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超抛物型方程的非线性奇摄动问题*

林苏榕[1], 莫嘉琪[2,3]

(1. 福建广播电视大学 计算机系, 福州 350003;
2. 安徽师范大学 数学系, 安徽 芜湖 241000;
3. 上海高校计算科学 E-研究院, 上海交通大学研究所, 上海 200240)

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摘要: 讨论了一类超抛物型方程的非线性奇摄动问题. 利用比较定理, 研究了问题解的存在性及其渐近性态.

关 键 词: 超抛物型方程; 奇摄动; 渐近性态; 比较定理

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引 言

许多自然现象的数学模型都涉及到超抛物型方程[1-2]:

$$\sum_{i=1}^{m} k_i(x, t) u_{t_i} - Lu = f(x, t, u),$$

其中, $x = (x_1, x_2, \cdots, x_n) \in R^n$, $t = (t_1, t_2, \cdots, t_m) \in R^m$; L 为椭圆型算子:

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x, t) \frac{\partial u}{\partial x_i} + c(x, t) u,$$

$$\sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \geqslant \lambda \sum_{i=1}^{n} \xi_i^2, \qquad \lambda > 0, \ \xi_i \in \mathbf{R}.$$

至今, 超抛物型方程理论研究一般尚未成熟. 近来非线性奇摄动问题在国际学术界已经有较多的研究. 近似方法也有进一步的发展和优化, 包括平均法、边界层法、匹配渐近展开法和多重尺度法[3-7]. 利用微分不等式和其它方法, 莫嘉琪等人也研究了一类非线性奇摄动问题[8-15]. 在文献[2]中, 讨论了一类线性超抛物型方程. 本文是涉及一类半线性奇摄动问题当 $n = 1$, $m = 2$, $k_1(x, t) = 1$ 时的超抛物型方程. 首先建立比较定理, 然后研究其解的渐近性态.

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作者简介: 林苏榕(1958—), 女, 福建福州人, 副教授(Tel: +86-553-87864970; E-mail: lsr@fjrtvu.edu.cn);
莫嘉琪(1937—), 男, 浙江德清人, 教授(联系人. Tel: +86-553-3869642; E-mail: mojiaqi@mail.ahnu.edu.cn).

考虑如下半线性超抛物型方程奇摄动问题:

$$\frac{\partial u}{\partial t}+k(x,\tau,t)\frac{\partial u}{\partial \tau}-\varepsilon\frac{\partial^2 u}{\partial \tau^2}-L_1 u=f(x,\tau,t,u),\qquad (x,\tau,t)\in Q_T, \tag{1}$$

$$u|_{x=0}=u|_{x=1},\qquad (u,u_\tau)|_{\tau=0}=(u,u_\tau)|_{\tau=1}, \tag{2}$$

$$u|_{t=0}=\Phi(x,\tau), \tag{3}$$

其中

$$L_1=a(x,\tau,t)\frac{\partial^2}{\partial x^2}+b(x,\tau,t)\frac{\partial}{\partial x}+c(x.\tau,t),$$

$$Q_T:=\{(x,\tau,t)\in[0,1]^2\times[0,T],$$

ε 是小的正参数, $a(x,\tau,t)\geqslant a_0>0$.

假设

[H_1] $k,a,b,c,f\in C^{2+\alpha,1+\alpha/2}_{\xi\ t}(Q_T)$, 其中, $\xi=(x,\tau)\in R^2$, $\Phi\in C^{2+\alpha}(Q_T)$;

[H_2] L_1 的系数、f 以及它们关于 τ 的偏导数在 Q_T 盅均为关于 τ 的单位周期的函数;

[H_3] $(k,f)|_{x=0}=(k,f)|_{x=1}=0$;

[H_4] f 关于 u 为充分光滑的函数, 且 $-N\leqslant f_u\leqslant -l$, 其中 N、l 为正常数.

定义 设 $\bar u(x,\tau,t)$, $\underline{u}(x,\tau,t)$ 为在 Q_T 上充分光滑的函数, 使得 $\underline{u}\leqslant \bar u$, 并成立

$$\frac{\partial \bar u}{\partial t}+k(x,\tau,t)\frac{\partial \bar u}{\partial \tau}-\varepsilon\frac{\partial^2 \bar u}{\partial \tau^2}-L_1\bar u-f(x,\tau,t,\bar u)\geqslant 0,\qquad (x,\tau,t)\in Q_T,$$

$$\frac{\partial \underline{u}}{\partial t}+k(x,\tau,t)\frac{\partial \underline{u}}{\partial \tau}-\varepsilon\frac{\partial^2 \underline{u}}{\partial \tau^2}-L_1\underline{u}-f(x,\tau,t,\underline{u})\leqslant 0,\qquad (x,\tau,t)\in Q_T,$$

$$(\underline{u}|_{x=0},\underline{u}|_{x=1})\leqslant(\bar u|_{x=0},\bar u|_{x=1}),\ (\bar u,\bar u_\tau)|_{\tau=0}=(\bar u,\bar u_\tau)|_{\tau=1},$$

$$(\underline{u},\underline{u}_\tau)|_{\tau=0}=(\underline{u},\underline{u}_\tau)|_{\tau=1},\ \underline{u}(x,\tau,0)\leqslant\Phi(x,\tau)\leqslant\bar u(x,\tau,0).$$

则 $\bar u$ 和 $\underline{u}$ 分别称为问题(1)~(3)的上解和下解.

定理 1 在假设[H_1]~[H_3]下, 对于 $\forall\varepsilon\in(0,\varepsilon_0]$, 若问题(1)~(3)有一个上解 $\bar u(x,\tau,t)$ 和下解 $\underline{u}(x,\tau,t)$, 则奇摄动问题(1)~(3)存在唯一的解 u, 并且

$$\underline{u}(x,\tau,t)\leqslant u(x,\tau,t)\leqslant \bar u(x,\tau,t),\qquad (x,\tau,t)\in Q_T.$$

证明 设 $\bar u^0=\bar u$, $\underline{u}^0=\underline{u}$ 为两个不同的初始迭代. 这时由文献[2]的定理2.2, 我们能由下列线性系统分别构造两个序列 $\{\bar u^k\}$、$\{\underline{u}^k\}$:

$$\frac{\partial \bar u^k}{\partial t}+k(x,\tau,t)\frac{\partial \bar u^k}{\partial \tau}-\varepsilon\frac{\partial^2 \bar u^k}{\partial \tau^2}-L_1\bar u^k+N\bar u^k=$$

$$N\bar u^{k-1}+f(x,\tau,t,\bar u^{k-1}),\qquad (x,\tau,t)\in Q_T, \tag{4}$$

$$\bar u^k|_{x=0}=\bar u^k|_{x=1}=0,\ (\bar u^k,\bar u^k_\tau)|_{\tau=0}=(\bar u^k,\bar u^k_\tau)|_{\tau=1}, \tag{5}$$

$$\bar u^k|_{t=0}=\Phi(x,t), \tag{6}$$

$$\frac{\partial \underline{u}^k}{\partial t}+k(x,\tau,t)\frac{\partial \underline{u}^k}{\partial \tau}-\varepsilon\frac{\partial^2 \underline{u}^k}{\partial \tau^2}-L_1\underline{u}^k+N\underline{u}^k=$$

$$N\underline{u}^{k-1}+f(x,\tau,t,\underline{u}^{k-1}),\qquad (x,\tau,t)\in Q_T, \tag{7}$$

$$\underline{u}^k|_{x=0}=\underline{u}^k|_{x=1}=0,\ (\underline{u}^k,\underline{u}^k_\tau)|_{\tau=0}=(\underline{u}^k,\underline{u}^k_\tau)|_{\tau=1}, \tag{8}$$

$$\underline{u}^k|_{t=0}=\Phi(x,\tau). \tag{9}$$

设 $w=\bar u^0-\bar u^1$. 由式(4)~(6), 有

$$\frac{\partial w}{\partial t}+k(x,\tau,t)\frac{\partial w}{\partial \tau}-\varepsilon\frac{\partial^2 w}{\partial \tau^2}-L_1 w+Nw\geqslant$$

$$N(u^0 - u^0) + f(x, \tau, t, u^0) - f(x, \tau, t, u^0) = 0, \qquad (x, \tau, t) \in Q_T,$$

$$w|_{x=0} = w|_{x=1} = 0, \ (w, w_\tau)|_{\tau=0} = (w, w_\tau)|_{\tau=1}, \ w|_{t=0} \geqslant 0.$$

于是 $w \geqslant 0$, 即

$$u^1 \leqslant u^0, \qquad (x, \tau, t) \in Q_T.$$

类似可得

$$\underline{u}^1 \geqslant \underline{u}^0, \qquad (x, \tau, t) \in Q_T.$$

现证明 $u^1 \geqslant \underline{u}^1$. 令 $w = u^1 - \underline{u}^1$. 由式(4) ~ (6) 和式(7) ~ (9) 有

$$\frac{\partial w}{\partial t} + k(x, \tau, t)\frac{\partial w}{\partial \tau} - \varepsilon\frac{\partial^2 w}{\partial \tau^2} - L_1 w + Nw =$$

$$N(u^0 - \underline{u}^0) + f(x, \tau, t, u^0) - f(x, \tau, t, u^0) \geqslant 0, \qquad (x, \tau, t) \in Q_T,$$

$$w|_{x=0} = w|_{x=1} = 0, \ (w, w_\tau)|_{\tau=0} = (w, w_\tau)|_{\tau=1}, \ w|_{t=0} \geqslant 0.$$

于是 $w \geqslant 0$, 即

$$\underline{u}^1 \leqslant u^1, \qquad (x, \tau, t) \in Q_T.$$

类似地有

$$\underline{u} = \underline{u}^0 \leqslant \underline{u}^1 \leqslant \cdots \leqslant \underline{u}^k \leqslant \cdots \leqslant u^k \leqslant \cdots \leqslant u^1 \leqslant u^0 = u, \qquad (x, \tau, t) \in Q_T.$$

由假设[H_1] ~ [H_3], 文献[2] 的定理 2.2 和不动点定理[16], 我们能证明:

$$\lim_{k\to\infty} \underline{u}^k = \lim_{k\to\infty} u^k = u, \qquad (x, \tau, t) \in Q_T,$$

且 u 为问题(1) ~ (3) 唯一的解. 定理 1 证毕.

现首先考虑问题(1) 当 $\varepsilon \to 0$ 时的极限方程:

$$\frac{\partial u}{\partial t} + k(x, \tau, t)\frac{\partial u}{\partial \tau} - L_1 u = f(x, \tau, t, u), \qquad (x, \tau, t) \in Q_T, \tag{10}$$

$$u|_{x=0} = u|_{x=1} = 0, \quad u|_{\tau=0} = u|_{\tau=1}, \tag{11}$$

$$u|_{t=0} = \phi(x, \tau). \tag{12}$$

我们还需要如下假设:

[H_5] 问题(10) ~ (12) 在 Q_T 上有唯一的光滑解 U_0.

其次, 考虑奇摄动问题(1) ~ (3) 的形式渐近解.

令原问题解 U 的形式展开式为

$$U \sim \sum_{i=0}^{\infty} U_i \varepsilon^i. \tag{13}$$

显然, 式(13) 中的 U_0 是超极限问题(10) ~ (12) 的解.

将式(13) 代入问题(1) ~ (3), 合并 ε 的同次幂, 对于 $i = 1, 2, \cdots$, 得

$$\frac{\partial U_i}{\partial t} + k(x, \tau, t)\frac{\partial U_i}{\partial \tau} - L_1 U_i - f_u(x, \tau, t, U_0) U_i =$$

$$F_i + \frac{\partial^2 U_{i-1}}{\partial \tau^2}, \qquad (x, \tau, t) \in Q_T, \tag{14}$$

$$U_i|_{\tau=0} = U_i|_{\tau=1}, \quad U_i|_{t=0} = 0, \tag{15}$$

其中, F_i 为关于 U_r, $r \leqslant j - 1$ 的已知函数.

于是由文献[2] 的定理 2.3, 能依次解出 U_i, $i = 1, 2, \cdots$. 由式(13), 我们便得到原问题(1) ~ (3) 解的形式渐近展开式:

$$u \sim \sum_{j=0}^{m} U_j \varepsilon^j + O(\varepsilon^{m+1}), \qquad 0 < \varepsilon \ll 1. \tag{16}$$

现证式(16)为一致有效的渐近展开式.

定理 2 在假设$[H_1]\sim[H_5]$下, 奇摄动超抛物型方程奇摄动问题(1)～(3)存在唯一的解u, 在Q_T中关于ε有一致有效的渐近展开式(16).

证明 首先构造辅助函数 α、β:

$$\alpha = Y_m - r\varepsilon^{m+1}, \quad \beta = Y_m + r\varepsilon^{m+1}. \tag{17}$$

其中, r为一个足够大的正常数, 它将在下面确定, 而 $Y_m = \sum_{j=0}^{m} U_j\varepsilon^j$.

显然由式(17)不难看出

$$\alpha \leqslant \beta, \tag{18}$$

$$(\alpha, \alpha_\tau)\,|_{\tau=0} = (\alpha, \alpha_\tau)\,|_{\tau=1}, \quad (\beta, \beta_\tau)\,|_{\tau=0} = (\beta, \beta_\tau)\,|_{\tau=1}, \tag{19}$$

$$(\alpha\,|_{x=0}, \alpha\,|_{x=1}) \leqslant (\beta\,|_{x=0}, \beta\,|_{x=1}), \quad \alpha(x, \tau, 0) \leqslant \Phi(x, \tau) \leqslant \beta(x, \tau, 0). \tag{20}$$

现证

$$\frac{\partial\alpha}{\partial t} + k(x, \tau, t)\frac{\partial\alpha}{\partial\tau} - \varepsilon\frac{\partial^2\alpha}{\partial\tau^2} - L_1\alpha - f(x, \tau, t, \alpha) \leqslant 0, \qquad (x, \tau, t) \in Q_T, \tag{21}$$

$$\frac{\partial\beta}{\partial t} + k(x, \tau, t)\frac{\partial\beta}{\partial\tau} - \varepsilon\frac{\partial^2\beta}{\partial\tau^2} - L_1\beta - f(x, \tau, t, \beta) \geqslant 0, \qquad (x, \tau, t) \in Q_T. \tag{22}$$

事实上, 存在一个正常数M, 使得

$$\begin{aligned}
&\frac{\partial\alpha}{\partial t} + k(x, \tau, t)\frac{\partial\alpha}{\partial\tau} - \varepsilon\frac{\partial^2\alpha}{\partial\tau^2} - L_1\alpha - f(x, \tau, t, \alpha) = \\
&\quad \frac{\partial Y_m}{\partial t} + k(x, \tau, t)\frac{\partial Y_m}{\partial\tau} - \varepsilon\frac{\partial^2 Y_m}{\partial\tau^2} - L_1 Y_m - f(x, \tau, t, Y_m) + \\
&\quad [f(x, \tau, t, Y_m) - f(x, \tau, t, Y_m - r\varepsilon^{m+1})] \leqslant \\
&\quad \frac{\partial U_0}{\partial t} + k(x, \tau, t)\frac{\partial U_0}{\partial\tau} - L_1 U_0 - f(x, \tau, t, U_0) + \\
&\quad \sum_{i=1}^{m}\left[\frac{\partial U_i}{\partial t} + k(x, \tau, t)\frac{\partial U_i}{\partial\tau} - L_1 U_i - f_u(x, \tau, t, U_0)U_i - F_i - \frac{\partial^2 U_{i-1}}{\partial\tau^2}\right]\varepsilon^i + \\
&\quad [f(x, \tau, t, Y_m) - f(x, \tau, t, Y_m - r\varepsilon^{m+1})] + M\varepsilon^{m+1} \leqslant (M - rl)\varepsilon^{m+1}.
\end{aligned}$$

选取 $r \geqslant M/l$, 这时我们有不等式(21).

同样我们能证明不等式(22).

由式(18)～(22)和定理 1, 非线性奇摄动问题(1)～(3)存在唯一的解u, 且成立

$$\alpha(x, \tau, t) \leqslant u(x, \tau, t) \leqslant \beta(x, \tau, t), \qquad (x, \tau, t) \in Q_T.$$

所以由式(17), 我们得到式(16). 定理 2 证毕.

注 超抛物型方程非线性奇摄动问题(1)～(3)不具有边界层或初始层.

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Nonlinear Singularly Perturbed Problems of Ultra Parabolic Equations

LIN Su-rong[1], MO Jia-qi[2,3]

(1. Department of Computer, Fujian Radio and TV University, Fuzhou 350003, P.R. China;

2. Department of Mathematics, Anhui Normal University, Wuhu, Anhui 241000, P.R. China;

3. Division of Computational Science, E-Institutes of Shanghai Universities at SJTU, Shanghai 200240, P.R. Chian)

Abstract: A class of nonlinear singularly perturbed problem with ultra parabolic equation are considered. Using the comparison theorem, the existence, uniqueness and its asymptotic behavior of solution for the problem are studied.

Key words: ultra parabolic equation; singular perturbation; asymptotic behavior; comparison theorem