文章编号:1000-0887(2015)09-0956-14

极坐标系下非分裂 PML 及时域有限元实现*

周凤玺[1,2], 曹小林[1], M·B·贾克萨[2]

(1. 兰州理工大学 土木工程学院, 兰州 730050;
2. 阿德莱德大学 土木环境与采矿工程学院, 阿德莱德 5005, 澳大利亚)

摘要: 在弹性波传播的数值模拟中,吸收边界被广泛应用于截取有限空间进行无限空间问题的分析.完全匹配层(perfect matched layer, PML)吸收边界较其它吸收边界条件具有更优越的吸收性能,已被成功应用于直角坐标系下的弹性波方程正演模拟.考虑极坐标系下二阶弹性波动方程,通过采用辅助函数的方法,提出了一种非分裂格式的完全匹配层吸收边界条件.并且基于 Galerkin 近似技术,给出了非对称以及轴对称条件下的时域有限元计算格式.通过数值算例分析了该极坐标系下分裂格式的完全匹配层吸收边界的有效性.

关 键 词: 完全匹配层; 极坐标系; 吸收边界条件; 时域有限元
中图分类号: O347.4 **文献标志码:** A
doi:10.3879/j.issn.1000-0887.2015.09.007

引 言

时域有限单元法是模拟弹性波传播特性的重要方法之一,在地震工程、岩土工程以及地球物理等领域有着十分广泛的应用.对于介质的空间尺度非常大,尤其是某些边界为无穷大的情况下,在计算机进行波动数值模拟时,由于受到计算机内存和计算耗时的限制,必须截取有限的空间区域进行模拟,这样就自然引入了人为的介质边界.当弹性波到达这些边界时产生的无实际意义的边界反射波场将会直接影响到对问题的分析和判断.为了减弱或消除上述影响,就必须在人工截止边界上附加一定的边界条件,通常被称为吸收边界条件,用来模拟外行波从近场有限域通过人工边界无反射地进入远场无限域这一波动单向传播的物理过程.迄今已有多种吸收边界被广泛应用[1-7],其中粘性边界条件[1-2]是最早的人工边界条件之一.由于粘性边界物理意义清楚,易于实现,尤其便于与商用有限元软件结合,因而被广泛应用于工程实践,但其模拟精度较低.Clayton-Engquist 边界[5]是最具代表性的局部人工边界条件之一,它采用 Padé 有理近似代替波动方程的单向频散关系,然后变换回时域,得到一系列描述外行波的高阶微分方程.理论和试验研究表明,外行波传播方向与人工边界外法向的夹角越小,其精度越高,并且其所能覆盖的透射角范围随着边界阶数的增加而增大;旁轴近似边界[6]比较容易实现且能完全吸收垂直入射波,但只在一定的角度和频率范围内能有效吸收反射波,且该方法对于 Ray-

* **收稿日期:** 2015-04-09; **修订日期:** 2015-05-06
基金项目: 国家自然科学基金(11162008;51368038);甘肃省环境保护厅科研基金(GSEP-2014-23);甘肃省教育厅研究生导师基金(1103-07)
作者简介: 周凤玺(1979—),男,甘肃人,副教授,博士,博士生导师(通讯作者. E-mail: geolut@163.com).

leigh(瑞利)面波的吸收效果不好[8].完全匹配层(PML)是由 Berenger[9]在计算电磁场波动问题中提出的一种新的吸收边界条件,在理论上该方法能完全吸收以任意角度和任意频率入射的波.因此,关于 PML 的研究取得了长足的进展,出现了不同形式和算法的完全匹配层吸收边界条件,如分裂形式(splitting PML, SPML)、非分裂形式(non-splitting PML, NPML)、卷积形式(convolutional PML, C-PML)和多轴形式(multiaxial PML, M-PML)等,且其已被成功应用到不同的波动模拟计算领域,如电磁波场波动方程、声波方程和弹性波方程等[10-13].

由于散射体结构的复杂性,在直角坐标中采用立方体或正方体网格单元来模拟复杂物体表面时会形成“阶梯”形边界,不仅会激励起表面波传播,引起数值频散,而且为了拟合小曲率半径,就要减小网格单元尺寸,相应地减少时间步长,从而大大地增加了计算存储量[14].因此,有必要将 PML 的研究从直角坐标系扩展到一般曲线坐标系.张洪欣等[14]、Teixeira 和 Chew[15-16]、He 和 Liu[17-18]将 PML 扩展到圆柱坐标系和球坐标系中,进行了电磁波的正演模拟.而关于弹性波在曲线坐标系内的 PML 吸收边界条件的研究成果并不多见.Liu[19]利用复伸展坐标变换,将传统的弹性波 PML 吸收条件由直角坐标系拓展到曲线坐标系,采用应力-速度分量,获得了分裂格式的 PML,并采用时域有限差分法求解了非均匀介质中弹性波的传播.同样利用复伸展坐标,Zheng 和 Huang[20]分析了圆柱坐标系和球坐标系下固体中弹性波传播的分裂格式的 PML 吸收边界条件,并给出了分别采用有限元法以及有限差分法的数值模拟算法.

本文以二维极坐标系下的非分裂 PML 为主要研究目标.从位移为基本未知量的弹性动力学控制方程出发,利用复伸展坐标变换,获得了含有辅助变量的 PML 方程,并给出了时域有限元计算格式.通过数值算例分析了该 PML 的有效性.

1 二维极坐标系下的非分裂 PML

1.1 基本方程

在极坐标系 (r,θ) 中,各向同性弹性介质的基本方程包括:

微分平衡方程

$$\begin{cases} \dfrac{\partial\sigma_r}{\partial r} + \dfrac{1}{r}\dfrac{\partial\tau_{r\theta}}{\partial\theta} + \dfrac{\sigma_r - \sigma_\theta}{r} + f_r = \dfrac{\partial^2 u_r}{\partial t^2}, \\ \dfrac{\partial\tau_{r\theta}}{\partial r} + \dfrac{1}{r}\dfrac{\partial\sigma_\theta}{\partial\theta} + 2\dfrac{\tau_{r\theta}}{r} + f_\theta = \dfrac{\partial^2 u_\theta}{\partial t^2}; \end{cases} \tag{1}$$

应力-应变关系

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \tau_{r\theta} \end{Bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \gamma_{r\theta} \end{Bmatrix} = \boldsymbol{D} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \gamma_{r\theta} \end{Bmatrix}; \tag{2}$$

几何关系

$$\begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \gamma_{r\theta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial r} & 0 \\ \dfrac{1}{r} & \dfrac{1}{r}\dfrac{\partial}{\partial\theta} \\ \dfrac{1}{r}\dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial r} - \dfrac{1}{r} \end{bmatrix} \begin{Bmatrix} u_r \\ u_\theta \end{Bmatrix} = \boldsymbol{L} \begin{Bmatrix} u_r \\ u_\theta \end{Bmatrix}, \tag{3}$$

式中，$\sigma_r,\sigma_\theta,\tau_{r\theta}$ 为应力分量；$\varepsilon_r,\varepsilon_\theta,\gamma_{r\theta}$ 为应变分量；u_r 和 u_θ 分别为位移分量；f_r,f_θ 表示体积力分量；λ 和 μ 表示 Lamé 常数。

结合方程(1)~(3)，略去体积力可得到由位移分量表示的控制方程为

$$\begin{cases}\dfrac{\partial^2 u_r}{\partial t^2}=\dfrac{\partial}{\partial r}\left[\lambda\left(\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial\theta}+\dfrac{u_r}{r}\right)+(\lambda+2\mu)\dfrac{\partial u_r}{\partial r}\right]+\\ \qquad\dfrac{\mu}{r}\dfrac{\partial}{\partial\theta}\left(\dfrac{1}{r}\dfrac{\partial u_r}{\partial\theta}+\dfrac{\partial u_\theta}{\partial r}-\dfrac{u_\theta}{r}\right)+\dfrac{2\mu}{r}\left(\dfrac{\partial u_r}{\partial r}-\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial\theta}-\dfrac{u_r}{r}\right),\\ \dfrac{\partial^2 u_\theta}{\partial t^2}=\dfrac{1}{r}\dfrac{\partial}{\partial\theta}\left[(\lambda+2\mu)\left(\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial\theta}+\dfrac{u_r}{r}\right)+\lambda\dfrac{\partial u_r}{\partial r}\right]+\\ \qquad\mu\left(\dfrac{\partial}{\partial r}+\dfrac{2}{r}\right)\left(\dfrac{1}{r}\dfrac{\partial u_r}{\partial\theta}+\dfrac{\partial u_\theta}{\partial r}-\dfrac{u_\theta}{r}\right).\end{cases}\tag{4}$$

1.2 时域内PML控制方程

吸收边界的基本思想是在研究区域的边界上加上一定厚度的吸收衰减层，使边界上传入吸收层的波随传播距离的增加而逐渐衰减，在边界处不产生反射，以此来达到消除边界反射的目的.PML吸收边界条件是将波动方程在层内进行复坐标变换，对于变换后的坐标方程及其解的形式是不变的，但是对于原坐标解是衰减的.传统的PML吸收边界条件中将Laplace变换域内的复伸展坐标定义为[12]

$$\tilde{r}=r+\frac{1}{s}\int_0^r\zeta_r(r')\,\mathrm{d}r'.\tag{5}$$

根据链式法则有

$$\frac{\partial}{\partial\tilde{r}}=\frac{s}{s+\zeta_r}\frac{\partial}{\partial r}=\frac{1}{\gamma_r}\frac{\partial}{\partial r},\tag{6}$$

式中，r 和 $\tilde{r}$ 分别为径向变换前、后的坐标；s 为Laplace变换参数；ζ_r 为 r 方向的衰减函数，且在截断的物理域内 $\zeta_r=0$，在PML区域 $\zeta_r>0$；γ_r 称为复拉伸函数.

与直角坐标下PML的处理不同，由于在极坐标系下，控制方程(4)中除了位移分量关于径向坐标 r 的导数项外，还含有坐标分量 r 的非导数项.为此，引入积分复变量

$$\bar{\zeta}_r=\frac{1}{r}\int_0^r\zeta_r(r')\,\mathrm{d}r'.\tag{7}$$

结合式(5)，则有

$$\tilde{r}=r\left(1+\frac{\bar{\zeta}_r}{s}\right)=r\bar{\gamma}_r.\tag{8}$$

将控制方程(4)进行Laplace变换后，考虑到式(6)和式(8)就可得到忽略了体积力后在复伸展坐标系下对应的控制方程：

$$\begin{aligned}s^2\hat{u}_r=&\frac{1}{\gamma_r}\frac{\partial}{\partial r}\left[(\lambda+2\mu)\frac{1}{\gamma_r}\frac{\partial\hat{u}_r}{\partial r}+\lambda\left(\frac{1}{\bar{\gamma}_r}\frac{1}{r}\frac{\partial\hat{u}_\theta}{\partial\theta}+\frac{1}{\bar{\gamma}_r}\frac{\hat{u}_r}{r}\right)\right]+\\&\frac{1}{\bar{\gamma}_r}\frac{\mu}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{\bar{\gamma}_r}\frac{1}{r}\frac{\partial\hat{u}_r}{\partial\theta}+\frac{1}{\gamma_r}\frac{\partial\hat{u}_\theta}{\partial r}-\frac{1}{\bar{\gamma}_r}\frac{\hat{u}_\theta}{r}\right)+\\&\frac{1}{\bar{\gamma}_r}\frac{2\mu}{r}\left(\frac{1}{\gamma_r}\frac{\partial\hat{u}_r}{\partial r}-\frac{1}{\bar{\gamma}_r}\frac{1}{r}\frac{\partial\hat{u}_\theta}{\partial\theta}-\frac{1}{\bar{\gamma}_r}\frac{\hat{u}_r}{r}\right),\end{aligned}\tag{9a}$$

$$s^2\hat{u}_\theta=\frac{1}{\bar{\gamma}_r}\frac{1}{r}\frac{\partial}{\partial\theta}\left[(\lambda+2\mu)\frac{1}{\bar{\gamma}_r}\frac{1}{r}\left(\frac{\partial\hat{u}_\theta}{\partial\theta}+\hat{u}_r\right)+\lambda\frac{1}{\gamma_r}\frac{\partial\hat{u}_r}{\partial r}\right]+$$
$$\mu\left(\frac{1}{\gamma_r}\frac{\partial}{\partial r}+\frac{1}{\bar{\gamma}_r}\frac{2}{r}\right)\left(\frac{1}{\bar{\gamma}_r}\frac{1}{r}\frac{\partial\hat{u}_r}{\partial\theta}+\frac{1}{\gamma_r}\frac{\partial\hat{u}_\theta}{\partial r}-\frac{1}{\bar{\gamma}_r}\frac{\hat{u}_\theta}{r}\right).\tag{9b}$$

对上式两边同乘复拉伸函数 γ_r，并且考虑到下列关系：

$$\begin{cases}s^2\gamma_r=s^2+s\zeta_r,\\ \dfrac{\gamma_z}{\gamma_r}=1+\dfrac{\zeta_z-\zeta_r}{s+\zeta_r},\\ \dfrac{\gamma_r}{\bar{\gamma}_r\bar{\gamma}_r}=1+\dfrac{\zeta_r-\bar{\zeta}_r}{s+\bar{\zeta}_r}+\dfrac{-\bar{\zeta}_r}{s+\bar{\zeta}_r}+\dfrac{\zeta_r-\bar{\zeta}_r}{s+\bar{\zeta}_r}\dfrac{-\bar{\zeta}_r}{s+\bar{\zeta}_r}.\end{cases}\tag{10}$$

则可以得到经过整理以后的控制方程：

$$(s^2+s\zeta_r)\hat{u}_r=\frac{\partial}{\partial r}\left[\lambda\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)+(\lambda+2\mu)\frac{\partial u_r}{\partial r}\right]+\frac{\mu}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)+$$
$$\frac{2\mu}{r}\left(\frac{\partial u_r}{\partial r}-\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}-\frac{u_r}{r}\right)+\frac{\partial}{\partial r}\left[(\lambda+2\mu)\frac{-\zeta_r}{s+\zeta_r}\frac{\partial\hat{u}_r}{\partial r}+\lambda\frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r}\frac{1}{r}\left(\frac{\partial\hat{u}_\theta}{\partial\theta}+\hat{u}_r\right)\right]+$$
$$\frac{\mu}{r}\frac{\partial}{\partial\theta}\left[\left(\frac{\zeta_r-\bar{\zeta}_r}{s+\bar{\zeta}_r}+\frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r}+\frac{\zeta_r-\bar{\zeta}_r}{s+\bar{\zeta}_r}\frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r}\right)\frac{1}{r}\left(\frac{\partial\hat{u}_r}{\partial\theta}-\hat{u}_\theta\right)+\frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r}\frac{\partial\hat{u}_\theta}{\partial r}\right]+$$
$$\frac{2\mu}{r}\left[\frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r}\frac{\partial\hat{u}_r}{\partial r}-\left(\frac{\zeta_r-\bar{\zeta}_r}{s+\bar{\zeta}_r}+\frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r}+\frac{\zeta_r-\bar{\zeta}_r}{s+\bar{\zeta}_r}\frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r}\right)\frac{1}{r}\left(\frac{\partial\hat{u}_\theta}{\partial\theta}+\hat{u}_r\right)\right],\tag{11a}$$

$$(s^2+s\zeta_r)\hat{u}_\theta=\frac{1}{r}\frac{\partial}{\partial\theta}\left[(\lambda+2\mu)\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)+\lambda\frac{\partial u_r}{\partial r}\right]+$$
$$\mu\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left[(\lambda+2\mu)\left(\frac{\zeta_r-\bar{\zeta}_r}{s+\bar{\zeta}_r}+\right.\right.$$
$$\left.\left.\frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r}+\frac{\zeta_r-\bar{\zeta}_r}{s+\bar{\zeta}_r}\frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r}\right)\frac{1}{r}\left(\frac{\partial\hat{u}_\theta}{\partial\theta}+\hat{u}_r\right)+\lambda\frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r}\frac{\partial\hat{u}_r}{\partial r}\right]+$$
$$\frac{2\mu}{r}\left[\left(\frac{\zeta_r-\bar{\zeta}_r}{s+\bar{\zeta}_r}+\frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r}+\frac{\zeta_r-\bar{\zeta}_r}{s+\bar{\zeta}_r}\frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r}\right)\frac{1}{r}\left(\frac{\partial\hat{u}_r}{\partial\theta}-\hat{u}_\theta\right)+\frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r}\frac{\partial\hat{u}_\theta}{\partial r}\right]+$$
$$\mu\frac{\partial}{\partial r}\left(\frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r}\frac{1}{r}\frac{\partial\hat{u}_r}{\partial\theta}-\frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r}\frac{\hat{u}_\theta}{r}+\frac{-\zeta_r}{s+\zeta_r}\frac{\partial\hat{u}_\theta}{\partial r}\right).\tag{11b}$$

为了得到时间域内非分裂格式的 PML 方程，引入下列 Laplace 域内的辅助变量 $\hat{\boldsymbol{\psi}}=\{\hat{\psi}_{r1},\hat{\psi}_{\theta1},\hat{\psi}_{r2},\hat{\psi}_{\theta2},\hat{\psi}_{r3},\hat{\psi}_{\theta3},\hat{\psi}_{r4},\hat{\psi}_{\theta4}\}$，其元素见附录 A．

将辅助函数(A1)~(A8)代入方程组(11)可得

$$(s^2+s\zeta_r)\rho\hat{u}_r=\frac{\partial}{\partial r}\left[\lambda\left(\frac{1}{r}\frac{\partial\hat{u}_\theta}{\partial\theta}+\frac{\hat{u}_r}{r}\right)+(\lambda+2\mu)\frac{\partial\hat{u}_r}{\partial r}\right]+$$
$$\frac{\mu}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\hat{u}_r}{\partial\theta}+\frac{\partial\hat{u}_\theta}{\partial r}-\frac{\hat{u}_\theta}{r}\right)+\frac{2\mu}{r}\left(\frac{\partial\hat{u}_r}{\partial r}-\frac{1}{r}\frac{\partial\hat{u}_\theta}{\partial\theta}-\frac{\hat{u}_r}{r}\right)+$$

$$\frac{\partial}{\partial r}[(\lambda+2\mu)\hat{\psi}_{r1}+\lambda\hat{\psi}_{r2}]+\mu\frac{\partial\hat{\psi}_{r3}}{\partial\theta}-2\mu\hat{\psi}_{r4}, \tag{12a}$$

$$(s^2+s\zeta_r)\rho\hat{u}_\theta=\frac{1}{r}\frac{\partial}{\partial\theta}\left[(\lambda+2\mu)\left(\frac{1}{r}\frac{\partial\hat{u}_\theta}{\partial\theta}+\frac{\hat{u}_r}{r}\right)+\lambda\frac{\partial\hat{u}_r}{\partial r}\right]+\mu\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\left(\frac{1}{r}\frac{\partial\hat{u}_r}{\partial\theta}+\frac{\partial\hat{u}_\theta}{\partial r}-\frac{\hat{u}_\theta}{r}\right)+\frac{\partial}{\partial\theta}[(\lambda+2\mu)\hat{\psi}_{\theta1}+\lambda\hat{\psi}_{\theta2}]+\mu\frac{\partial\hat{\psi}_{\theta3}}{\partial r}-2\mu\hat{\psi}_{\theta4}. \tag{12b}$$

对方程(12)进行 Laplace 逆变换,即可得到时间域内具有非分裂式 PML 吸收边界的控制方程:

$$\rho\frac{\partial^2u_r}{\partial t^2}+\rho\zeta_r\frac{\partial u_r}{\partial t}=\frac{\partial}{\partial r}\left[\lambda\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)+(\lambda+2\mu)\frac{\partial u_r}{\partial r}\right]+\frac{\mu}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)+\frac{2\mu}{r}\left(\frac{\partial u_r}{\partial r}-\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}-\frac{u_r}{r}\right)+\frac{\partial}{\partial r}[(\lambda+2\mu)\psi_{r1}+\lambda\psi_{r2}]+\mu\frac{\partial\psi_{r3}}{\partial\theta}-2\mu\psi_{r4}, \tag{13a}$$

$$\rho\frac{\partial^2u_\theta}{\partial t^2}+\rho\zeta_r\frac{\partial u_\theta}{\partial t}=\frac{1}{r}\frac{\partial}{\partial\theta}\left[(\lambda+2\mu)\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)+\lambda\frac{\partial u_r}{\partial r}\right]+\mu\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)+\frac{\partial}{\partial\theta}[(\lambda+2\mu)\psi_{\theta1}+\lambda\psi_{\theta2}]+\mu\frac{\partial\psi_{\theta3}}{\partial r}-2\mu\psi_{\theta4}. \tag{13b}$$

可以看出,当在截取的物理域内 $\zeta_r=0$ 时,控制方程(13)即退化为极坐标系下的控制方程(4).关于辅助函数 $\hat{\boldsymbol{\psi}}$ 的时间域变换见附录 A.

1.3 时域有限元实现

根据有限单元法,对弹性体进行空间离散,离散后各单元内任一点的位移可用节点位移表示为[21]

$$\boldsymbol{u}(r,\theta,t)=\boldsymbol{N}(r,\theta)\boldsymbol{d}^{\mathrm{e}}(t). \tag{14}$$

式中, $\boldsymbol{N}=[N_1\boldsymbol{I},\cdots,N_m\boldsymbol{I}]$ 为形函数; $\boldsymbol{d}^{\mathrm{e}}=\{\boldsymbol{d}^1,\cdots,\boldsymbol{d}^m\}^{\mathrm{T}}$ 为单元各节点位移矢量,其中 $\boldsymbol{d}^k=\{u_r^k\quad u_\theta^k\}^{\mathrm{T}}$ 表示单元体第 k 个节点上的径向位移和环向位移.

将控制方程(13)进行 Galerkin 有限元法近似后,通过整理可以得到时域内的动力有限元格式为

$$\boldsymbol{M}\ddot{\boldsymbol{d}}+\boldsymbol{Z}\dot{\boldsymbol{d}}+\boldsymbol{K}\boldsymbol{d}=\boldsymbol{F}+\boldsymbol{Q}, \tag{15}$$

其中, $\boldsymbol{M}$, $\boldsymbol{Z}$ 和 $\boldsymbol{K}$ 分别为整体质量矩阵、因 PML 引起的整体阻尼矩阵以及整体刚度矩阵; $\boldsymbol{F}$ 和 $\boldsymbol{Q}$ 分别为节点荷载矢量以及由于辅助函数 $\boldsymbol{\psi}$ 引起的类似于结点荷载矢量.它们分别由各单元的质量矩阵、阻尼矩阵、刚度矩阵、荷载矢量组装而成.

$$\begin{cases}\boldsymbol{M}^{\mathrm{e}}=\int_{A^{\mathrm{e}}}\boldsymbol{N}^{\mathrm{T}}\rho\boldsymbol{N}r\mathrm{d}r\mathrm{d}\theta,\ \boldsymbol{Z}^{\mathrm{e}}=\int_{A^{\mathrm{e}}}\boldsymbol{N}^{\mathrm{T}}\rho\zeta_r\boldsymbol{N}r\mathrm{d}r\mathrm{d}\theta,\ \boldsymbol{K}^{\mathrm{e}}=\int_{A^{\mathrm{e}}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{B}r\mathrm{d}r\mathrm{d}\theta,\\ \boldsymbol{F}^{\mathrm{e}}=\int_{A^{\mathrm{e}}}\boldsymbol{N}^{\mathrm{T}}\boldsymbol{q}r\mathrm{d}r\mathrm{d}\theta,\ \boldsymbol{Q}^{\mathrm{e}}=\int_{A^{\mathrm{e}}}\boldsymbol{H}r\mathrm{d}r\mathrm{d}\theta\bar{\boldsymbol{\psi}}^{\mathrm{e}},\end{cases} \tag{16}$$

其中，$\boldsymbol{q}$ 表示计算域内弹性体受到的荷载；$\boldsymbol{B}=\boldsymbol{LN}$. 值得指出的是方程(13)在推导过程中考虑了计算区域外侧为 Dirichlet 边界条件，这里取 $\boldsymbol{d}=\boldsymbol{0}$，这样的假设将有利于长时过程中对时间积分的稳定性[12].

式(16)中

$$\boldsymbol{H}=-\begin{bmatrix}(\lambda+2\mu)\frac{\partial N_1}{\partial r} & 0 & \lambda\frac{\partial N_1}{\partial r} & 0 & \mu\frac{\partial N_1}{\partial\theta} & 0 & -2\mu N_1 & 0\\ 0 & (\lambda+2\mu)\frac{\partial N_1}{\partial\theta} & 0 & \lambda\frac{\partial N_1}{\partial\theta} & 0 & \mu\frac{\partial N_1}{\partial r} & 0 & -2\mu N_1\\ (\lambda+2\mu)\frac{\partial N_2}{\partial r} & 0 & \lambda\frac{\partial N_2}{\partial r} & 0 & \mu\frac{\partial N_2}{\partial\theta} & 0 & -2\mu N_2 & 0\\ 0 & (\lambda+2\mu)\frac{\partial N_2}{\partial\theta} & 0 & \lambda\frac{\partial N_2}{\partial\theta} & 0 & \mu\frac{\partial N_2}{\partial r} & 0 & -2\mu N_2\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ (\lambda+2\mu)\frac{\partial N_m}{\partial r} & 0 & \lambda\frac{\partial N_m}{\partial r} & 0 & \mu\frac{\partial N_m}{\partial\theta} & 0 & -2\mu N_m & 0\\ 0 & (\lambda+2\mu)\frac{\partial N_m}{\partial\theta} & 0 & \lambda\frac{\partial N_m}{\partial\theta} & 0 & \mu\frac{\partial N_m}{\partial r} & 0 & -2\mu N_m\end{bmatrix},$$

$\bar{\boldsymbol{\psi}}^{\mathrm{e}}=\{\bar{\psi}_{r1}^{\mathrm{e}},\bar{\psi}_{\theta1}^{\mathrm{e}},\cdots,\bar{\psi}_{r4}^{\mathrm{e}},\bar{\psi}_{\theta4}^{\mathrm{e}}\}^{\mathrm{T}}$，$\bar{\psi}_i^{\mathrm{e}}=\sum_{k=1}^{m}\psi_i^{\mathrm{e}}/m$ 为辅助函数在单元内的平均值.

2　轴对称情形下非分裂 PML 方程

极坐标系下轴对称问题是一种特殊情形，由于轴对称性，应力和位移仅是 r 的函数，与 θ 无关.则问题的基本方程变为：

运动平衡方程

$$\frac{\partial^2 u_r}{\partial t^2}=\frac{\partial\sigma_r}{\partial r}+\frac{\sigma_r-\sigma_\theta}{r}+f_r;\tag{17}$$

应力应变关系

$$\begin{Bmatrix}\sigma_r\\ \sigma_\theta\end{Bmatrix}=\begin{bmatrix}\lambda+2\mu & \lambda\\ \lambda & \lambda+2\mu\end{bmatrix}\begin{Bmatrix}\varepsilon_r\\ \varepsilon_\theta\end{Bmatrix}=\boldsymbol{D}\begin{Bmatrix}\varepsilon_r\\ \varepsilon_\theta\end{Bmatrix};\tag{18}$$

几何关系

$$\begin{Bmatrix}\varepsilon_r\\ \varepsilon_\theta\end{Bmatrix}=\begin{bmatrix}\frac{\partial}{\partial r}\\ \frac{1}{r}\end{bmatrix}\{u_r\}=\boldsymbol{L}\{u_r\}.\tag{19}$$

通过基本方程(17)~(19)，可以得到用位移表示的控制方程：

$$\frac{\partial^2 u_r}{\partial t^2}=\frac{\partial}{\partial r}\left[(\lambda+2\mu)\frac{\partial u_r}{\partial r}+\lambda\frac{u_r}{r}\right]+\frac{2\mu}{r}\left(\frac{\partial u_r}{\partial r}-\frac{u_r}{r}\right).\tag{20}$$

同样，结合式(6)、(8)和(10)，对经过 Laplace 变换后的控制方程(20)两边同乘复拉伸函数 γ_r，并通过引入辅助函数 $\hat{\boldsymbol{\psi}}=\{\hat{\psi}_{r1},\hat{\psi}_{r2},\hat{\psi}_{r3}\}$（各元素见附录 B）后，最后可得到变换到时间

域的具有 PML 边界条件的控制方程为

$$\left(\rho\frac{\partial^2 u_r}{\partial t^2}+\rho\zeta_r\frac{\partial u_r}{\partial t}\right)=\frac{\partial}{\partial r}\left[(\lambda+2\mu)\frac{\partial u_r}{\partial r}+\lambda\frac{u_r}{r}\right]+\frac{2\mu}{r}\left(\frac{\partial u_r}{\partial r}-\frac{u_r}{r}\right)+\frac{\partial}{\partial r}[(\lambda+2\mu)\psi_{r1}+\lambda\psi_{r2}]-2\mu\psi_{r3}. \tag{21}$$

通过 Galerkin 有限元法近似后可以得到与式(15)形式相同的用位移表示的经典有限元格式.对于轴对称问题,空间离散后单元的质量矩阵、阻尼矩阵、刚度矩阵和荷载矢量分别为

$$\begin{cases}\boldsymbol{M}^{\mathrm e}=2\pi\int_{L^{\mathrm e}}\boldsymbol{N}^{\mathrm T}\rho\boldsymbol{N}r\mathrm{d}r,\ \boldsymbol{Z}^{\mathrm e}=2\pi\int_{L^{\mathrm e}}\boldsymbol{N}^{\mathrm T}\rho\zeta_r\boldsymbol{N}r\mathrm{d}r,\ \boldsymbol{K}^{\mathrm e}=2\pi\int_{L^{\mathrm e}}\boldsymbol{B}^{\mathrm T}\boldsymbol{D}\boldsymbol{B}r\mathrm{d}r,\\ \boldsymbol{F}^{\mathrm e}=2\pi\int_{L^{\mathrm e}}\boldsymbol{N}^{\mathrm T}\boldsymbol{q}r\mathrm{d}r,\ \boldsymbol{Q}^{\mathrm e}=2\pi\int_{L^{\mathrm e}}\boldsymbol{H}r\mathrm{d}r\bar{\boldsymbol{\psi}}^{\mathrm e},\end{cases} \tag{22}$$

其中

$$\boldsymbol{H}=-\begin{bmatrix}(\lambda+2\mu)\frac{\partial N_1}{\partial r} & \lambda\frac{\partial N_1}{\partial r} & -2\mu N_1\\ (\lambda+2\mu)\frac{\partial N_2}{\partial r} & \lambda\frac{\partial N_2}{\partial r} & -2\mu N_2\\ \vdots & \vdots & \vdots\\ (\lambda+2\mu)\frac{\partial N_m}{\partial r} & \lambda\frac{\partial N_m}{\partial r} & -2\mu N_m\end{bmatrix},$$

$\bar{\boldsymbol{\psi}}^{\mathrm e}=\{\bar\psi^{\mathrm e}_{r1},\bar\psi^{\mathrm e}_{r2},\bar\psi^{\mathrm e}_{r3}\}^{\mathrm T}$,$\bar\psi^{\mathrm e}_{ri}=\sum_{k=1}^{m}\psi^{\mathrm e}_{ri}/m$ 为辅助函数在单元内的平均值.

3 数值算例

关于衰减函数 ζ_r 可以为一次、二次或者高次的多种经验公式.本文采用 Collino 和 Tsogka[22] 使用的多项式函数来设置衰减函数 ζ_r:

$$\zeta_r(r)=\begin{cases}0, & |r|<a,\\ \dfrac{3c_{\max}\ln R}{2L}\left(\dfrac{r-a}{L}\right)^m, & a\leqslant|r|\leqslant a+L,\end{cases} \tag{23}$$

式中,L 为 PML 区域的厚度,m 为多项式阶数,a 为规则计算区域的半径,R 为理论发射系数,$c_{\max}$ 为弹性波最大波速.本文在算例中取 $m=2$,$R=10^{-8}$.

这里采用 4 节点扇形单元进行空间离散(如图 1),对于半径差为 Δr,圆心角为 $\Delta\theta$ 的单元采用如下二次线性函数进行插值,即

$$\boldsymbol{N}=[N_1\boldsymbol{I}\quad N_2\boldsymbol{I}\quad N_3\boldsymbol{I}\quad N_4\boldsymbol{I}], \tag{24}$$

其中

$$\begin{cases}N_1(r,\theta)=\dfrac{(r-r_1)(\theta_2-\theta)}{\Delta r\Delta\theta},\ N_2(r,\theta)=\dfrac{(r-r_1)(\theta-\theta_1)}{\Delta r\Delta\theta},\\ N_3(r,\theta)=\dfrac{(r_2-r)(\theta-\theta_1)}{\Delta r\Delta\theta},\ N_4(r,\theta)=\dfrac{(r_2-r)(\theta_2-\theta)}{\Delta r\Delta\theta}.\end{cases}$$

3.1 含圆孔的无限大平板受内压作用

考虑如图 2 所示的含有圆孔的无限大平板在内压作用下的轴对称平面应变问题.材料的

弹性模量为 $E = 2 \times 10^6$ N/m²，Poisson（泊松）比为 $\nu = 0.3$，材料密度为 $\rho = 2\,000$ kg/m³．假定圆孔的半径为 0.1 m，规则计算区域的半径为 4.1 m，PML 厚度设置为 2 m．内径处受到 Ricker 子波形式的压力为

$$p(t) = [1 - 2(\pi f_0 t)^2]\exp[-(\pi f_0 t)^2], \tag{25}$$

其中，f_0 为 Ricker 子波主频率，取 30 Hz．其时域和频域变化如图 3 所示．

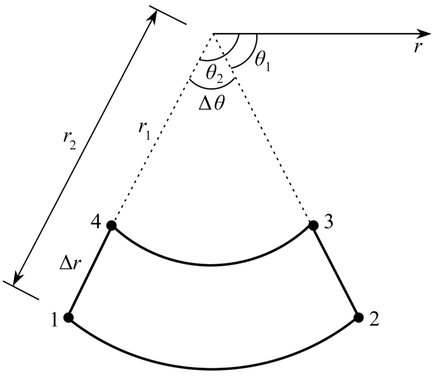


图 1　二维极坐标下标准的 4 结点有限单元

Fig. 1　A normal 4-node finite element in 2D polar coordinates

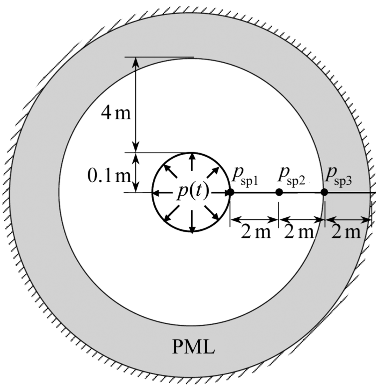


图 2　含圆孔的平板及 PML 示意图

Fig. 2　An infinite plate with a circular hole and the corresponding PML

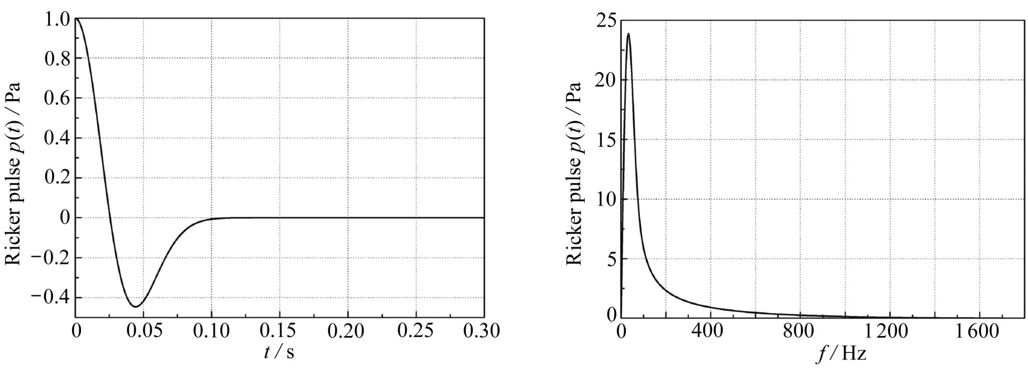


图 3　动荷载示意图

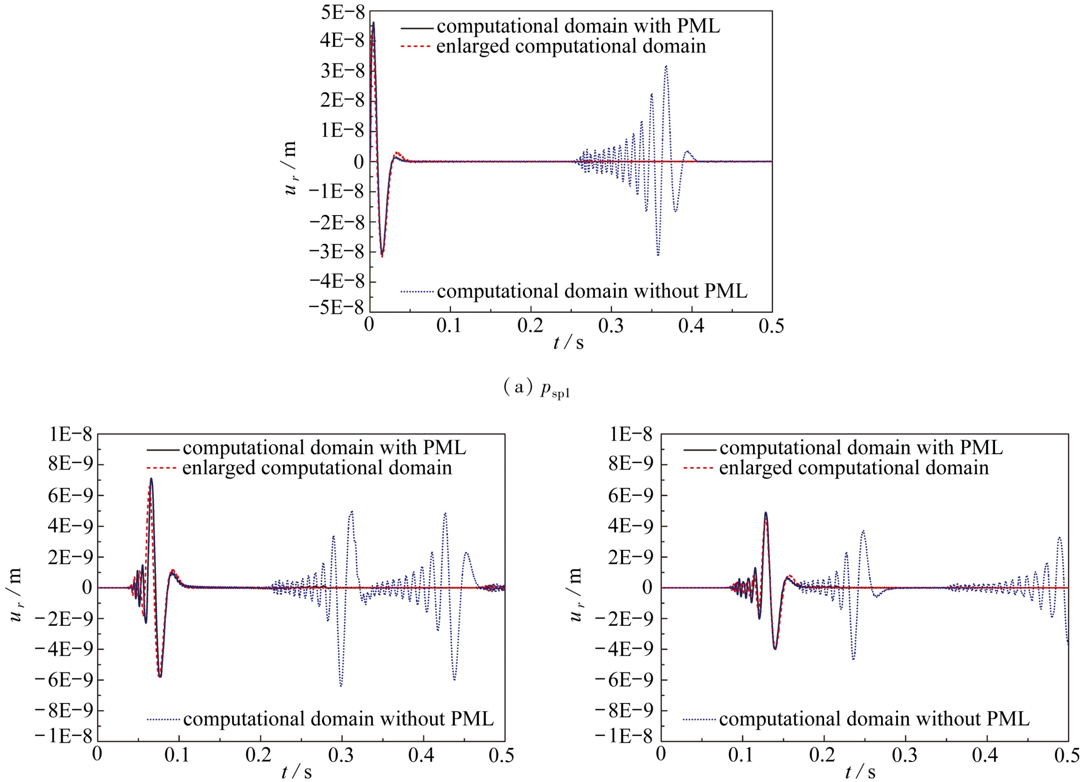
Fig. 3　The variation curves of excitation

在算例中取空间步长和时间步长分别为 $\Delta r = 0.05$ m，$\Delta t = 0.000\,2$ s．为了验证计算结果的正确性以及 PML 吸收边界条件的有效性，将计算区扩大 10 倍后用 ANSYS 有限元软件进行计算．对于考虑 PML 边界、无 PML 边界以及扩大区域的情形，图 4 分别给出了在不同条件下计算区域中 p_{sp1}，p_{sp2} 和 p_{sp3} 这 3 点处的径向位移随时间的变化曲线．从图中可以看出本文提出的非分裂 PML 有着很好的吸收性能．

为了进一步说明 PML 的有效性，图 5 绘出了截断物理域内，即计算区域中不含 PML 的能量随时间的变化．其中能量 E_{RD} 按下式进行计算[23]：

$$E_{RD} = \frac{1}{2}\dot{\boldsymbol{u}}^{T}\boldsymbol{M}\dot{\boldsymbol{u}} + \frac{1}{2}\boldsymbol{u}^{T}\boldsymbol{K}\boldsymbol{u}, \tag{26}$$

式中，$\boldsymbol{M}$,$\boldsymbol{K}$ 分别表示截断物理域内的总体质量矩阵和刚度矩阵.



(a) p_{sp1}

(b) p_{sp2}

(c) p_{sp3}

图 4 不同点处的位移变化曲线

Fig. 4 Comparison of displacement time histories at various sampling points

理论上，P 波的波速为 36.690 m/s，因此可以计算得出在 0.109 s 后，P 波到达 PML 区域，能量开始衰减.从图 5 中可以明显地看出能量的衰减，并且没有任何发射现象.

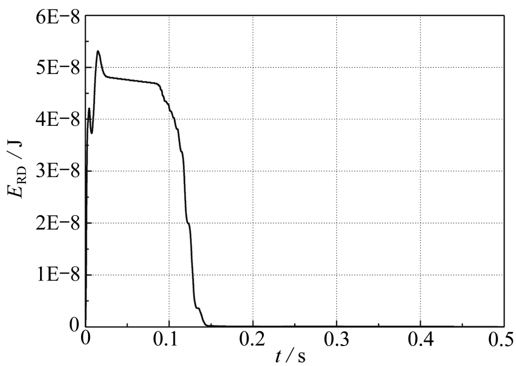


图 5 物理区内能量的变化

Fig. 5 Energy decay in the physical domain

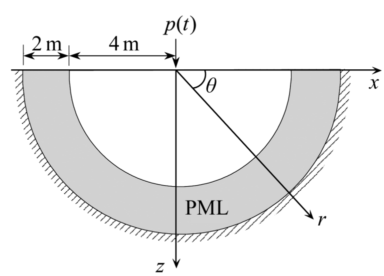
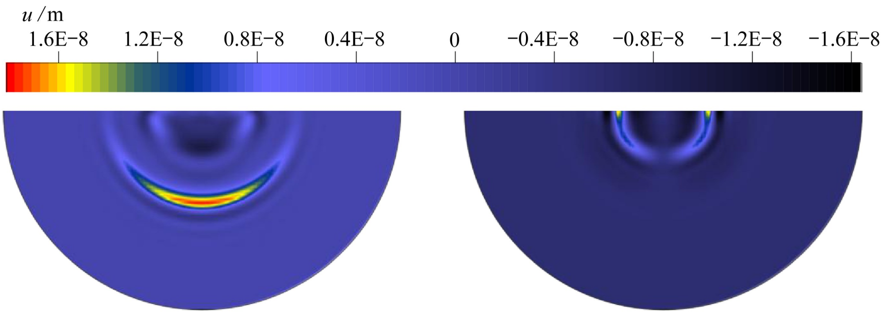


图 6 二维弹性地基以及 PML 截断示意图

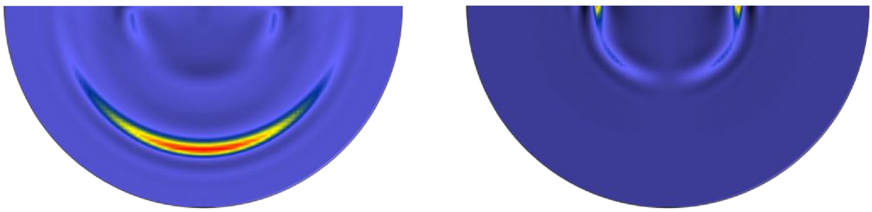
Fig. 6 The 2D elastic foundation and the PML

3.2 受集中荷载作用下弹性半平面的动力响应

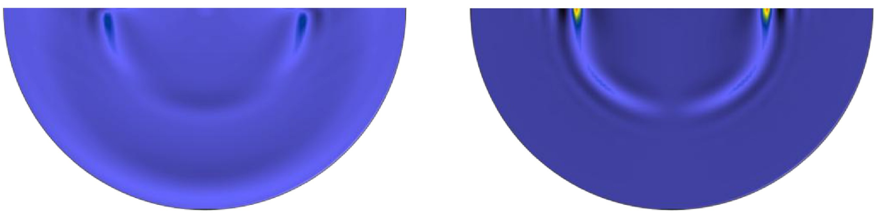
考虑如图 6 所示的二维弹性地基受到单位竖向集中力作用的平面应变问题，材料参数以及荷载形式与算例 3.1 中的相同.图 7 给出了 t = 0.08，0.12，0.16，0.20，0.30 s 不同时间下径向



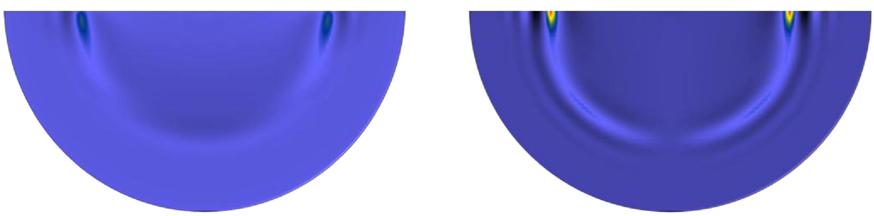
(a) t = 0.08 s



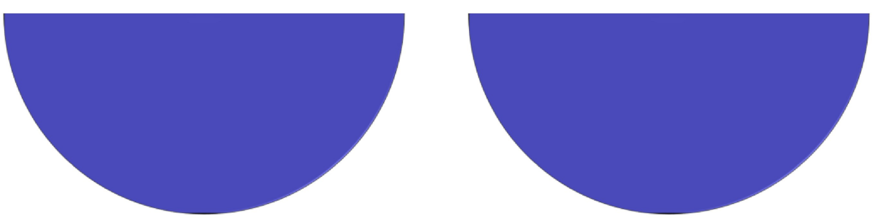
(b) t = 0.12 s



(c) t = 0.16 s



(d) t = 0.20 s



(e) t = 0.30 s

图 7　不同时刻的位移波场快照

Fig. 7　Snapshots of the displacement at different times

位移(左侧)和环向位移(右侧)的波场图.从图中可以看出 P 波和 S 波逐渐向外扩散,而在

PML 边界层处没有产生反射波场.图 8 给出了截断的物理区域内能量随时间的变化曲线.同样与 P 波类似,当 S 波大约在 0.204 s 到达 PML 局域,PML 开始吸收 S 波携带的能量.因此本文提出的非分裂 PML 对向外传播的弹性波具有很好的吸收性能.

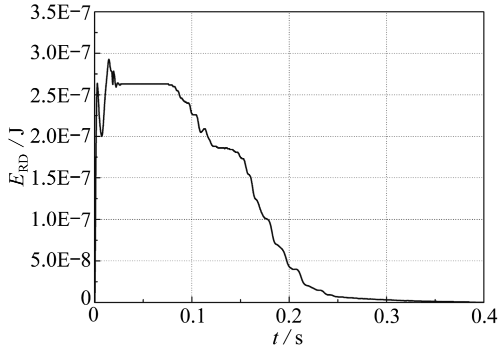


图 8 物理区内能量的变化

Fig. 8 Energy decay in the physical domain

4 结 论

利用复伸展坐标变换,将直角坐标系下的 PML 吸收边界推广到极坐标系下,借助于辅助函数,获得了以位移为基本未知量的非分裂 PML 弹性波动方程,并利用 Galerkin 近似方法,给出了时域内的有限元格式.通过数值算例从位移时程曲线、波场以及能量衰减的角度分析了该 PML 吸收边界的有效性.该方法在研究截断区域为圆形的弹性动力学问题时会带来较大的方便.

附 录 A

非轴对称问题的辅助函数 $\hat{\boldsymbol{\psi}}$ 的元素为

$$\hat{\psi}_{r1} = \frac{-\zeta_r}{s+\zeta_r}\frac{\partial \hat{u}_r}{\partial r}, \tag{A1}$$

$$\hat{\psi}_{r2} = \frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r}\frac{1}{r}\left(\frac{\partial \hat{u}_\theta}{\partial \theta} + \hat{u}_r\right), \tag{A2}$$

$$\hat{\psi}_{r3} = \frac{1}{r^2}\left(\frac{\zeta_r - \bar{\zeta}_r}{s+\bar{\zeta}_r} + \frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r} + \frac{\zeta_r - \bar{\zeta}_r}{s+\zeta_r}\frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r}\right)\left(\frac{\partial \hat{u}_r}{\partial \theta} - \hat{u}_\theta\right) + \frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r}\frac{1}{r}\frac{\partial \hat{u}_\theta}{\partial r}, \tag{A3}$$

$$\hat{\psi}_{r4} = -\left(\frac{\zeta_r - \bar{\zeta}_r}{s+\bar{\zeta}_r} + \frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r} + \frac{\zeta_r - \bar{\zeta}_r}{s+\bar{\zeta}_r}\frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r}\right)\frac{1}{r^2}\left(\frac{\partial \hat{u}_\theta}{\partial \theta} + \hat{u}_r\right) + \frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r}\frac{1}{r}\frac{\partial \hat{u}_r}{\partial r}, \tag{A4}$$

$$\hat{\psi}_{\theta 1} = \frac{1}{r^2}\left(\frac{\zeta_r - \bar{\zeta}_r}{s+\bar{\zeta}_r} + \frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r} + \frac{\zeta_r - \bar{\zeta}_r}{s+\zeta_r}\frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r}\right)\left(\frac{\partial \hat{u}_\theta}{\partial \theta} + \hat{u}_r\right), \tag{A5}$$

$$\hat{\psi}_{\theta 2} = \frac{1}{r}\frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r}\frac{\partial \hat{u}_r}{\partial r}, \tag{A6}$$

$$\hat{\psi}_{\theta 3} = \frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r}\frac{1}{r}\frac{\partial \hat{u}_r}{\partial \theta} - \frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r}\frac{\hat{u}_\theta}{r} + \frac{-\zeta_r}{s+\zeta_r}\frac{\partial \hat{u}_\theta}{\partial r}, \tag{A7}$$

$$\hat{\psi}_{\theta 4} = \hat{\psi}_{r3}. \tag{A8}$$

以式(A1)为例,将辅助函数 $\hat{\psi}_{r1}$ 变换到时间域内的过程:

式(A1)等价于

$$(s+\zeta_r)\hat{\psi}_{r1} = -\zeta_r \frac{\partial \hat{u}_r}{\partial r}, \tag{A9}$$

根据 Laplace 逆变换可得

$$\frac{\partial \psi_{r1}}{\partial t} = -\zeta_r \psi_{r1} - \zeta_r \frac{\partial u_r}{\partial r}. \tag{A10}$$

考虑式(14),上式采用单元节点位移可以表示为

$$\frac{\partial \psi_{r1}}{\partial t} = -\zeta_r \psi_{r1} - \zeta_r \sum_{i=1}^{m} \frac{\partial N_i}{\partial r} u_r^i. \tag{A11}$$

方程(A11)可采用 Runge-Kutta 法进行求解.

附　录　B

轴对称问题的辅助函数 $\hat{\boldsymbol{\psi}}$ 的元素为

$$\hat{\psi}_{r1} = \frac{-\zeta_r}{s+\zeta_r} \frac{\partial \hat{u}_r}{\partial r}, \tag{B1}$$

$$\hat{\psi}_{r2} = \frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r} \frac{\hat{u}_r}{r}, \tag{B2}$$

$$\hat{\psi}_{r3} = \frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r} \frac{1}{r} \frac{\partial \hat{u}_r}{\partial r} - \left(\frac{\zeta_r - \bar{\zeta}_r}{s+\bar{\zeta}_r} + \frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r} + \frac{\zeta_r - \bar{\zeta}_r}{s+\bar{\zeta}_r} \frac{-\bar{\zeta}_r}{s+\bar{\zeta}_r} \right) \frac{\hat{u}_r}{r^2}. \tag{B3}$$

参考文献(References):

[1] Lysmer J, Kuhlemeyer R L. Finite dynamic model for infinite media[J]. *Journal of the Engineering Mechanics Division*, 1969, **95**(4): 869-878.

[2] White W, Valliappan S, Lee I K. Unified boundary for finite dynamic model[J]. *Journal of the Engineering Mechanics Division*, 1977, **103**(5): 949-964.

[3] Marfurt K J. Accuracy of finite-difference and finite-element modeling of the scalar and elastic wave equations[J]. *Geophysics*, 1984, **49**(5): 533-549.

[4] Clayton R, Engquist B. Absorbing boundary conditions for acoustic and elastic wave equations [J]. *Bulletin of the Seismological Society of America*, 1977, **67**(6): 1529-1540.

[5] Engquist B, Majda A. Absorbing boundary conditions for numerical simulation of waves[J]. *Proceedings of the National Academy of Sciences*, 1977, **74**(5): 1765-1766.

[6] LIAO Zhen-feng, HUANG Kong-liang, YANG Bai-po, YUAN Yi-fan. A transmitting boundary for transient wave analyses[J]. *Science China*: *Mathematics*, 1984, **27**(10): 1063-1076.

[7] Higdon R L. Numerical absorbing boundary conditions for the wave equation[J]. *Mathematics of Computation*, 1987, **49**(179): 65-90.

[8] 熊章强, 唐圣松, 张大洲. 瑞利面波数值模拟中的 PML 吸收边界条件[J]. 物探与化探, 2009, **33**(4): 453-457.(XIONG Zhang-qiang, TANG Sheng-song, ZHANG Da-zhou. PML absorbing boundary condition for numerical modeling of Rayleigh wave[J]. *Geophysical and Geochemical Exploration*, 2009, **33**(4): 453-457.(in Chinese))

[9] Berenger J P. Three-dimensional perfectly matched layer for the absorption of electromagnetic waves[J]. *Journal of Computational Physics*, 1996, **127**(2): 363-379.

[10] Udagedara I, Premaratne M, Rukhlenko I D, Hattori H T, Agrawal G P. Unified perfectly

matched layer for finite-difference time-domain modeling of dispersive optical materials[J]. *Optics Express*, 2009, **17**(23): 21179-21190.

[11] MA You-neng, YU Jin-hua, WANG Yuan-yuan. A novel unsplit perfectly matched layer for the second-order acoustic wave equation[J]. *Ultrasonics*, 2014, **54**(6): 1568-1574.

[12] Komatitsch D, Tromp J. A perfectly matched layer absorbing boundary condition for the second-order seismic wave equation[J]. *Geophysical Journal International*, 2003, **154**(1): 146-153.

[13] PING Ping, ZHANG Yu, XU Yi-xian. A multiaxial perfectly matched layer (M-PML) for the long-time simulation of elastic wave propagation in the second-order equations[J]. *Journal of Applied Geophysics*, 2014, **101**: 124-135.

[14] 张洪欣, 吕英华, 黄永明. 柱坐标系下 PML-FDTD 及总场-散射场区连接条件[J]. 重庆邮电学院学报, 2003, **15**(3): 4-8.(ZHANG Hong-xin, LÜ Ying-hua, HUANG Yong-ming. The study on PML-FDTD and boundary connecting conditions of total-scattering fields in 3-D cylindrical coordinates[J]. *Journal of Chongqing University of Posts and Telecommunications*, 2003, **15**(3): 4-8.(in Chinese))

[15] Teixeira F L, Chew W C. PML-FDTD in cylindrical and spherical grids[J]. *Microwave and Guided Wave Letters, IEEE*, 1997, **7**(9): 285-287.

[16] Teixeira F L, Chew W C. Finite-difference computation of transient electromagnetic waves for cylindrical geometries in complex media[J]. *Geoscience and Remote Sensing, IEEE Transactions on*, 2000, **38**(4): 1530-1543.

[17] HE Jiang-qi, LIU Qing-huo. A nonuniform cylindrical FDTD algorithm with improved PML and quasi-PML absorbing boundary conditions[J]. *Geoscience and Remote Sensing, IEEE Transactions on*, 1999, **37**(2): 1066-1072.

[18] LIU Qing-huo, HE Jiang-qi. An efficient PSTD algorithm for cylindrical coordinates[J]. *IEEE Transactions on Antennas and Propagation*, 2001, **49**(9): 1349-1351.

[19] LIU Qing-huo. Perfectly matched layers for elastic waves in cylindrical and spherical coordinates[J]. *The Journal of the Acoustical Society of America*, 1999, **105**(4): 2075-2084.

[20] ZHENG Yi-bing, HUANG Xiao-jun. Anisotropic perfectly matched layers for elastic waves in Cartesian and curvilinear coordinates[R]. Earth Resources Laboratory Industry Consortia Annual Report. Earth Resources Laboratory, Massachusetts Institute of Technology, 2002: 1-18.

[21] SHI Dong-yang, LIAO Xin, TANG Qi-li. Highly efficient H^1-Galerkin mixed finite element method(MFEM) for parabolic integro-differential equation[J]. *Applied Mathematics and Mechanics(English Edition)*, 2014, **35**(7): 897-912.

[22] Collino F, Tsogka C. Application of the perfectly matched absorbing layer model to the linear elastodynamic problem in anisotropic heterogeneous media[J]. *Geophysics*, 2001, **66**(1): 294-307.

[23] Matzen R. An efficient finite element time-domain formulation for the elastic second-order wave equation: a non-split complex frequency shifted convolutional PML[J]. *International Journal for Numerical Methods in Engineering*, 2011, **88**(10): 951-973.

A Non-Splitting PML for Elastic Waves in Polar Coordinates and Its Finite Element Implementation

ZHOU Feng-xi[1,2], CAO Xiao-lin[1], Mark B. Jaksa[2]

(1. *School of Civil Engineering, Lanzhou University of Technology, Lanzhou* 730050, *P.R.China*;

2. *School of Civil, Environmental and Mining Engineering, University of Adelaide, South Australia* 5005, *Australia*)

Abstract: In the solving of the elastic wave equations with the numerical approximation techniques, the absorbing boundary conditions had been widely used to truncate the infinite-space simulation to a finite-space one. The perfect matched layer (PML) technique as an absorbing boundary condition had exhibited excellent absorbing efficiency in the forward simulation of the elastic wave equation formulated in rectangular coordinates. Based on the stretched coordinate concept, an advanced non-splitting-field perfect matched layer (non-splitting PML) equation for elastic waves was formulated in the polar coordinate system. Through the introduction of integrated complex variables in the radial direction into the auxiliary functions, the PML formulation was extended in polar coordinates in view of the 2nd-order elastic wave equation with displacements as basic unknowns. In addition, aimed at the time-domain cases and with the finite-element method for space discretization, the finite-element time-domain (FETD) scheme in standard displacement-based formulation was presented. The scheme for the special cases in axisymmetric polar coordinates was also given. The effectiveness and validity of the present non-splitting PML formulation are demonstrated with several numerical examples.

Key words: perfect matched layer; polar coordinate system; absorbing boundary condition; time-domain finite element

Foundation item: The National Natural Science Foundation of China(11162008;51368038)