同阶近似时的收敛性

Table 1　Convergence of the HAM solutions for dif ferent order of approximations when $K=0.1$, $\zeta=1.0$, $Sc=\gamma=1$ and $c=0.5$

Order of approximations	$-f''(0)$	$-g''(0)$	$-\phi'(0)$
1	1.237 917	0.552 291	1.070 833
5	1.272 742	0.566 459	1.057 425
10	1.272 797	0.566 604	1.056 421
15	1.272 796	0.566 603	1.056 340
20	1.272 796	0.566 603	1.056 319
25	1.272 796	0.566 603	1.056 310
30	1.272 796	0.566 603	1.056 310
35	1.272 796	0.566 603	1.056 310
40	1.272 796	0.566 603	1.056 310

表 2　稳定状态（$\zeta=0$）下，$f''(0)$ 和 $g''(0)$ 的 HAM 解和文献[17]的数值解比较

Table 2　Comparison of values of $f''(0)$ and $g''(0)$ with those in references [17] for the steady state case ($\zeta=0$)

$-K$	$-f''(0)$, $c=0.5$		$-g''(0)$, $c=0.5$		$-f''(0)$, $c=0.0$	
	HAM	Numerical[17]	HAM	Numerical[17]	HAM	Numerical[17]
0.0	1.093 095	1.093 5	0.465 204	0.465 6	1.000 000	1.001 3
0.1	1.000 699	1.001 2	0.418 367	0.433 7	0.953 463	0.955 7
0.2	0.929 309	0.930 6	0.348 360	0.407 6	0.912 870	0.916 3
0.4	0.823 869	0.827 8	0.336 581	0.367 8	0.845 154	0.851 5
0.6	0.748 149	0.755 9	0.303 570	0.338 9	0.790 569	0.800 6
0.8	0.690 238	0.702 3	0.278 862	0.317 1	0.745 356	0.759 7

表 3 K, ζ, Sc 和 γ 不同取值时的面传质值 $-\phi'(0)$

Table 3 values of surface mass transfer $-\phi'(0)$ for dif ferent values of K, ζ, Sc and γ

K	ζ	Sc	γ	$-\phi'(0)$
0.0	0.1	1.0	1.0	1.072 76
0.1				1.056 31
0.2				1.032 25
0.3				0.968 92
0.1	0.0			1.078 85
	0.1			1.056 31
	0.2			1.032 13
	0.3			1.012 44
	0.1	1.2		1.171 23
		1.4		1.277 52
		1.6		1.376 85
		2.0		1.559 08
		1.0	1.0	1.056 31
			1.2	1.118 31
			1.4	1.177 03
			1.6	1.232 93

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Time-Dependent Three-Dimensional Flow and Mass Transfer of an Elastico-Viscous Fluid Over an Unsteady Stretching Sheet

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Abstract: The three-dimensional boundary layer flow of an elastico-viscous fluid over a stretching surface was looked at. Velocity of the stretching sheet was assumed to be time-dependent. Effect of mass transfer with higher order chemical reaction was further considered. Computations were made by homptopy analysis method(HAM). The convergence of the obtained series solutions was explicitly analyzed. The variations of embedding parameters on the velocity and concentration were graphically discussed. Numerical computations of surface mass transfer were reported. Comparison of the present results with the numerical solutions was also seen.

Key words: unsteady flow; mass transfer; viscoelastic fluid; HAM solution