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对边简支矩形薄板方程的算子半群方法*

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摘要: 考虑弹性理论中对边简支矩形薄板方程,用算子半群方法求解问题.首先,将方程转换成抽象 Cauchy 问题.其次,构造空间框架并证明对应的算子矩阵生成压缩半群.最后,经 Fourier 变换,采用一致连续半群做逼近,进而给出对边简支矩形薄板方程的解析解.该方法自然蕴含着解的存在唯一性.

关 键 词: 矩形板; 抽象 Cauchy 问题; C_0 半群; 解析解

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引　言

弹性力学在建筑、机械、航天等很多领域具有非常重要的作用.虽然理论建立较早,但是弹性力学问题通常是求解非常困难的高阶多变量偏微分方程,所以一般很难得到解析解.直接解法有位移法、应力法及混合解法.因数学上的困难,在解决实际问题时很少用直接解法.传统解法——半逆解法缺乏理性和一般性,所以具有局限性.此外,学者们开发了很多数值方法求近似解,如有限差分法、变分法、变分原理为基础的 Ritz 法、Galerkin 法及有限单元法等.但是,数值方法得到的只是问题的数值解而不是解析解[1].20 世纪末,钟万勰院士将无穷维 Hamilton 系统引入弹性力学,并建立了弹性力学求解新方法[2].该方法实际上是 Hamilton 算子矩阵本征函数展开法[3-7].后来,其它一些算子矩阵的本征函数展开法也相继应用到了弹性力学[8-10].

然而,我们需要更多有效通用的方法去研究并解决弹性力学问题.众所周知,算子半群理论已经发展成应用广泛、理论成熟的一门学科.微分方程定解问题可以化成抽象 Cauchy 问题来研究,而算子半群则是解决抽象 Cauchy 问题的有力工具[11-13].因此,半群方法在弹性力学问题的研究中具有一定的应用前景.

本文中 I 表示单位算子,$\mathscr{D}(T)$,$\mathscr{R}(T)$,$R(\lambda;T)$,T^*,T_λ 分别表示线性算子 T 的定义域、值域、预解算子、共轭算子和 Yosida 逼近; $\mathrm{Re}\,z$ 为复数 z 的实部.为了避免混淆,不同内积空间中

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的内积均由下标标明.

1 问题导出的算子矩阵及空间结构

考虑对边简支矩形薄板的弯曲问题

$$\begin{cases} D\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)^2 w=f(x,y), & 0\leqslant x\leqslant h,\ 0\leqslant y\leqslant 1,\ h>0, \\ w=0,\ \dfrac{\partial w}{\partial y}=c,\ \dfrac{\partial^2 w}{\partial y^2}=0, & y=0,\ y=1, \\ w=0,\ \dfrac{\partial w}{\partial x}=\varphi(y),\ \dfrac{\partial^2 w}{\partial x^2}=0, & x=0,\ x=h, \end{cases} \tag{1}$$

其中, D 是抗弯刚度, w 为挠度, $f(x,y)$ 是在区域 $\{(x,y)\mid 0\leqslant x\leqslant h, 0\leqslant y\leqslant 1\}$ 上的横向荷载, c 是常数, $\varphi(y)$ 是四阶可微函数.

将上述问题转换成抽象 Cauchy 问题[14]:

$$\frac{\partial}{\partial x}\begin{pmatrix} p\\ 0\\ q\\ 0\end{pmatrix}=\begin{pmatrix} 0 & 0 & I & 0\\ 0 & 0 & 0 & I\\ \dfrac{\partial^2}{\partial y^2} & 0 & 2\mathrm{i}\dfrac{\partial}{\partial y} & I\\ 0 & \dfrac{\partial^2}{\partial y^2} & 0 & 2\mathrm{i}\dfrac{\partial}{\partial y}\end{pmatrix}\begin{pmatrix} p\\ 0\\ q\\ 0\end{pmatrix}+\begin{pmatrix} 0\\ 0\\ f(x,y)\\ 0\end{pmatrix},$$

其中, I 为单位算子,

$$p=D\left(\frac{\partial^2 w}{\partial x^2}+2\mathrm{i}\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial^2 w}{\partial y^2}\right),\ q=D\left(\frac{\partial^3 w}{\partial x^3}+2\mathrm{i}\frac{\partial^3 w}{\partial x^2\partial y}-\frac{\partial^3 w}{\partial x\partial^2 y}\right). \tag{2}$$

定义 $\mathscr{H}=L^2(0,1)$ 中的微分算子

$$P=Q=\frac{\partial^2}{\partial y^2},\ \mathscr{D}(Q)=\mathscr{D}(P)=\{p\in\mathscr{H}:\ p',\ p''\in\mathscr{H},\ p'(0)=p'(1)=0\},$$

$$S=T=2\mathrm{i}\frac{\partial}{\partial y},\mathscr{D}(T)=\mathscr{D}(S)=\{q\in\mathscr{H}:\ q'\in\mathscr{H},\ q(0)=q(1)\}.$$

易知 $-P$, $-Q$ 是自伴正定算子, 进而 $(-P)^{1/2}$ 和 $(-Q)^{1/2}$ 存在,且

$$\mathscr{D}((-P)^{1/2})\subset\mathscr{D}(S),\ \mathscr{D}((-Q)^{1/2})\subset\mathscr{D}(T)^{[15]}.$$

由于 $(-P)^{1/2}$ 和 $(-Q)^{1/2}$ 也是自伴正定算子,定义 Hilbert 空间

$$\mathscr{H}_P=\mathscr{D}((-P)^{1/2}),$$

其范数为内积

$$(x,y)_{\mathscr{H}_P}=((-P)^{1/2}x,(-P)^{1/2}y)_{\mathscr{H}},\qquad x,y\in\mathscr{H}_P$$

诱导的范数.同样定义 Hilbert 空间 $\mathscr{H}_Q=\mathscr{D}((-Q)^{1/2})$.于是,问题(1)可以转换成 Hilbert 空间 $\mathscr{X}=\mathscr{H}_P\times\mathscr{H}_Q\times\mathscr{H}\times\mathscr{H}$ 中的抽象 Cauchy 问题:

$$\begin{cases}\dfrac{\mathrm{d}\boldsymbol{V}(x)}{\mathrm{d}x}=\boldsymbol{M}\boldsymbol{V}(x)+\boldsymbol{F}(x),\\ \boldsymbol{V}(0)=(2\mathrm{i}D\varphi'(y),0,D\varphi''(y),0)^{\mathrm{T}},\end{cases} \tag{3}$$

其中

$$\boldsymbol{M}=\begin{pmatrix}0&0&I&0\\0&0&0&I\\P&0&S&I\\0&Q&0&T\end{pmatrix}:\mathcal{D}(\boldsymbol{M})=\mathcal{D}(P)\times\mathcal{D}(Q)\times\mathcal{H}_P\times\mathcal{H}_Q,$$

$$\boldsymbol{V}(x)=(p,0,q,0)^{\mathrm{T}},\ \boldsymbol{F}(x)=(0,0,f(x,y),0)^{\mathrm{T}}.$$

2 算子矩阵 $\boldsymbol{M}$ 的半群生成性质

定理 2.1 算子矩阵 $\boldsymbol{M}$ 能生成 $\mathcal{Z}$ 上的压缩半群.

证明 显然, $\boldsymbol{M}$ 是稠定闭算子.将 $\boldsymbol{M}$ 分解成 $\boldsymbol{M}=\boldsymbol{M}_0+\boldsymbol{M}_1$, 其中

$$\boldsymbol{M}_0=\begin{pmatrix}0&0&I&0\\0&0&0&I\\P&0&0&0\\0&Q&0&0\end{pmatrix},\ \boldsymbol{M}_1=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&S&I\\0&0&0&T\end{pmatrix}.$$

对任意的 $\boldsymbol{x}=(x_1,x_2,x_3,x_4)^{\mathrm{T}}$, $\boldsymbol{y}=(y_1,y_2,y_3,y_4)^{\mathrm{T}}\in\mathcal{D}(\boldsymbol{M}_0)=\mathcal{D}(\boldsymbol{M})$,

$$(\boldsymbol{M}_0\boldsymbol{x},\boldsymbol{y})_{\mathcal{Z}}=\left(\begin{pmatrix}0&0&I&0\\0&0&0&I\\P&0&0&0\\0&Q&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix},\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}\right)_{\mathcal{Z}}=$$

$$((-P)^{1/2}x_3,(-P)^{1/2}y_1)_{\mathcal{H}}+((-Q)^{1/2}x_4,(-Q)^{1/2}y_2)_{\mathcal{H}}+$$
$$(Px_1,y_3)_{\mathcal{H}}+(Qx_2,y_4)_{\mathcal{H}}=$$
$$-(x_3,Py_1)_{\mathcal{H}}-(x_4,Qy_2)_{\mathcal{H}}-((-P)^{1/2}x_1,(-P)^{1/2}y_3)_{\mathcal{H}}-$$
$$((-Q)^{1/2}x_2,(-Q)^{1/2}y_4)_{\mathcal{H}}=$$
$$-\left(\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix},\begin{pmatrix}0&0&I&0\\0&0&0&I\\P&0&0&0\\0&Q&0&0\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}\right)_{\mathcal{Z}}=(\boldsymbol{x},-\boldsymbol{M}_0\boldsymbol{y})_{\mathcal{Z}}.$$

因此, $\boldsymbol{M}_0$ 是空间 $\mathcal{Z}$ 中的反自伴算子,即 $\boldsymbol{M}_0^*=-\boldsymbol{M}_0$.根据 Stone 定理[12], $\boldsymbol{M}_0$ 生成 $\mathcal{Z}$ 上的酉算子群.由 S,T 的定义可知,$\boldsymbol{M}_1$对 $\boldsymbol{M}_0$ 相对有界且相对界小于 1.因此,由 C_0 半群生成元的扰动定理可知,$\boldsymbol{M}$ 能生成 $\mathcal{Z}$ 上的压缩半群.

3 板弯曲问题的半群方法

首先,给出 $\boldsymbol{M}$ 所生成压缩半群的具体表达式,进而给出抽象 Cauchy 问题(3)的解.

令 $\lambda>0$.对任意的 $\boldsymbol{u}=(u_1,u_2,u_3,u_4)^{\mathrm{T}}\in\mathcal{Z}$, 考虑方程

$$(\lambda-\boldsymbol{M})\boldsymbol{v}=\boldsymbol{u},\ \boldsymbol{v}=(v_1,v_2,v_3,v_4)^{\mathrm{T}}\in\mathcal{D}(\boldsymbol{M}),$$

即

$$\begin{pmatrix} \lambda & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 \\ -\dfrac{\partial^2}{\partial y^2} & 0 & \lambda - 2\mathrm{i}\dfrac{\partial}{\partial y} & -1 \\ 0 & -\dfrac{\partial^2}{\partial y^2} & 0 & \lambda - 2\mathrm{i}\dfrac{\partial}{\partial y} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}. \tag{4}$$

将方程(4)中的函数 $\boldsymbol{u}(y)$ 和 $\boldsymbol{v}(y)$ 在 $(-\infty, +\infty)$ 上作延拓(不妨仍用原来的记号记延拓后的函数):

$$\boldsymbol{u}(y) = \begin{cases} \boldsymbol{u}(y), & y \in (0,1), \\ \boldsymbol{0}, & y \notin (0,1), \end{cases} \quad \boldsymbol{v}(y) = \begin{cases} \boldsymbol{v}(y), & y \in (0,1), \\ \boldsymbol{0}, & y \notin (0,1), \end{cases} \tag{5}$$

并以 $\mathscr{F}(\boldsymbol{v}(y)) = \boldsymbol{V}(\omega)$, $\mathscr{F}(\boldsymbol{u}(y)) = \boldsymbol{U}(\omega)$ 分别表示 $\boldsymbol{v}(y)$, $\boldsymbol{u}(y)$ 的 Fourier 变换.对方程(4)两端作 Fourier 变换可得

$$\begin{pmatrix} \lambda & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 \\ \omega^2 & 0 & \lambda + 2\omega & -1 \\ 0 & \omega^2 & 0 & \lambda + 2\omega \end{pmatrix} \begin{pmatrix} V_1(\omega) \\ V_2(\omega) \\ V_3(\omega) \\ V_4(\omega) \end{pmatrix} = \begin{pmatrix} U_1(\omega) \\ U_2(\omega) \\ U_3(\omega) \\ U_4(\omega) \end{pmatrix},$$

即

$$\mathscr{F}[(R(\lambda;\boldsymbol{M})\boldsymbol{u})(y)] = \boldsymbol{V}(\omega) = \frac{1}{(\lambda+\omega)^2} \begin{pmatrix} \lambda + 2\omega & -\dfrac{\omega^2}{(\lambda+\omega)^2} & 1 & \dfrac{\lambda}{(\lambda+\omega)^2} \\ 0 & \lambda + 2\omega & 0 & 1 \\ -\omega^2 & -\dfrac{\lambda\omega^2}{(\lambda+\omega)^2} & \lambda & \dfrac{\lambda^2}{(\lambda+\omega)^2} \\ 0 & -\omega^2 & 0 & \lambda \end{pmatrix} \boldsymbol{U}(\omega).$$

进一步可得

$$\mathscr{F}[(R(\lambda;\boldsymbol{M})^n\boldsymbol{u})(y)] = \frac{1}{(\lambda+\omega)^{2n}} \begin{pmatrix} \lambda + 2\omega & -\dfrac{\omega^2}{(\lambda+\omega)^2} & 1 & \dfrac{\lambda}{(\lambda+\omega)^2} \\ 0 & \lambda + 2\omega & 0 & 1 \\ -\omega^2 & -\dfrac{\lambda\omega^2}{(\lambda+\omega)^2} & \lambda & \dfrac{\lambda^2}{(\lambda+\omega)^2} \\ 0 & -\omega^2 & 0 & \lambda \end{pmatrix}^n \boldsymbol{U}(\omega),$$

进而

$$\begin{aligned} &\mathscr{F}[(\mathrm{e}^{x\boldsymbol{M}_\lambda}\boldsymbol{u})(y)] = \\ &\quad \mathrm{e}^{-\lambda x}\mathscr{F}[(\mathrm{e}^{x\lambda^2 R(\lambda;\boldsymbol{M})}\boldsymbol{u})(y)] = \\ &\quad \mathrm{e}^{-\lambda x}\mathscr{F}\left[\left(\sum_{n=0}^{\infty}\frac{x^n\lambda^{2n}}{n!}R(\lambda;\boldsymbol{M})^n\boldsymbol{u}\right)(y)\right] = \end{aligned}$$

$$\exp\begin{pmatrix} \frac{(\lambda+2\omega)x\lambda^2}{(\lambda+\omega)^2}-\lambda x & \frac{-x\lambda^2\omega^2}{(\lambda+\omega)^4} & \frac{x\lambda^2}{(\lambda+\omega)^2} & \frac{x\lambda^3}{(\lambda+\omega)^4} \\ 0 & \frac{(\lambda+2\omega)x\lambda^2}{(\lambda+\omega)^2}-\lambda x & 0 & \frac{x\lambda^2}{(\lambda+\omega)^2} \\ \frac{-x\lambda^2\omega^2}{(\lambda+\omega)^2} & \frac{-x\lambda^3\omega^2}{(\lambda+\omega)^4} & \frac{x\lambda^3}{(\lambda+\omega)^2}-\lambda x & \frac{x\lambda^4}{(\lambda+\omega)^4} \\ 0 & \frac{-x\lambda^2\omega^2}{(\lambda+\omega)^2} & 0 & \frac{x\lambda^3}{(\lambda+\omega)^2}-\lambda x \end{pmatrix}\boldsymbol{U}(\omega).$$

又因为

$$\mathcal{F}[(T(x)\boldsymbol{u})(y)] = \lim_{\lambda\to\infty}\mathcal{F}[(e^{x\boldsymbol{M}_\lambda}\boldsymbol{u})(y)] =$$

$$\exp\begin{pmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ -x\omega^2 & 0 & -2x\omega & x \\ 0 & -x\omega^2 & 0 & -2x\omega \end{pmatrix}\boldsymbol{U}(\omega) =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -\omega & 1 & 0 & 0 \\ 0 & 0 & -\omega & 0 \end{pmatrix}\exp\begin{pmatrix} -x\omega & x & 0 & 0 \\ 0 & -x\omega & -x\omega & 0 \\ 0 & 0 & -x\omega & x\omega \\ 0 & 0 & 0 & -x\omega \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ \omega & 0 & 1 & 0 \\ 0 & 0 & 0 & -1/\omega \\ 0 & 1 & 0 & 1/\omega \end{pmatrix}\boldsymbol{U}(\omega) =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -\omega & 1 & 0 & 0 \\ 0 & 0 & -\omega & 0 \end{pmatrix}\begin{pmatrix} e^{-x\omega} & xe^{-x\omega} & -\frac{x^2\omega}{2}e^{-x\omega} & -\frac{x^3\omega^2}{6}e^{-x\omega} \\ 0 & e^{-x\omega} & -x\omega e^{-x\omega} & -\frac{x^2\omega^2}{2}e^{-x\omega} \\ 0 & 0 & e^{-x\omega} & x\omega e^{-x\omega} \\ 0 & 0 & 0 & e^{-x\omega} \end{pmatrix}\times$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \omega & 0 & 1 & 0 \\ 0 & 0 & 0 & -1/\omega \\ 0 & 1 & 0 & 1/\omega \end{pmatrix}\boldsymbol{U}(\omega) =$$

$$
\begin{pmatrix}
e^{-x\omega}+x\omega e^{-x\omega} & -\dfrac{x^3\omega^2}{6}e^{-x\omega} & xe^{-x\omega} & \dfrac{x^2}{2}e^{-x\omega}-\dfrac{x^3\omega}{6}e^{-x\omega} \\
0 & x\omega e^{-x\omega}+e^{-x\omega} & 0 & x\omega e^{-x\omega} \\
-x\omega^2 e^{-x\omega} & \dfrac{x^3\omega^3}{6}e^{-x\omega}-\dfrac{x^2\omega^2}{2}e^{-x\omega} & -x\omega e^{-x\omega}+e^{-x\omega} & xe^{-x\omega}-x^2\omega e^{-x\omega}+\dfrac{x^3\omega^2}{6}e^{-x\omega} \\
0 & -x\omega^2 e^{-x\omega} & 0 & e^{-x\omega}-x\omega e^{-x\omega}
\end{pmatrix}\boldsymbol{U}(\omega)
$$

且 $\mathcal{F}(e^{-\omega x})=x/(\pi(x^2+y^2))$，采用卷积公式可得 $(T(x)\boldsymbol{u})(y)=(T_1,T_2,T_3,T_4)^{\mathrm{T}}$，其中

$$
\begin{aligned}
T_1 = {} & \int_0^1 \frac{xu_1(\xi)}{\pi(x^2+(y-\xi)^2)}d\xi + \int_0^1 \frac{-ix^2u_1'(\xi)}{\pi(x^2+(y-\xi)^2)}d\xi + \\
& \int_0^1 \frac{x^4u_2''(\xi)}{6\pi(x^2+(y-\xi)^2)}d\xi + \int_0^1 \frac{x^2u_3(\xi)}{\pi(x^2+(y-\xi)^2)}d\xi + \\
& \int_0^1 \frac{x^3u_4(\xi)}{2\pi(x^2+(y-\xi)^2)}d\xi + \int_0^1 \frac{ix^4u_4'(\xi)}{6\pi(x^2+(y-\xi)^2)}d\xi,
\end{aligned}
$$

$$
\begin{aligned}
T_2 = {} & \int_0^1 \frac{-ix^2u_2'(\xi)}{\pi(x^2+(y-\xi)^2)}d\xi + \int_0^1 \frac{xu_2(\xi)}{\pi(x^2+(y-\xi)^2)}d\xi + \\
& \int_0^1 \frac{-ix^2u_4'(\xi)}{\pi(x^2+(y-\xi)^2)}d\xi,
\end{aligned}
$$

$$
\begin{aligned}
T_3 = {} & \int_0^1 \frac{x^2u_1''(\xi)}{\pi(x^2+(y-\xi)^2)}d\xi + \int_0^1 \frac{ix^4u_2''(\xi)}{6\pi(x^2+(y-\xi)^2)}d\xi + \\
& \int_0^1 \frac{x^3u_2''(\xi)}{2\pi(x^2+(y-\xi)^2)}d\xi + \int_0^1 \frac{ix^2u_3'(\xi)}{\pi(x^2+(y-\xi)^2)}d\xi + \\
& \int_0^1 \frac{xu_3(\xi)}{\pi(x^2+(y-\xi)^2)}d\xi + \int_0^1 \frac{x^2u_4(\xi)}{\pi(x^2+(y-\xi)^2)}d\xi + \\
& \int_0^1 \frac{ix^3u_4'(\xi)}{\pi(x^2+(y-\xi)^2)}d\xi + \int_0^1 \frac{-x^4u_4''(\xi)}{6\pi(x^2+(y-\xi)^2)}d\xi,
\end{aligned}
$$

$$
\begin{aligned}
T_4 = {} & \int_0^1 \frac{x^2u_2''(\xi)}{\pi(x^2+(y-\xi)^2)}d\xi + \int_0^1 \frac{xu_4(\xi)}{\pi(x^2+(y-\xi)^2)}d\xi + \\
& \int_0^1 \frac{ix^2u_4'(\xi)}{\pi(x^2+(y-\xi)^2)}d\xi.
\end{aligned}
$$

当 $\boldsymbol{V}(0)\in\mathcal{D}(\boldsymbol{M})$ 时，方程(3)的解为

$$
\boldsymbol{V}(x)=T(x)\boldsymbol{V}(0)+\int_0^x T(x-s)\boldsymbol{F}(s)ds,
$$

即

$$\boldsymbol{V}(x)=$$

$$D\begin{pmatrix}\int_0^1\frac{2\mathrm{i}x\varphi'(\xi)}{\pi(x^2+(y-\xi)^2)}\mathrm{d}\xi+\int_0^1\frac{2x^2\varphi''(\xi)}{\pi(x^2+(y-\xi)^2)}\mathrm{d}\xi+\\ \int_0^1\frac{x^2\varphi''(\xi)}{\pi(x^2+(y-\xi)^2)}\mathrm{d}\xi\\ 0\\ \int_0^1\frac{2\mathrm{i}x^2\varphi'''(\xi)}{\pi(x^2+(y-\xi)^2)}\mathrm{d}\xi+\int_0^1\frac{\mathrm{i}x^2\varphi'''(\xi)}{\pi(x^2+(y-\xi)^2)}\mathrm{d}\xi+\\ \int_0^1\frac{x\varphi''(\xi)}{\pi(x^2+(y-\xi)^2)}\mathrm{d}\xi\\ 0\end{pmatrix}+$$

$$\begin{pmatrix}\int_0^x\int_0^1\frac{s^2f(x-s,\xi)}{\pi(s^2+(y-\xi)^2)}\mathrm{d}\xi\mathrm{d}s\\ 0\\ \int_0^x\int_0^1\frac{\mathrm{i}s^2f'(x-s,\xi)}{\pi(s^2+(y-\xi)^2)}\mathrm{d}\xi\mathrm{d}s+\int_0^x\int_0^1\frac{sf(x-s,\xi)}{\pi(s^2+(y-\xi)^2)}\mathrm{d}\xi\mathrm{d}s\\ 0\end{pmatrix}.$$

因此

$$p=D\int_0^1\frac{2\mathrm{i}x\varphi'(\xi)}{\pi(x^2+(y-\xi)^2)}\mathrm{d}\xi+D\int_0^1\frac{2x^2\varphi''(\xi)}{\pi(x^2+(y-\xi)^2)}\mathrm{d}\xi+$$
$$D\int_0^1\frac{x^2\varphi''(\xi)}{\pi(x^2+(y-\xi)^2)}\mathrm{d}\xi+\int_0^x\int_0^1\frac{s^2f(x-s,\xi)}{\pi(s^2+(y-\xi)^2)}\mathrm{d}\xi\mathrm{d}s.$$

显然, p 满足关系式(2).因此,将方程

$$\frac{\partial^2 w}{\partial x^2}+2\mathrm{i}\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial^2 w}{\partial y^2}-\frac{1}{D}p=0 \tag{6}$$

转换为如下微分系统:

$$\frac{\partial}{\partial x}\begin{pmatrix}w\\ \hat{w}\end{pmatrix}=\begin{pmatrix}-\mathrm{i}\frac{\partial}{\partial y} & I\\ 0 & -\mathrm{i}\frac{\partial}{\partial y}\end{pmatrix}\begin{pmatrix}w\\ \hat{w}\end{pmatrix}+\begin{pmatrix}0\\ \frac{1}{D}p\end{pmatrix},$$

其中 $\hat{w}=\partial w/\partial x+\mathrm{i}\partial w/\partial y$.定义 $\mathscr{X}=L^2(0,1)$ 中的微分算子

$$A=D=-\mathrm{i}\frac{\partial}{\partial y},\ \mathscr{D}(A)=\mathscr{D}(D)=\{v\in\mathscr{X}:\ v'\in\mathscr{X},v(0)=v(1)\},$$

并记 $\hat{\boldsymbol{V}}(x)=(w,\ \hat{w})^{\mathrm{T}}$, $\boldsymbol{P}(x)=(0,p/D)^{\mathrm{T}}$, 则方程(6) 可以转换成空间 $\mathscr{X}\times\mathscr{X}$ 中的抽象 Cauchy 问题:

$$\begin{cases}\dfrac{\mathrm{d}\hat{\boldsymbol{V}}(x)}{\mathrm{d}x}=\boldsymbol{M}_2\hat{\boldsymbol{V}}(x)+\boldsymbol{P}(x),\\ \hat{\boldsymbol{V}}(0)=(0,\varphi(y))^{\mathrm{T}},\end{cases} \tag{7}$$

其中

$$\boldsymbol{M}_2 = \begin{pmatrix} A & I \\ 0 & D \end{pmatrix}: \mathscr{D}(\boldsymbol{M}_2) = \mathscr{D}(A) \oplus \mathscr{D}(D) \to \mathscr{X} \oplus \mathscr{X},$$

$$\hat{\boldsymbol{V}}(x) = (w, \hat{w})^{\mathrm{T}},\ \boldsymbol{P}(x) = \left(0, \frac{1}{D} p\right)^{\mathrm{T}}.$$

定理 3.1 算子矩阵 $\boldsymbol{M}_2$ 生成 $\mathscr{X} \oplus \mathscr{X}$ 上的压缩半群.

证明 显然, $A = A^*$, $D = D^*$ 且

$$(Av_1, v_1)_{\mathscr{X}} = (Dv_2, v_2)_{\mathscr{X}} = 0,\ \boldsymbol{v} = (v_1, v_2)^{\mathrm{T}} \in \mathscr{D}(\boldsymbol{M}_2),$$

进而 A 和 D 能生成 $\mathscr{X}$ 上的压缩半群.因为 I 是单位算子,由 C_0 半群生成元的扰动定理可知,$\boldsymbol{M}_2$ 生成 $\mathscr{X} \oplus \mathscr{X}$ 上的压缩半群.

其次,给出问题(7)的解,进而给出问题(1)的解.

令 $\lambda > 0$.对任意的 $\boldsymbol{u} = (u_1, u_2)^{\mathrm{T}} \in \mathscr{X} \times \mathscr{X}$, 考虑方程

$$(\lambda I - \boldsymbol{M}_2)\boldsymbol{v} = \boldsymbol{u},\ \boldsymbol{v} = (v_1, v_2)^{\mathrm{T}} \in \mathscr{D}(\boldsymbol{M}_2),$$

即

$$\begin{cases} \lambda v_1 + \mathrm{i}v_1' - v_2 = u_1, \\ \lambda v_2 + \mathrm{i}v_2' = u_2. \end{cases} \tag{8}$$

将方程(8)中函数 $\boldsymbol{u}(y)$ 和 $\boldsymbol{v}(y)$ 在 $(-\infty, +\infty)$ 上作延拓(如式(5)),并对方程(8)两端作 Fourier 变换可得

$$\begin{cases} \lambda V_1(\omega) - \omega V_1(\omega) - V_2(\omega) = U_1(\omega), \\ \lambda V_2(\omega) - \omega V_2(\omega) = U_2(\omega). \end{cases} \tag{9}$$

由式(9)可得

$$\boldsymbol{V}(\omega) = \frac{1}{\lambda - \omega}\begin{pmatrix} 1 & \dfrac{1}{\lambda - \omega} \\ 0 & 1 \end{pmatrix} \boldsymbol{U}(\omega),$$

即

$$\mathscr{F}[(R(\lambda; \boldsymbol{M}_2)\boldsymbol{u})(y)] = \frac{1}{\lambda - \omega}\begin{pmatrix} 1 & \dfrac{1}{\lambda - \omega} \\ 0 & 1 \end{pmatrix} \boldsymbol{U}(\omega).$$

进一步可得

$$\mathscr{F}[(R(\lambda; \boldsymbol{M}_2)^n\boldsymbol{u})(y)] = \frac{1}{(\lambda - \omega)^n}\begin{pmatrix} 1 & \dfrac{1}{\lambda - \omega} \\ 0 & 1 \end{pmatrix}^n \boldsymbol{U}(\omega),$$

进而

$$\mathscr{F}[(\mathrm{e}^{x\boldsymbol{M}_{2,\lambda}}\boldsymbol{u})(y)] = \mathrm{e}^{-\lambda x}\mathscr{F}[(\mathrm{e}^{x\lambda^2 R(\lambda; \boldsymbol{M}_2)}\boldsymbol{u})(y)] =$$

$$\mathrm{e}^{-\lambda x}\mathscr{F}\left[\left(\sum_{n=0}^{\infty} \frac{x^n \lambda^{2n}}{n!} R(\lambda; \boldsymbol{M}_2)^n \boldsymbol{u}\right)(y)\right] =$$

$$\exp\begin{pmatrix} \dfrac{x\omega\lambda}{\lambda - \omega} & \dfrac{x\lambda^2}{(\lambda - \omega)^2} \\ 0 & \dfrac{x\omega\lambda}{\lambda - \omega} \end{pmatrix} \boldsymbol{U}(\omega).$$

因为

$$\mathcal{F}[(T(x)\boldsymbol{u})(y)]=\lim_{\lambda\to\infty}\mathcal{F}[(\mathrm{e}^{x\boldsymbol{M}_{2,\lambda}}\boldsymbol{u})(y)]=$$

$$\exp\begin{pmatrix} x\omega & x \\ 0 & x\omega \end{pmatrix}\boldsymbol{U}(\omega)=$$

$$\begin{pmatrix} \mathrm{e}^{x\omega} & x\mathrm{e}^{x\omega} \\ 0 & \mathrm{e}^{x\omega} \end{pmatrix}\boldsymbol{U}(\omega)=\begin{pmatrix} \mathrm{e}^{x\omega}U_1(\omega)+x\mathrm{e}^{x\omega}U_2(\omega) \\ \mathrm{e}^{x\omega}U_2(\omega) \end{pmatrix}$$

且 $\mathcal{F}[\mathrm{e}^{x\omega}]=-x/(\pi(x^2+y^2))$，采用卷积公式可得

$$(T(x)\boldsymbol{u})(y)=\begin{pmatrix} \int_0^1\frac{-x}{\pi(x^2+(y-\xi)^2)}u_1(\xi)\mathrm{d}\xi+\int_0^1\frac{-x^2}{\pi(x^2+(y-\xi)^2)}u_2(\xi)\mathrm{d}\xi \\ \int_0^1\frac{-x}{\pi(x^2+(y-\xi)^2)}u_2(\xi)\mathrm{d}\xi \end{pmatrix}.$$

当 $\hat{\boldsymbol{V}}(0)\in\mathcal{D}(M_2)$ 时，方程(7)的解为

$$\hat{\boldsymbol{V}}(x)=T(x)\hat{\boldsymbol{V}}(0)+\int_0^x T(x-s)\boldsymbol{P}(s)\mathrm{d}s,$$

即

$$\hat{\boldsymbol{V}}(x)=\begin{pmatrix} \int_0^1\frac{-x^2}{\pi(x^2+(y-\xi)^2)}\varphi(\xi)\mathrm{d}\xi \\ \int_0^1\frac{-x}{\pi(x^2+(y-\xi)^2)}\varphi(\xi)\mathrm{d}\xi \end{pmatrix}+\frac{1}{D}\begin{pmatrix} \int_0^x\int_0^1\frac{-s^2}{\pi(s^2+(y-\xi)^2)}p\Big|_{\substack{y\to\xi,\\ x\to x-s}}\mathrm{d}\xi\mathrm{d}s \\ \int_0^x\int_0^1\frac{-s}{\pi(s^2+(y-\xi)^2)}p\Big|_{\substack{y\to\xi,\\ x\to x-s}}\mathrm{d}\xi\mathrm{d}s \end{pmatrix},$$

其中 $p\Big|_{\substack{y\to\xi,\\ x\to x-s}}$ 是将 p 中的 y,x 分别用 $\xi,x-s$ 代换.因此，方程(1)的解为

$$w=\int_0^1\frac{-x^2}{\pi(x^2+(y-\xi)^2)}\varphi(\xi)\mathrm{d}\xi+\frac{1}{D}\int_0^x\int_0^1\frac{-s^2}{\pi(s^2+(y-\xi)^2)}p\Big|_{\substack{y\to\xi,\\ x\to x-s}}\mathrm{d}\xi\mathrm{d}s,$$

其中

$$p=D\int_0^1\frac{2\mathrm{i}x\varphi'(\xi)}{\pi(x^2+(y-\xi)^2)}\mathrm{d}\xi+D\int_0^1\frac{2x^2\varphi''(\xi)}{\pi(x^2+(y-\xi)^2)}\mathrm{d}\xi+$$

$$D\int_0^1\frac{x^2\varphi''(\xi)}{\pi(x^2+(y-\xi)^2)}\mathrm{d}\xi+\int_0^x\int_0^1\frac{s^2f(x-s,\xi)}{\pi(s^2+(y-\xi)^2)}\mathrm{d}\xi\mathrm{d}s.$$

4 结 论

本文运用算子半群方法求解了弹性理论中对边简支矩形薄板方程.证明了问题导出的算子矩阵在所构造的 Hilbert 空间中生成压缩半群.通过 Fourier 变换，采用一致连续半群逼近所生成的 C_0 半群，进而得到方程的解析解.与其它方法相比，该方法计算量不是很大，并且在给出方程解析解的同时也确保所得解的存在唯一性.板方程可以等价地转换成无穷多个抽象 Cauchy 问题[14].因此，该方法的关键点是所导出的算子矩阵能否在相应的空间中生成 C_0 半群.这与板方程的边界条件和算子矩阵的结构有紧密的联系.文中得到的算子矩阵和构造的空间框架，对求解板方程具有非常重要的应用价值.该方法为求解板方程提供了一个有效的途径.

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An Operator Semigroup Method for Rectangular Plates With 2 Opposite Sides Simply Supported

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Abstract: The problem of solving a rectangular thin plate with 2 opposite sides simply supported in elasticity theory by means of the operator semigroup method was addressed. First, the plate equations were transformed into the abstract Cauchy problem. Then, the Hilbert space was defined and it was proved that the corresponding operator matrix generates contraction semigroups. Finally, the uniformly continuous semigroup approximation was applied through the Fourier transform, and the analytical solutions to the equations were given. The method naturally implies the existence and uniqueness of the solution.

Key words: rectangular plate; abstract Cauchy problem; C_0 semigroup; analytical solution

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