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具有边界摄动弱非线性反应扩散方程的奇摄动*

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摘要: 在适当的条件下研究了一类具有边界摄动的非线性反应扩散方程奇摄动初始边值问题. 首先,借助正规摄动方法,得到了原问题的外部解. 其次,利用伸长变量和幂级数展开理论,构造了解的初始层项. 然后,利用微分不等式理论,研究了初始边值问题解的渐近性态. 最后,利用一些相关的不等式,讨论了原问题解的存在、唯一性及其一致有效的渐近估计.

关 键 词: 非线性; 反应扩散; 奇摄动; 边界摄动

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引　言

研究非线性奇摄动问题是数学界非常关注的一个问题[1]. 近来,渐近方法被发展和优化,包括平均法、边界层法、匹配渐近展开法和多重尺度法. 许多学者,诸如Ni和Wei[2],Zhang[3],Khasminskii和Yin[4],Marques[5]以及Bobkova[6]做了大量的工作. 利用微分不等式等方法,莫嘉琪等人也研究了一类奇摄动非线性常微分方程边值问题[7],如反应扩散方程[8-9]、椭圆型边值问题[10]、奇摄动问题的激波层解[11]、大气物理问题[12-16]和生态问题[17-18]等. 本文是用一个特殊而简单的奇摄动方法,研究一类带有边界摄动的奇摄动非线性反应扩散问题.

现讨论如下弱非线性奇摄动问题:

$$\varepsilon \frac{\partial u}{\partial t} - Lu = \varepsilon f(x, u, u_x), \qquad (t, x) \in (0, T] \times \Omega_\varepsilon, \tag{1}$$

$$u = g(x, \varepsilon), \qquad x \in \partial\Omega_\varepsilon, \tag{2}$$

$$u = h(x, \varepsilon), \qquad t = 0, \tag{3}$$

其中,ε为小的正参数,$x \equiv (x_1, x_2, \cdots, x_n) \in \Omega_\varepsilon \subset R^n$,$u_x \equiv (u_{x_1}, u_{x_2}, \cdots, u_{x_n})$,$\Omega_\varepsilon$表示具有光

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滑的摄动边界 $\partial\Omega$ 的有界凸域，L 为一致椭圆型算子，

$$L=\sum_{i,j=1}^{n}a_{ij}\frac{\partial^2}{\partial x_i\partial x_j},\ \sum_{i,j=1}^{n}a_{ij}\xi_i\xi_j\geqslant\lambda\sum_{i=1}^{n}\xi_i^2,\qquad\lambda>0.$$

问题(1)~(3)是一个带有边界摄动的初始边值问题，我们来构造其解的渐近展开式并讨论它的渐近性态.

假设

[H_1] L 的系数和 f,g 及 h 在它们的自变量对应的区域内为充分光滑的函数；

[H_2] $f_u(x,u,u_x)\leqslant -c_1<0$，其中 c_1 为正常数；

[H_3] $\Omega_0\supset\Omega_\varepsilon$，其中 Ω_0 为 Ω_ε 当 $\varepsilon=0$ 时的区域.

1 外部解

设原问题(1)~(3)的外部解 U 的形式渐近展开式为

$$U(x,\varepsilon)\sim\sum_{i=0}^{\infty}U_i(x)\varepsilon^i. \tag{4}$$

将式(4)代入式(1)，按 ε 展开 f，合并 ε 同次幂的系数，并对 $i=0,1,2,\cdots$ 等式两边的系数相等，我们得到

$$-LU_0=0, \tag{5}$$

$$-LU_i=F_{i-1},\qquad i=1,2,\cdots, \tag{6}$$

其中，$F_i(i=1,2,\cdots)$ 为关于 $\{U_j\}_{j=0}^{i-1}$ 为已知的函数.

考虑边界摄动. 将式(4)代入式(2)，得

$$U(x,\varepsilon)\mid_{x\in\partial\Omega_\varepsilon}=\sum_{i=0}^{\infty}[U_i(x)]_{x\in\partial\Omega_\varepsilon}\varepsilon^i=$$

$$\sum_{i=0}^{\infty}\left[[U_i]_{x\in\partial\Omega_0}+\sum_{j=1}^{i}\left[\frac{1}{j!}\left[\frac{\partial^jU_{i-j}}{\partial\varepsilon^j}\right]_{x\in\partial\Omega_0}\right]\right]\varepsilon^i,$$

$$g(x,\varepsilon)\mid_{x\in\partial\Omega_\varepsilon}=\sum_{i=0}^{\infty}\left[\sum_{j=0}^{i}\left[\frac{1}{j!}\left[\frac{\partial^jg_{i-j}}{\partial\varepsilon^j}\right]_{x\in\partial\Omega_0}\right]\right]\varepsilon^i.$$

于是得到如下 U_i 的边界条件：

$$[U_0]_{x\in\partial\Omega_0}=g(x,0)\mid_{x\in\partial\Omega_0}, \tag{7}$$

$$[U_i]_{x\in\partial\Omega_0}=\sum_{j=1}^{i}\left[\frac{1}{j!}\left[\frac{\partial^jg_{i-j}}{\partial\varepsilon^j}\right]_{x\in\partial\Omega_0}\right]-\sum_{j=1}^{i}\left[\frac{1}{j!}\left[\frac{\partial^jU_{i-j}}{\partial\varepsilon^j}\right]_{x\in\partial\Omega_0}\right],\qquad i=1,2,\cdots. \tag{8}$$

由上面的线性问题(5)~(8)，能依次解出 $U_i(i=0,1,2,\cdots)$. 再由式(4)，我们得到了原问题的外部解 $U(x,\varepsilon)$. 但它未必满足初始条件(3)，故尚需构造初始层校正项 V.

2 初始层校正

引入伸长变量[1] $\tau=t/\varepsilon$，并设原问题(1)~(3)的解 u 为

$$u=U(x,\varepsilon)+V(\tau,x,\varepsilon). \tag{9}$$

将式(9)代入式(1)~(3)，得

$$V_\tau-LV=\varepsilon[f(x,U+V,U_x+V_x)-f(x,U,U_x)],\qquad x\in\Omega_\varepsilon, \tag{10}$$

$$V=0,\qquad x\in\partial\Omega_\varepsilon, \tag{11}$$

$$V(0,x,\varepsilon) = h(x,\varepsilon) - U(x,\varepsilon), \qquad x \in \Omega_\varepsilon. \tag{12}$$

令

$$V \sim \sum_{i=0}^{\infty} v_i(\tau,x)\varepsilon^i. \tag{13}$$

将式(13)、(4)和式(9)代入式(10)~(12),展开非线性项,并使等式两边 ε 的同次幂相等,我们有

$$(v_0)_\tau - Lv_0 = 0, \qquad x \in \Omega_0, \tag{14}$$

$$v_0 = 0, \qquad x \in \partial\Omega_0, \tag{15}$$

$$v_0(0,x) = h(x,0) - U_0(x), \qquad x \in \Omega_0. \tag{16}$$

对于 $i = 1,2,\cdots$, 有

$$(v_i)_\tau - L v_i = \bar{F}_i, \qquad x \in \Omega_0, \tag{17}$$

$$v_i \mid_{\partial\Omega_0} = -\sum_{j=1}^{i}\left[\frac{1}{j!}\left[\frac{\partial^i(U_{i-j}+v_{i-j})}{\partial\varepsilon^j}\right]_{x\in\partial\Omega_0}\right], \qquad x \in \partial\Omega_0, \tag{18}$$

$$v_i(0,x) = \left[\frac{1}{i!}\left[\frac{\partial^i h}{\partial\varepsilon^i}\right]_{\varepsilon=0}\right]_{\tau=0} - U_i(x), \qquad x \in \Omega_0. \tag{19}$$

其中 $\bar{F}_i(i = 1,2,\cdots)$,为关于$\{U_j\}_{j=0}^{i}$和$\{v_j\}_{j=0}^{i-1}$的已知函数.

由问题(14)~(16)和式(17)~(19),我们可得到 v_0 和 $v_i(i = 1,2,\cdots)$,且能构造如下的原问题(1)~(3)解 u 的形式渐近展开式

$$u \sim \sum_{i=0}^{\infty}[U_i + v_i]\varepsilon^i, \qquad 0 < \varepsilon \ll 1. \tag{20}$$

3 最后的结果

现证式(20)为 Ω_ε 中的一致有效的渐近展开式. 有如下定理:

定理 在假设$[H_1]$~$[H_3]$下,奇摄动问题(1)~(3)存在1个解 u,并在$[0,T]\times(\Omega_\varepsilon + \partial\Omega_\varepsilon)$上有一致有效的渐近展开式(20).

证明 首先构造辅助函数 α 和β:

$$\alpha = Y_m - \bar{r}\varepsilon^m, \tag{21}$$

$$\beta = Y_m + \bar{r}\varepsilon^m, \tag{22}$$

其中, $\bar{r}$ 为足够大的正常数,它将在下面决定,且

$$Y_m \equiv \sum_{i=0}^{m}[U_i + v_i]\varepsilon^i.$$

显然

$$\alpha \leqslant \beta,\ (t,x) \in [0,T]\times(\Omega_\varepsilon + \partial\Omega_\varepsilon), \tag{23}$$

及关于 $x \in \partial\Omega_\varepsilon$ 存在一个正常数M_1, 使得

$$\alpha = Y_m - \bar{r}^m\varepsilon^m = \sum_{i=0}^{m}U_i\varepsilon^i + \sum_{i=0}^{m}v_i\varepsilon^i - \bar{r}\varepsilon^m =$$

$$g(x,0) + \sum_{i=1}^{m}\left[\sum_{j=1}^{i}\left[\frac{1}{j!}\left[\frac{\partial^j g_{i-j}}{\partial\varepsilon^j}\right]_{x\in\partial\Omega_0}\right] - \sum_{j=1}^{i}\left[\frac{1}{j!}\left[\frac{\partial^j U_{i-j}}{\partial\varepsilon^j}\right]_{x\in\partial\Omega_0}\right]\right]\varepsilon^i -$$

$$\sum_{i=0}^{m}\left[\sum_{j=1}^{i}\left[\frac{1}{j!}\left[\frac{\partial^j(U_{i-j}+v_{i-j})}{\partial\varepsilon^j}\right]_{x\in\partial\Omega_0}\right]\right]\varepsilon^i \leqslant$$

$$g(x,\varepsilon)+(M_1-\bar{r})\varepsilon^m.$$

于是选择 $\bar{r}\geqslant M_1$，我们有

$$\alpha\leqslant g(x,\varepsilon),\qquad x\in\partial\Omega_\varepsilon. \tag{24}$$

同理可得

$$\beta\geqslant g(x,\varepsilon),\qquad x\in\partial\Omega_\varepsilon. \tag{25}$$

并且由假设，也存在一个正常数 M_2，使得

$$\alpha(0,x,\varepsilon)=Y_m|_{t=0}-\bar{r}\varepsilon^m=\sum_{i=0}^m U_i\varepsilon^i+\sum_{i=0}^m v_i|_{\tau=0}\varepsilon^i-\bar{r}\varepsilon^m\leqslant$$
$$\sum_{i=0}^m U_i\varepsilon^i+[h(x,0)-U_0(x)]+$$
$$\sum_{i=1}^m\left[\sum_{j=1}^i\left[\frac{1}{j!}\left[\frac{\partial^j h}{\partial\varepsilon^j}\right]_{\varepsilon=0}\right]_{\tau=0}-U_i(x)\right]\varepsilon^i-\bar{r}\varepsilon^m\leqslant$$
$$h(x,\varepsilon)+(M_2-\bar{r})\varepsilon^m.$$

其中，ε_1 为足够小的正常数，选择 $\bar{r}\geqslant M_2$，我们有

$$\alpha(0,x,\varepsilon)\leqslant h(x,\varepsilon),\qquad x\in\Omega_\varepsilon. \tag{26}$$

同理对于 $\bar{r}\geqslant M_2$，也有

$$\beta(0,x,\varepsilon)\geqslant h(x,\varepsilon),\qquad x\in\Omega_\varepsilon. \tag{27}$$

现证

$$\varepsilon\alpha_t-L\alpha-f(x,\alpha,\alpha_x)\leqslant 0,\qquad (t,x)\in(0,T)\times\Omega_\varepsilon, \tag{28}$$
$$\varepsilon\beta_t-L\beta-f(x,\beta,\beta_x)\geqslant 0,\qquad (t,x)\in(0,T)\times\Omega_\varepsilon. \tag{29}$$

由假设，对于足够小的 ε，存在一个正常数 M_3，使得

$$\varepsilon\alpha_t-L\alpha-f(x,\alpha,\alpha_x)=$$
$$\varepsilon(Y_m-\bar{r}\varepsilon^m)_t-L[Y_m-\bar{r}\varepsilon^m]-f(x,\alpha,\alpha_x)=$$
$$\varepsilon(Y_m)_t-LY_m-f(x,Y_m,Y_{mx})+[f(x,Y_m,Y_{mx})-f(x,\alpha,\alpha_x)]\leqslant$$
$$-LU_0-\sum_{i=1}^m[LU_i+F_{i-1}]\varepsilon^i+(v_0)_\tau-Lv_0+\sum_{i=1}^m[(v_i)_\tau-Lv_i-\bar{F}_i]\varepsilon^i+$$
$$M_3\varepsilon^m-c_1\bar{r}\varepsilon^m\leqslant$$
$$(M_3-c_1\bar{r})\varepsilon^m,$$

选择 $\bar{r}\geqslant M_3/c_1$，我们便证明了不等式(28).

同理可证不等式(29)也成立.

于是由式(23)～(29)，利用微分不等式理论，我们得到[19-20]

$$\alpha(t,x,\varepsilon)\leqslant u(t,x,\varepsilon)\leqslant\beta(t,x,\varepsilon),$$
$$(t,x,\varepsilon)\in[0,T]\times(\Omega_\varepsilon+\partial\Omega_\varepsilon)\times[0,\varepsilon_1],$$

其中，ε_1 为足够小的正常数，且由式(21)和(22)，便得到了最后的结果

$$u=\sum_{i=0}^m[U_i+v_i]\varepsilon^i+O(\varepsilon^m),\qquad 0<\varepsilon\ll 1,$$

定理证毕.

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Singular Perturbation for the Weakly Nonlinear Reaction Diffusion Equation With Boundary Perturbation

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Abstract: A class of nonlinear singularly perturbed initial boundary value problems for reaction diffusion equations with boundary perturbation is considered under suitable conditions. Firstly, by dint of regular perturbation method, the outer solution of the original problem was obtained. Secondly, by using the stretched variable and the expanding theory of power series, the initial layer term of solution was constructed. And then, by using the theory of differential inequalities the asymptotic behavior of solutions for the initial boundary value problems was studied. Finally, using some relational inequalities the existence and uniqueness of solution for the original problem and the uniformly valid asymptotic estimation were discussed.

Key words: nonlinear; reaction diffusion; singular perturbation; boundary perturbation