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基于状态方程矩形层合板多种边界条件下的解析解*

卿光辉, 张小欢

(中国民航大学 航空工程学院, 天津 300300)

摘要: 以边界位移函数方法为基础,推导了矩形层合板多种边界条件下的非齐次状态方程和定解条件.将非齐次状态方程增维齐次化,可避免积分时可能出现的数值病态问题,并简化了计算过程.边界位移沿厚度方向非线性分布假设可以适当减少数值结果收敛要求的薄层数.数值结果可作为其它数值法或半解析法的标准解.该文的方法可为分析更加复杂的边界条件问题提供参考.

关 键 词: 状态空间; 矩形层合板; 边界位移函数; 解析解

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引 言

基于状态方程理论的半解析法[1-5]处理板的侧面边界条件没有原则上的困难,但复杂边界条件下板壳结构的解析解问题却一直是一个富有挑战性的问题.因此,基于三维状态方程理论关于四边简支矩形层合板或开口壳的解析解的文献[6-11]较多,而关于其它复杂边界条件问题的研究[7,12-14]却不多见.

文献[7]结合三角级数展开待求函数的理论,应用 δ 函数方法研究了一对边简支另一对边固支的层合板壳的解析解.文献[12]采用相同的方法研究了含固支边的压电材料层合板的解析解.文献[7]在研究含自由边的层合板的解析解时,采用了边界位移函数方法.文献[13-14]采用边界位移函数方法分别对含固支和含自由边的层合板解析解进行了更详细的理论分析和数值实例研究.同时,文献[13]指出矩形板侧面边界的位移分布函数可有多种形式的多项式.对比边界位移函数法和 δ 函数法,边界位移函数法的收敛快,数值分析相对简单[13-14].

如果矩形板一对边简支,另一对边是其它情况,则矩形板侧面边界条件的基本类型共有 8 种[15-16].当然,若考虑两对边边界条件的任意组合,那么矩形板的边界条件种类就更多了.迄今为止,基于状态空间理论,有 3 种基本边界条件的解析解得到了研究: 1) 四边简支问题[6-11]; 2) 一对边简支另一对边固支[7,12-13]的问题(文献[7,12-13]中的方法也适于处理四边固支的问题); 3) 一对边简支另一对边自由的边界问题[7,14].

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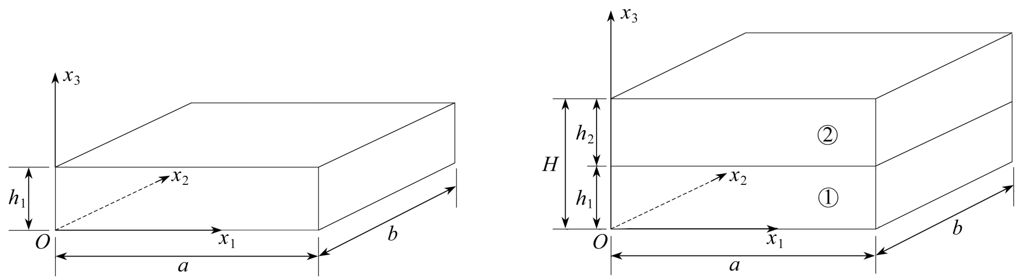
作者简介: 卿光辉(1968—),男,湖南人,教授,博士(E-mail: qingluke@126.com);
张小欢(1989—),男,安徽人,硕士生(通讯作者. E-mail: huan10595@126.com).

文献[16]基于二维的 Stroh 公式理论,将其扩展至三维问题,除研究了文献[15]指出的矩形板的 8 种基本边界条件的解析解外,还就其它更加复杂的边界条件进行了分析讨论.因此,在状态空间理论的基础上,有待对更多的边界条件下矩形板壳结构的解析解进行深入研究.

1　矩形板的坐标系统和边界条件

图 1 是矩形单板(图 1(a))、矩形层合板(图 1(b))和坐标系统的描述和说明.假定与 x_2 轴垂直的对边简支,则表 1 中的第 2 列与 x_2 方向的简支边组合就是 8 种基本的边界条件[15-16].假设材料是正交异性的,若表 1 中的第 2 列和第 3 列任意组合,则无刚体位移的边界条件类型共有 63 种.

为了方便起见,本文采用文献[16]的方法标识 8 种基本边界条件,当在边界 $x_1 = 0$ 和 $x_1 = a$ 上满足表 1 中 Bn 的边界条件,且在边界 $x_2 = 0$ 和 $x_2 = b$ 简支时,我们称这种边界条件为 BnBn($n = 1,2,\cdots,8$), 即 B1B1, B2B2, B3B3, B4B4, B5B5, B6B6, B7B7 和 B8B8.



(a) 单层板　　　　(b) 多层板

(a) The single-layer plate　　　　(b) The laminated plate

图 1　矩形单板、层合板和坐标系统

Fig. 1　The rectangular single-layer plate, the laminated plate and the coordinate systems

表 1　基本边界条件

Table 1　Basic boundary conditions

notation	BC: $x_1 = 0$ or $x_1 = a$	BC: $x_2 = 0$ or $x_2 = b$	description
B1	$u_1 = 0,\ u_2 = 0,\ u_3 = 0$	$u_1 = 0,\ u_2 = 0,\ u_3 = 0$	clamped surface
B2	$u_1 = 0,\ u_2 = 0,\ \sigma_{13} = 0$	$u_1 = 0,\ u_2 = 0,\ \sigma_{23} = 0$	–
B3	$u_1 = 0,\ \sigma_{12} = 0,\ u_3 = 0$	$\sigma_{12} = 0,\ u_2 = 0,\ u_3 = 0$	–
B4	$u_1 = 0,\ \sigma_{12} = 0,\ \sigma_{13} = 0$	$\sigma_{12} = 0,\ u_2 = 0,\ \sigma_{23} = 0$	smooth surface
B5	$\sigma_{11} = 0,\ u_2 = 0,\ u_3 = 0$	$u_1 = 0,\ \sigma_{22} = 0,\ u_3 = 0$	simply supported
B6	$\sigma_{11} = 0,\ u_2 = 0,\ \sigma_{13} = 0$	$u_1 = 0,\ \sigma_{22} = 0,\ \sigma_{23} = 0$	–
B7	$\sigma_{11} = 0,\ \sigma_{12} = 0,\ u_3 = 0$	$\sigma_{12} = 0,\ \sigma_{22} = 0,\ u_3 = 0$	–
B8	$\sigma_{11} = 0,\ \sigma_{12} = 0,\ \sigma_{13} = 0$	$\sigma_{12} = 0,\ \sigma_{22} = 0,\ \sigma_{23} = 0$	traction-free surface

2　弹性材料的状态方程

在直角坐标系中(如图 1),根据平衡方程和材料本构关系或修正后的变分原理[4-5,8,17],忽略体积力,可导出图 1(a)中所示矩形板的状态方程:

$$\frac{\partial}{\partial x_3}\begin{Bmatrix}\sigma_{13}\\ \sigma_{23}\\ \sigma_{33}\\ u_1\\ u_2\\ u_3\end{Bmatrix}=\begin{bmatrix}0 & 0 & s_4\alpha & -s_6\alpha^2-s_9\beta^2 & -(s_7+s_9)\alpha\beta & 0\\ 0 & 0 & s_5\beta & -(s_7+s_9)\alpha\beta & -s_9\alpha^2-s_8\beta^2 & 0\\ -\alpha & -\beta & 0 & 0 & 0 & 0\\ s_1 & 0 & 0 & 0 & 0 & -\alpha\\ 0 & s_2 & 0 & 0 & 0 & -\beta\\ 0 & 0 & s_3 & s_4\alpha & s_5\beta & 0\end{bmatrix}\begin{Bmatrix}\sigma_{13}\\ \sigma_{23}\\ \sigma_{33}\\ u_1\\ u_2\\ u_3\end{Bmatrix}, \tag{1}$$

式中

$\alpha=\partial/\partial x_1$, $\beta=\partial/\partial x_2$, $s_1=1/C_{55}$, $s_2=1/C_{44}$, $s_3=1/C_{33}$, $s_4=-C_{13}/C_{33}$,

$s_5=-C_{23}/C_{33}$, $s_6=C_{11}-C_{13}^2/C_{33}$, $s_7=C_{12}-C_{13}C_{23}/C_{33}$,

$s_8=C_{22}-C_{23}^2/C_{33}$, $s_9=C_{66}$.

平面内的 3 个应力分量可表示为

$$\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{Bmatrix}=\begin{bmatrix}s_6\alpha & s_7\beta & -s_4\\ s_7\alpha & s_8\beta & -s_5\\ s_9\beta & s_9\alpha & 0\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\\ \sigma_{33}\end{Bmatrix}. \tag{2}$$

对于四边简支(B5B5)的矩形板问题,可设

$$\begin{cases}(\sigma_{13},u_1)=\sum\limits_m\sum\limits_n(\sigma_{13}^{mn}(x_3),u_1^{mn}(x_3))\cos(\zeta x_1)\sin(\eta x_2),\\ (\sigma_{23},u_2)=\sum\limits_m\sum\limits_n(\sigma_{23}^{mn}(x_3),u_2^{mn}(x_3))\sin(\zeta x_1)\cos(\eta x_2),\\ (\sigma_{33},u_3)=\sum\limits_m\sum\limits_n(\sigma_{33}^{mn}(x_3),u_3^{mn}(x_3))\sin(\zeta x_1)\sin(\eta x_2),\end{cases} \tag{3}$$

式中, $\zeta=m\pi/a$, $\eta=n\pi/b$.

将方程(3)代入方程(1)中,对每对 (m,n) 有

$$\frac{\mathrm{d}}{\mathrm{d}x_3}\begin{Bmatrix}\sigma_{13}^{mn}(x_3)\\ \sigma_{23}^{mn}(x_3)\\ \sigma_{33}^{mn}(x_3)\\ u_1^{mn}(x_3)\\ u_2^{mn}(x_3)\\ u_3^{mn}(x_3)\end{Bmatrix}=\begin{bmatrix}0 & 0 & s_4\zeta & s_6\zeta^2+s_9\eta^2 & (s_7+s_9)\zeta\eta & 0\\ 0 & 0 & s_5\eta & (s_7+s_9)\zeta\eta & s_9\zeta^2+s_8\eta^2 & 0\\ \zeta & \eta & 0 & 0 & 0 & 0\\ s_1 & 0 & 0 & 0 & 0 & -\zeta\\ 0 & s_2 & 0 & 0 & 0 & -\eta\\ 0 & 0 & s_3 & -s_4\zeta & -s_5\eta & 0\end{bmatrix}\begin{Bmatrix}\sigma_{13}^{mn}(x_3)\\ \sigma_{23}^{mn}(x_3)\\ \sigma_{33}^{mn}(x_3)\\ u_1^{mn}(x_3)\\ u_2^{mn}(x_3)\\ u_3^{mn}(x_3)\end{Bmatrix}. \tag{4}$$

如果通过求解方程(4)得到 3 个位移 u_1,u_2,u_3 和 3 个平面外应力 $\sigma_{13},\sigma_{23},\sigma_{33}$,则可通过方程(2) 得到 3 个平面内应力 $\sigma_{11},\sigma_{22},\sigma_{12}$ 的值.

3 非简支边界的边界位移函数、非齐次项和定解条件

假定与 x_2 轴垂直的对边简支(如图 1),并假设本节公式中各边界位移函数 $g_1(x_1)$, $h_1(x_1)=1-x_1/a$, $g_2(x_1)$, $h_2(x_1)=x_1/a$,其中 a 代表矩形板 x_1 方向的长度.为了满足边界条件,设

$$\hat{u}_i=u_i+\bar{u}_i \qquad (i=1,2,3), \tag{5}$$

其中, $u_i(i=1,2,3)$ 的展开式与方程(3) 相同, $\bar{u}_i$ 为边界位移函数.

值得说明的是,除了符号表达有所区别外,本文关于 5 种基本边界条件的公式推导方法和步骤与文献[13-14]基本类似.由于篇幅所限,这里省略了关于 $m=0$ 或 $n=0$ 的非齐次项公式.

3.1 B1B1 的边界位移函数、非齐次项和定解条件[13]

为了在 $x_1=0,a$ 处满足 $u_1=0$,设方程(5) 中的边界位移函数 $\bar{u}_1$ 为

$$\bar{u}_1=\sum_n \bar{u}_{10}^n(x_3)g_1(x_1)\sin(\eta x_2)+\sum_n \bar{u}_{1a}^n(x_3)g_2(x_1)\sin(\eta x_2). \tag{6}$$

将方程(5)的 $\hat{u}_1$ 视为 u_1, 代入方程(1),则方程(4)增加的非齐次项可表示为

$$\begin{Bmatrix} f_1\\ f_2\\ f_3\\ f_4\\ f_5\\ f_6 \end{Bmatrix}=\begin{Bmatrix} \sin(\eta x_2)(s_9\eta^2(g_1(x_1)\bar{u}_{10}(x_3)+g_2(x_1)\bar{u}_{1a}(x_3))-\\ s_6(g_1''(x_1)\bar{u}_{10}(x_3)+g_2''(x_1)\bar{u}_{1a}(x_3)))\\ -\cos(\eta x_2)(s_7+s_9)\eta(g_1'(x_1)\bar{u}_{10}(x_3)+g_2'(x_1)\bar{u}_{1a}(x_3))\\ 0\\ -\sin(\eta x_2)(g_1(x_1)\bar{u}_{10}'(x_3)+g_2(x_1)\bar{u}_{1a}'(x_3))\\ 0\\ \sin(\eta x_2)s_4(g_1'(x_1)\bar{u}_{10}(x_3)+g_2'(x_1)\bar{u}_{1a}(x_3)) \end{Bmatrix}. \tag{7}$$

考虑矩形板的对称性,并将式(7)中的 $g_1(x_1)$,$g_2(x_1)$ 及其导数非零的项展开成必要的级数后化简,则方程(7)就变为

$$\begin{Bmatrix} f_1\\ f_2\\ f_3\\ f_4\\ f_5\\ f_6 \end{Bmatrix}=\begin{Bmatrix} 2s_9\eta^2\phi\bar{u}_{10}(x_3)\\ (2(s_7+s_9)\eta\psi/a)\bar{u}_{10}(x_3)\\ 0\\ -2\phi\bar{u}_{10}'(x_3)\\ 0\\ -(2s_4\psi/a)\bar{u}_{10}(x_3) \end{Bmatrix}, \tag{8}$$

其中

$$\phi=2(1-\cos(m\pi))/(m^2\pi^2),\ \psi=2(1-\cos(m\pi))/(m\pi).$$

根据文献[13-14]中的方法,由 $x_1=0,a$ 处的边界条件 $u_1=0$ 可求得上式中的 $\bar{u}_{10}$,其定解条件为

$$\bar{u}_{10}^n(x_3)+\sum_m u_1^{mn}(x_3)=0 \qquad (m=1,3,5,\cdots;\ n=1,3,5,\cdots). \tag{9}$$

将方程(5)的 $\hat{u}_1$ 视为 u_1,并与 u_2,σ_{33} 一同代入方程(2),可得平面内应力的解.

3.2 B2B2 的边界位移函数、非齐次项和定解条件

具体的推导方法和步骤与 B1B1 相同.设方程(5)中的边界位移函数 $\bar{u}_1$ 和 $\bar{u}_3$ 为

$$\begin{cases} \bar{u}_1=\sum_n \bar{u}_{10}^n(x_3)g_1(x_1)\sin(\eta x_2)+\sum_n \bar{u}_{1a}^n(x_3)g_2(x_1)\sin(\eta x_2),\\ \bar{u}_3=\sum_n \bar{u}_{30}^n(x_3)h_1(x_1)\sin(\eta x_2)+\sum_n \bar{u}_{3a}^n(x_3)h_2(x_1)\sin(\eta x_2). \end{cases} \tag{10}$$

将方程(5)的 $\hat{u}_i(i=1,3)$ 视为 $u_i(i=1,3)$, 代入方程(1),则方程(4)增加的非齐次项为

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{Bmatrix} = \begin{Bmatrix} 2s_9\eta^2\phi\bar{u}_{10}(x_3) \\ (2(s_7+s_9)\eta\psi/a)\bar{u}_{10}(x_3) \\ 0 \\ -2\phi\bar{u}'_{10}(x_3) \\ -\eta(\kappa+\lambda)\ \bar{u}_{30}(x_3) \\ -(2s_4\psi/a)\ \bar{u}_{10}(x_3)-(\kappa+\lambda)\bar{u}'_{30}(x_3) \end{Bmatrix}, \tag{11}$$

其中，ϕ 和 ψ 与式(8)相同，这里 $\kappa=2/(m\pi)$，$\lambda=2(-1)^{m-1}/(m\pi)$。

由 $x_1=0,a$ 处的边界条件 $u_1=0$ 和 $\sigma_{13}=0$ 可求得上式中 $\bar{u}_{10}$ 和 $\bar{u}_{30}$，其定解条件为

$$\begin{cases} \bar{u}^n_{10}(x_3)+\sum\limits_m u^{mn}_1(x_3)=0 & (m=1,3,5,\cdots;\ n=1,3,5,\cdots), \\ \sum\limits_m \sigma^{mn}_{13}(x_3)=0 & (m=1,3,5,\cdots;\ n=1,3,5,\cdots). \end{cases} \tag{12}$$

将方程(5)中的 $\hat{u}_1$ 视为 u_1，并与 u_2,σ_{33} 一同代入方程(2)，可得平面内应力的解。

3.3 B3B3 的边界位移函数、非齐次项和定解条件

考虑到在 $x_1=0,a$ 处 $\sigma_{12}=0$ 的情况，设方程(5)中的边界位移函数 $\bar{u}_1$ 和 $\bar{u}_2$ 分别为

$$\begin{cases} \bar{u}_1=\sum\limits_n \bar{u}^n_{10}(x_3)g_1(x_1)\sin(\eta x_2)+\sum\limits_n \bar{u}^n_{1a}(x_3)g_2(x_1)\sin(\eta x_2), \\ \bar{u}_2=\sum\limits_n \bar{u}^n_{10}(x_3)(\eta a g^2_1(x_1)/2)\cos(\eta x_2)- \\ \qquad \sum\limits_n \bar{u}^n_{1a}(x_3)(\eta a g^2_2(x_1)/2)\cos(\eta x_2). \end{cases} \tag{13}$$

将方程(5)的 $\hat{u}_i(i=1,2)$ 视为 $u_i(i=1,2)$，代入方程(1)，则方程(4)增加的非齐次项为

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{Bmatrix} = \begin{Bmatrix} -2s_7\eta^2\phi\bar{u}_{10}(x_3) \\ (\eta(a^2s_8\eta^2(\theta+\vartheta)+4s_7\psi)/(2a))\bar{u}_{10}(x_3) \\ 0 \\ -2\phi\bar{u}'_{10}(x_3) \\ -(a(\theta+\vartheta)/2)\eta\ \bar{u}'_{10}(x_3) \\ -((a^2s_5\eta^2(\theta+\vartheta)+4s_4\psi)/(2a))\bar{u}_{10}(x_3) \end{Bmatrix}, \tag{14}$$

其中

$$\theta=2(\pi^2m^2+2(\cos(m\pi)-1))/(\pi^3m^3),$$

$$\vartheta=2(-1)^{m-1}(\pi^2m^2+2(\cos(m\pi)-1))/(\pi^3m^3).$$

由 $x_1=0,a$ 处的边界条件 $u_1=0$ 可求得上式中的 $\bar{u}_{10}$，其定解条件为

$$\bar{u}^n_{10}(x_3)+\sum_m u^{mn}_1(x_3)=0 \qquad (m=1,3,5,\cdots;\ n=1,3,5,\cdots). \tag{15}$$

将方程(5)的 $\hat{u}_i(i=1,2)$ 视为 $u_i(i=1,2)$，并与 σ_{33} 一同代入方程(2)，可得平面内应力的解。

3.4 B4B4 的边界位移函数、非齐次项和定解条件

考虑到在 $x_1=0,a$ 处 $\sigma_{12}=0$ 的情况，设方程(5)中的边界位移函数 $\bar{u}_1,\bar{u}_2,\bar{u}_3$ 分别为

$$\begin{cases}\bar{u}_1 = \sum_n \bar{u}_{10}^n(x_3)g_1(x_1)\sin(\eta x_2) + \sum_n \bar{u}_{1a}^n(x_3)g_2(x_1)\sin(\eta x_2),\\ \bar{u}_2 = \sum_n \bar{u}_{10}^n(x_3)(\eta a g_1^2(x_1)/2)\cos(\eta x_2) - \\ \qquad \sum_n \bar{u}_{1a}^n(x_3)(\eta a g_2^2(x_1)/2)\cos(\eta x_2),\\ \bar{u}_3 = \sum_n \bar{u}_{30}^n(x_3)h_1(x_1)\sin(\eta x_2) + \sum_n \bar{u}_{3a}^n(x_3)h_2(x_1)\sin(\eta x_2).\end{cases} \tag{16}$$

将方程(5)的 $\hat{u}_i(i=1,2,3)$ 视为 $u_i(i=1,2,3)$，代入方程(1)，则方程(4)增加的非齐次项为

$$\begin{Bmatrix} f_1\\ f_2\\ f_3\\ f_4\\ f_5\\ f_6\end{Bmatrix} = \begin{Bmatrix} -2s_7\eta^2\phi\bar{u}_{10}(x_3)\\ (\eta(a^2s_8\eta^2(\theta+\vartheta)+4s_7\psi)/(2a))\bar{u}_{10}(x_3)\\ 0\\ -2\phi\bar{u}'_{10}(x_3)\\ -\eta(\kappa+\lambda)\bar{u}_{30}(x_3)+(a(\theta+\vartheta)/2)\bar{u}'_{10}(x_3)\\ -((a^2s_5\eta^2(\theta+\vartheta)+4s_4\psi)/(2a))\bar{u}_{10}(x_3)+(\kappa+\lambda)\bar{u}'_{30}(x_3)\end{Bmatrix}. \tag{17}$$

由 $x_1=0,a$ 处的边界条件 $u_1=0$ 和 $\sigma_{13}=0$ 可求得上式中 $\bar{u}_{10}$ 和 $\bar{u}_{30}$，其定解条件为

$$\begin{cases}\bar{u}_{10}^n(x_3) + \sum_m u_1^{mn}(x_3) = 0 \qquad (m=1,3,5,\cdots;\ n=1,3,5,\cdots),\\ \sum_m \sigma_{13}^{mn}(x_3) = 0 \qquad (m=1,3,5,\cdots;\ n=1,3,5,\cdots).\end{cases} \tag{18}$$

将方程(5)中的 $\hat{u}_i(i=1,2)$ 视为 $u_i(i=1,2)$，并与 σ_{33} 一同代入方程(2)，可得平面内应力的解.

3.5 B6B6 的边界位移函数、非齐次项和定解条件

设方程(5)中的边界位移函数 $\bar{u}_3$ 为

$$\bar{u}_3 = \sum_n \bar{u}_{30}^n(x_3)g_1(x_1)\sin(\eta x_2) + \sum_n \bar{u}_{3a}^n(x_3)g_2(x_1)\sin(\eta x_2). \tag{19}$$

将方程(5)左边 $\hat{u}_3$ 视为 u_3，代入方程(1)，则方程(4)增加的非齐次项为

$$\begin{Bmatrix} f_1\\ f_2\\ f_3\\ f_4\\ f_5\\ f_6\end{Bmatrix} = \begin{Bmatrix} 0\\ 0\\ 0\\ 0\\ -\eta(\kappa+\lambda)\bar{u}_{30}(x_3)\\ -(\kappa+\lambda)\bar{u}'_{30}(x_3)\end{Bmatrix}. \tag{20}$$

由 $x_1=0,a$ 处的边界条件 $\sigma_{13}=0$ 可求得上式中的 $\bar{u}_{30}$，其定解条件为

$$\sum_m \sigma_{13}^{mn}(x_3) = 0 \qquad (m=1,3,5,\cdots;\ n=1,3,5,\cdots). \tag{21}$$

将 u_1,u_2,σ_{33} 的展开式代入方程(2)，可得平面内应力的解.

3.6 B7B7 的边界位移函数、非齐次项和定解条件

考虑到在 $x_1=0,a$ 处 $\sigma_{12}=0$ 的情况，设方程(5)中的边界位移函数 $\bar{u}_1,\bar{u}_2$ 分别为

$$\begin{cases}\bar{u}_1 = \sum_n \bar{u}_{10}^n(x_3)g_1(x_1)\sin(\eta x_2) + \sum_n \bar{u}_{1a}^n(x_3)g_2(x_1)\sin(\eta x_2),\\ \bar{u}_2 = \sum_n \bar{u}_{10}^n(x_3)(\eta a g_1^2(x_1)/2)\cos(\eta x_2) - \\ \qquad \sum_n \bar{u}_{1a}^n(x_3)(\eta a g_2^2(x_1)/2)\cos(\eta x_2).\end{cases} \tag{22}$$

将方程(5)的 $\hat{u}_i(i = 1,2,3)$ 视为 $u_i(i = 1,2,3)$，代入方程(1)，则方程(4)增加的非齐次项为

$$\begin{Bmatrix} f_1\\ f_2\\ f_3\\ f_4\\ f_5\\ f_6\end{Bmatrix} = \begin{Bmatrix} -2s_7\eta^2\phi\bar{u}_{10}(x_3)\\ (\eta(a^2s_8\eta^2(\theta+\vartheta)+4s_7\psi)/(2a))\bar{u}_{10}(x_3)\\ 0\\ -2\phi\bar{u}'_{10}(x_3)\\ -(a\eta(\theta+\vartheta)/2)\bar{u}'_{10}(x_3)\\ -((a^2s_5\eta^2(\theta+\vartheta)+4s_4\psi)/(2a))\bar{u}_{10}(x_3)\end{Bmatrix}. \tag{23}$$

根据方程(2)，由 $x_1 = 0, a$ 处的边界条件 $\sigma_{11} = 0$ 可求得上式中的 $\bar{u}_{10}$，其定解条件为

$$\bar{\sigma}_{11}^n(x_3) + \sum_m \sigma_{11}^{mn}(x_3) = 0 \qquad (m = 1,3,5,\cdots;\ n = 1,3,5,\cdots), \tag{24}$$

其中

$$\bar{\sigma}_{11}^n = (a^2s_7\eta^2(\theta+\vartheta)+4s_6\psi)/(2a)\bar{u}_{10}^n - s_4\sigma_{33}^n.$$

将方程(5)的 $\hat{u}_1$ 视为 u_1，并与 u_2，σ_{33} 一同代入方程(2)，可得平面内应力的解.

3.7 B8B8 的边界位移函数、非齐次项和定解条件

考虑到在 $x_1 = 0, a$ 处 $\sigma_{12} = 0$ 的情况，这里设方程(5)中的边界位移函数 $\bar{u}_1$，$\bar{u}_2$，$\bar{u}_3$ 分别为

$$\begin{cases}\bar{u}_1 = \sum_n \bar{u}_{10}^n(x_3)g_1(x_1)\sin(\eta x_2) + \sum_n \bar{u}_{1a}^n(x_3)g_2(x_1)\sin(\eta x_2),\\ \bar{u}_2 = \sum_n \bar{u}_{10}^n(x_3)(\eta a g_1^2(x_1)/2)\cos(\eta x_2) - \\ \qquad \sum_n \bar{u}_{1a}^n(x_3)(\eta a g_2^2(x_1)/2)\cos(\eta x_2),\\ \bar{u}_3 = \sum_n \bar{u}_{30}^n(x_3)h_1(x_1)\sin(\eta x_2) + \sum_n \bar{u}_{3a}^n(x_3)h_2(x_1)\sin(\eta x_2).\end{cases} \tag{25}$$

将方程(5)的 $\hat{u}_i(i = 1,2)$ 视为 $u_i(i = 1,2)$，代入方程(1)，则方程(4)增加的非齐次项为

$$\begin{Bmatrix} f_1\\ f_2\\ f_3\\ f_4\\ f_5\\ f_6\end{Bmatrix} = \begin{Bmatrix} -2s_7\eta^2\phi\bar{u}_{10}(x_3)\\ (\eta(a^2s_8\eta^2(\theta+\vartheta)+4s_7\psi)/(2a))\bar{u}_{10}(x_3)\\ 0\\ -2\phi\bar{u}'_{10}(x_3)\\ -(a\eta(\theta+\vartheta)/2)\bar{u}'_{10}(x_3)\\ -((a^2s_5\eta^2(\theta+\vartheta)+4s_4\psi)/(2a))\bar{u}_{10}(x_3)\end{Bmatrix}. \tag{26}$$

根据方程(2)，由 $x_1 = 0, a$ 处的边界条件 $\sigma_{11} = 0$ 和 $\sigma_{13} = 0$ 可求得上式中的 $\bar{u}_{10}$ 和 $\bar{u}_{30}$，其定解条件为

$$\begin{cases} \bar{\sigma}_{11}^{n}(x_3) + \sum_{m} \sigma_{11}^{mn}(x_3) = 0 & (m = 1,3,5,\cdots;\ n = 1,3,5,\cdots), \\ \sum_{m} \sigma_{13}^{mn}(x_3) = 0 & (m = 1,3,5,\cdots;\ n = 1,3,5,\cdots), \end{cases} \tag{27}$$

其中

$$\bar{\sigma}_{11}^{n} = (a^2 s_7 \eta^2 (\theta + \vartheta) + 4 s_6 \psi)/(2a) \bar{u}_{10}^{n} - s_4 \sigma_{33}^{n}.$$

将方程(5)的 $\hat{u}_i(i = 1,2)$ 视为 $u_i(i = 1,2)$，并与 σ_{33} 一同代入方程(2)，可得平面内应力的解.

4 数值实例

已知两层板[0°/90°](图 1(b)),假定与 x_2 轴垂直的对边简支.$a = b = 1$，$H = 0.2$，$h_1 = 0.5H$，$h_2 = 0.5H$.材料工程弹性常数为[13]

$$E_{11} = 25E_{22} = 25E_{33},\ G_{12} = G_{22} = 0.5E_{33},\ G_{23} = 0.2E_{33},\ \mu_{12} = \mu_{13} = \mu_{23} = 0.25.$$

板的上表面受垂直向上的正弦载荷 $q = q_0 \sin(\pi x_1/a)\sin(\pi x_2/b)$ 的作用.求其变形.

解 首先要说明,在求解方法上,本文与文献[7, 12-14]有所不同:

1) 精细积分法可给出达到计算机精度的数值结果,并且不涉及复数及其求逆的运算.所以本文首先应用文献[18]的方法对方程(4)增加的非齐次项进行了齐次化处理,然后采用精细积分法求解.

2) 文献[7, 12-14]假设边界位移沿厚度方向的分布为线性表达式.本文假设边界位移沿厚度方向的分布为非线性表达式(28),这一假设可以适当减少数值结果收敛要求的薄层数.

$$U(x_3) = U_{nj1} - (U_{nj1} - U_{nj2})(\exp(x_3) - 1)/(\exp(d_j) - 1), \quad x_3 \in [0, d_j], \tag{28}$$

其中，$U(x_3)$ 代表 $\bar{u}_1(x_3)$，$\bar{u}_2(x_3)$ 或 $\bar{u}_3(x_3)$，d_j 是薄层的厚度.

除此以外,求解过程与文献[7, 12-14]基本相同.因为板的上表面受垂直向上的正弦载荷,所以在计算过程中,当 $m = n = 1$ 时,取 $q_0 \neq 0$;当 $m = 3,5,\cdots,99$,$n = 1$ 时,$q_0 = 0$.本文分别对各层的薄层数为 3,4 和 5 的情况进行了数值比较,当 $m = 1,3,5,\cdots,99$,$n = 1$ 时,其数值结果就有明显的变化.

具体的数值结果见表 2(各层的薄层数均为 4 层),其中圆括号中的数据为文献[16]的结果(参见无量纲式(29)),方括号中给出的数据为误差(|(本文解−文献解)/ 文献解|×100%).

$$\begin{cases} [\hat{u}_1(x_3), \hat{u}_2(x_3)] = \\ \quad (100E_{\mathrm{T}}H^2/(q_0 L_1^3))[u_1(L_1/4, L_2/2, x_3), u_2(L_1/2, L_2/4, x_3)], \\ [\hat{u}_3(x_3)] = (100E_{\mathrm{T}}H^3/(q_0 L_1^4)) u_3(L_1/2, L_2/2, x_3), \\ [\hat{\sigma}_{11}(x_3), \hat{\sigma}_{22}(x_3)] = \\ \quad (10H^2/(q_0 L_1^2))[\sigma_{11}(L_1/2, L_2/2, x_3), \sigma_{22}(L_1/2, L_2/2, x_3)], \\ \hat{\sigma}_{12}(x_3) = (10H^2/(q_0 L_1^2))\sigma_{12}(L_1/8, 0, x_3), \\ [\hat{\sigma}_{23}(x_3), \hat{\sigma}_{31}(x_3)] = (10H/(q_0 L_1))[\sigma_{23}(L_1/2, 0, x_3), \sigma_{31}(L_1/8, L_2/2, x_3)], \\ [\hat{\sigma}_{33}(x_3)] = \sigma_{33}(L_1/2, L_2/2, x_3)/q_0. \end{cases} \tag{29}$$

从总体上分析,最小误差对应于 B5B5 型;对应于 B4B4 型,本文的解与文献[16]的解相

差较大,最大误差是3.84%(参见表2中的平面外应力 $\widehat{\sigma}_{33}(H/2)$).因此,本文的结果与文献[16]的结果是比较吻合的.

表2　8种边界条件下的位移和应力

Table 2　Displacements and stresses under the 8 boundary conditions

variable	B1B1	B2B2	B3B3	B4B4	B5B5	B6B6	B7B7	B8B8
$\widehat{u}_1(H)$	−1.071 1 (−1.047) [2.29%]	−0.523 6 (−0.512) [2.17%]	−1.072 3 (−1.050) [2.15%]	−0.354 5 (−0.343) [3.41%]	−1.892 4 (−1.870) [1.20%]	−0.865 5 (−0.847) [2.18%]	−1.964 4 (−1.924) [2.11%]	−0.577 1 (−0.565) [2.13%]
$\widehat{u}_2(0)$	1.370 4 (1.341) [2.18%]	2.688 4 (2.632) [2.16%]	1.387 6 (1.360) [2.03%]	3.361 4 (3.247) [3.52%]	1.920 8 (1.899) [1.14%]	2.774 2 (2.717) [2.12%]	2.001 8 (1.961) [2.08%]	3.359 6 (3.291) [2.09%]
$\widehat{u}_3(H/2)$	1.236 3 (1.217) [1.56%]	2.279 3 (2.246) [1.48%]	1.246 0 (1.229) [1.38%]	2.812 5 (2.708) [3.85%]	1.733 0 (1.712) [1.22%]	2.373 9 (2.327) [2.02%]	1.793 2 (1.758) [2.01%]	2.809 7 (2.753) [2.06%]
$\widehat{\sigma}_{11}(0)$	−4.705 2 (−4.630) [1.52%]	−2.537 2 (−2.499) [1.49%]	−4.732 4 (−4.667) [1.40%]	−2.060 4 (−1.994) [3.33%]	−7.761 5 (−7.671) [1.18%]	−3.528 6 (−3.457) [2.07%]	−8.105 0 (−7.913) [2.43%]	−2.723 4 (−2.660) [2.38%]
$\widehat{\sigma}_{22}(H)$	5.822 8 (5.723) [1.74%]	10.746 3 (10.568) [1.68%]	5.858 7 (5.771) [1.52%]	13.183 2 (12.705 0) [3.76%]	7.976 2 (7.894) [1.04%]	11.151 2 (10.888) [2.42%]	8.105 1 (8.096) [2.73%]	13.261 1 (12.877) [2.98%]
$\widehat{\sigma}_{33}(H/2)$	0.590 1 (0.579) [1.92%]	0.440 3 (0.432) [1.86%]	0.587 7 (0.577) [1.87%]	0.382 1 (0.368) [3.84%]	0.500 5 (0.495) [1.12%]	0.426 9 (0.416) [2.62%]	0.501 3 (0.489) [2.51%]	0.368 6 (0.359) [2.68%]
$\widehat{\sigma}_{12}(0)$	0.320 4 (0.313) [2.35%]	0.498 2 (0.487) [2.29%]	0.251 9 (0.247) [2.01%]	0.067 5 (0.065) [3.81%]	0.532 8 (0.527 0) [1.09%]	0.553 2 (0.539) [2.65%]	0.434 3 (0.424) [2.43%]	0.110 6 (0.108) [2.39%]
$\widehat{\sigma}_{23}(H/2)$	0.895 9 (0.875) [2.39%]	1.382 0 (1.351) [2.32%]	0.889 8 (0.871) [2.14%]	1.509 2 (1.499) [3.74%]	1.224 7 (1.211) [1.13%]	1.453 3 (1.416) [2.63%]	1.255 7 (1.225) [2.51%]	1.578 8 (1.541) [2.46%]
$\widehat{\sigma}_{13}(H/2)$	1.587 6 (1.550) [2.41%]	0.652 80 (0.638) [2.32%]	1.602 0 (1.567) [2.21%]	0.435 1 (0.419) [3.83%]	1.230 4 (1.216) [1.15%]	0.655 4 (0.638) [2.71%]	1.255 5 (1.195) [2.49%]	0.425 3 (0.416) [2.22%]

5　结　　论

基于三维的状态空间理论体系,通过引入边界位移函数,对一对边简支另一对边为其它情况的层合板建立了非齐次的状态方程,并给出了各种基本边界条件下静力学问题的精确解析解.精确解析解可方便地展示状态变量(即位移变量和应力变量)沿厚度变化的情况.由物理方程求出的各种边界条件下的平面外应力都是连续的.

为了验证本文的理论,将数值结果与相关文献的结果进行了比较,数值结果可作为其它数值法或半解析法的标准解.本文在计算方法上采用了精细积分法,并假设边界位移沿厚度方向的分布为非线性的.非线性的分布假设可以适当减少数值结果收敛要求的薄层数,提高了计算效率.本文的工作进一步说明用状态空间理论求解非简支边界的板壳问题是有效的,本文的方

法可为分析更复杂的边界条件问题提供参考.

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Analytical Solutions of Rectangular Laminated Plates Under Various Boundary Conditions in the State Space

QING Guang-hui, ZHANG Xiao-huan

(*College of Aeronautical Engineering*, *Civil Aviation University of China*, *Tianjin* 300300, *P.R.China*)

Abstract: Based upon the method of boundary displacement functions, the non-homogeneous state equations and the control equations for the solutions of rectangular laminated plates under various boundary conditions were derived. Through dimension expansion, the non-homogeneous equations were converted to homogeneous equations, thus the possible problems of numerical ill-conditions were avoided, and the calculation process was simplified for the homogeneous equations. An introduction of the hypothesis of nonlinear boundary displacement distribution along the thickness direction reasonably decreased the number of sub-layers for the convergence of numerical results. The numerical results in the examples provide standard referential solutions for other numerical or semi-analytical methods. The present method is suitable for the problems under more complex boundary conditions.

Key words: state space; rectangular laminated plate; boundary displacement function; analytical solution

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