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弹性混合板的振动和稳定问题*

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摘要: 通过将非线性应变位移关系引入 Hellinger-Reissner 变分原理, 推导出了各向异性板基于弹性理论下的振动和屈曲控制方程. 用精细积分法研究了四边简支混合矩形板. 精细积分法与传统的有限差分法相比, 可以给出计算机精度允许的很精确的数值结果. 所以, 给出的混合板的振动和稳定的结果可以看作是近似解析的. 这些结果可以作为衡量各种板理论准确性的一个标准. 而且, 非对称层合板中出现的各种耦合影响, 比如弯扭耦合, 拉弯耦合等在同一组控制方程里被考虑了.

关 键 词: 振动; 屈曲; 精细积分; 混合板; 状态空间

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引 言

振动和稳定问题是层合板研究中最为重要的问题. 因此, 这些问题在过去几十年已经引起了很多研究者的注意. 这个课题通常用经典板理论或者 Mindlin 一阶板理论进行研究[1-]然而, 层合板的精确解[4-5]表明经典板理论和一阶板理论不能充分模拟层合板的行为. 因此, 为了改进这两种理论的精确性, 许多不同的高阶理论[6-11]应运而生. 这些理论中, Lo 的高阶板理论[11]非常准确地预测了非线性横向剪切应力分布, 这已经在高阶板理论[12]的评估中得到了证实.

以上这些只是研究了同一种材料铺设的层合板. 然而, 由不同材料铺设的混合板已被广泛的应用. Barai 等[1]用一阶理论研究了混合板的振动和屈曲. 阐述了曲率, 长宽比, 叠层顺序和铺层方向的影响. Benjeddou 等[14]给出了简支压电夹层板自由振动的二维闭合解. 由纤维增强复合材料和金属叠加而成的混合板已经被广泛应用于高温环境. Harras 等[15]给出了基于哈密顿原理的理论模型, 研究了由金属和纤维增强复合材料组成的混合板的非线性自由振动. Lee 等[16]用基于经典板理论的 Lagrange 方程研究了混合板的振动. 研究了铺层顺序, 长宽比和层数对振动行为的影响. Chen 等[17]基于 Mindlin 板理论介绍了弹性混合板的振动行为. 发现基于 Mindlin 板理论的振动频率比(非线性振动频率与线性振动频率的比) 大于基于经典板理论的振动频率比. 这是因为在经典板理论中没有考虑横向剪切和转动惯量的影响.

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在制造过程结束后,层合板中总会存在残余应力．因此,带有残余应力的板的研究已经吸引了研究者的注意．Brunell 等[18-19]用平均应力和振动方法研究了任意初应力对各向同性板的振动的影响．Yamaki 等[20]基于一阶理论研究了初应力层合板的振动行为．Rogerson 等[21]基于不同的理论,介绍了预应力层合板小幅振动的任意应变能函数．Cheung 等[22]用有限条法分析了初应力板的非线性振动．Chen 等[2-25],Doong 等[26-27]和 Chen 等[17, 28-0]也研究了初应力对振动和稳定的影响．

本文用精细积分法[1]基于弹性理论研究了弹性混合板在任意初应力下振动和稳定的解析解,这在以前从来没有报导过．精细积分法与有限差分法相比,可以给出计算机所允许的很精确的数值结果．因此,本文弹性混合板振动和稳定的结果可以认为是近似解析的．这些结果可以作为衡量其它板理论准确性的一个标准．Chen 等[0]研究了基于高阶理论的弹性混合板并给出了很详细的计算结果．然而,一些结果似乎存在疑问．本文对此做出了修正．非对称层合板中出现的各种耦合影响,比如弯扭耦合,拉弯耦合等在同一组控制方程里被考虑了．

将非线性应变位移关系引入 Hellinger-Reissner 变分原理[2-],推导出了状态空间体系下的振动和屈曲控制方程[4]．文献[5]从大变形理论也得到类似的方程．由于控制方程中的未知变量是状态空间向量,所以可以满足层合板中层间位移和应力的连续．

1 控 制 方 程

用 Gallagher[4]的方法推导出控制方程．二维 Hellinger-Reissner 广义势能[2-]为

$$U(u, v, \sigma_x, \sigma_y, \tau_{xy}) = \iint_\Omega \Big[\sigma_x \frac{\partial u}{\partial x} + \sigma_y \frac{\partial v}{\partial y} + \tau_{xy}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) - \frac{1}{2E}(\sigma_x^2 + \sigma_y^2) + \frac{\nu}{E}\sigma_x\sigma_y - \frac{1}{2G}\tau_{xy}^2 - f_x u - f_y v\Big] \mathrm{d}\Omega - \int_{S_u} [p_x(u - \bar{u}) + p_y(v - \bar{v})] \mathrm{d}S - \int_{S_\sigma} [(p_x - \bar{p}_x)u + (p_y - \bar{p}_y)v] \mathrm{d}S, \tag{1}$$

这里 S_u 表示位移边界条件, S_σ 表示力的边界条件.

状态空间法的优点在于分析层合结构并严格满足层间位移和应力连续． y 等于常数时状态空间向量为$(u, v, \sigma_y, \tau_{xy})$． 由 Hooke 定律得

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y),$$

这里 $\varepsilon_x = \dfrac{\partial u}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x}\right)^2, \quad \varepsilon_y = \dfrac{\partial v}{\partial y}.$

令位移条件自然满足 $u = \bar{u}, v = \bar{v}$． 将

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y)$$

代入方程(1),得到 $(u, v, \sigma_y, \tau_{xy})$ 的泛函． 这里为了考虑 σ_x 对 y 方向法线平衡的贡献引入了 ε_x 的非线性项．修正后的泛函表示为

$$U(u, v, \sigma_y, \tau_{xy}) = \iint_\Omega \Big\{ \tau_{xy}\frac{\partial u}{\partial y} + \sigma_y\frac{\partial v}{\partial y} + \frac{E}{2}\Big[\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{\partial v}{\partial x}\right)^4\Big] + \tau_{xy}\frac{\partial v}{\partial x} + \nu\sigma_y\Big[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2\Big] - \frac{1-\nu^2}{2E}\sigma_y^2 - \frac{1}{2G}\tau_{xy}^2 - f_x u - f_y v\Big\} \mathrm{d}\Omega -$$

$$\int_{S_u}[p_x(u-u)+p_y(v-v)]\mathrm{d}S-\int_{S_\sigma}[(p_x-p_x)u+(p_y-p_y)v]\mathrm{d}S. \tag{2}$$

$U(u, v, \sigma_y, \tau_{xy})$ 的变分表示为

$$\delta U(u,v,\sigma_y,\tau_{xy})=\iint_\Omega\Bigg[\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}-\frac{\tau_{xy}}{G}\right)\delta\tau_{xy}+\left(\frac{\partial v}{\partial y}+\nu\varepsilon_x-\frac{1-\nu^2}{E}\sigma_y\right)\delta\sigma_y-\left(\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+f_x\right)\delta u-\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+f_y+\sigma_x\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial\sigma_x}{\partial x}\right)\delta v\Bigg]\mathrm{d}\Omega+\int_{S_\sigma}[\delta u(p_x-p_x)+\delta v(p_y-p_y)]\mathrm{d}S=0. \quad (\ )$$

由方程( )得到下面的方程:

$$\frac{\tau_{xy}}{G}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}, \tag{4}$$

$$\sigma_y=\frac{E}{1-\nu^2}\left\{\frac{\partial v}{\partial y}+\nu\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2\right]\right\}, \tag{5}$$

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+f_x=0, \tag{6}$$

$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+f_y+\sigma_x\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial\sigma_x}{\partial x}=0. \tag{7}$$

由于小变形, $|\partial v/\partial x|\ll 1$, 所以

$$\frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 \text{和} \left|\frac{\partial v}{\partial x}\frac{\partial\sigma_x}{\partial x}\right|$$

可以被忽略. 因此得到下面的方程:

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}+\frac{1}{G}\tau_{xy}, \tag{8}$$

$$\frac{\partial\sigma_y}{\partial y}=-\frac{\partial\tau_{xy}}{\partial x}-f_y-\sigma_x\frac{\partial^2 v}{\partial x^2}, \tag{9}$$

$$\frac{\partial\tau_{xy}}{\partial y}=-E\frac{\partial^2 u}{\partial x^2}-\nu\frac{\partial\sigma_y}{\partial x}-f_x, \tag{10}$$

$$\frac{\partial v}{\partial y}=-\nu\frac{\partial u}{\partial x}+\frac{1-\nu^2}{E}\sigma_y. \tag{11}$$

方程(9)意味着由于变形 v, σ_x 对 y 方向法线平衡的贡献. 令 $\sigma_x=\sigma_x^0$, 作为初应力作用于弹性体. 令 $f_x=-\rho\partial^2u/\partial t^2, f_y=-\rho\partial^2v/\partial t^2$. ρ 代表材料密度. 方程(8)~(11) 可以作为计算弹性体在初应力 σ_x^0 下固有频率的控制方程, 即

$$\frac{\partial}{\partial y}\begin{Bmatrix}u\\ \sigma_y\\ \tau_{xy}\\ v\end{Bmatrix}=\begin{bmatrix}0 & 0 & \frac{1}{G} & -\frac{\partial}{\partial x}\\ 0 & 0 & -\frac{\partial}{\partial x} & -\sigma_x^0\frac{\partial^2}{\partial x^2}-\rho\frac{\partial^2}{\partial t^2}\\ -E\frac{\partial^2}{\partial x^2}-\rho\frac{\partial^2}{\partial t^2} & -\nu\frac{\partial}{\partial x} & 0 & 0\\ -\nu\frac{\partial}{\partial x} & \frac{1-\nu^2}{E} & 0 & 0\end{bmatrix}\begin{Bmatrix}u\\ \sigma_y\\ \tau_{xy}\\ v\end{Bmatrix}, \tag{12}$$

这里应用混合状态空间, 不仅保证了二维层合结构层间位移 (u, v) 的连续, 也保证了层间应力 (σ_y, τ_{xy}) 的连续.

对于各向异性材料, 在材料主方向 1, 2, 和 上的应力应变关系为

$$\varepsilon = \boldsymbol{S} \cdot \sigma_L \qquad (1\ )$$

或者

$$\sigma_L = \boldsymbol{C} \cdot \varepsilon, \qquad (14)$$

这里

$$\varepsilon = (\varepsilon_1 \quad \varepsilon_2 \quad \varepsilon \quad \gamma_2 \quad \gamma_1 \quad \gamma_{12})^{\mathrm{T}}, \qquad (15)$$

$$\sigma_L = (\sigma_1 \quad \sigma_2 \quad \sigma \quad \tau_2 \quad \tau_1 \quad \tau_{12})^{\mathrm{T}}, \qquad (16)$$

$$\boldsymbol{S} = \begin{bmatrix} 1/E_1 & -\nu_{12}/E_2 & -\nu_1/E & 0 & 0 & 0 \\ -\nu_{21}/E_1 & 1/E_2 & -\nu_2/E & 0 & 0 & 0 \\ -\nu_1/E_1 & -\nu_2/E_2 & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{12} \end{bmatrix}, \qquad (17)$$

$$\boldsymbol{C} = \boldsymbol{S}^{-1}, \qquad (18)$$

这里 $\nu_{ij}(i, j = 1, 2, \ )$ 表示 Possion 比, 即在 j 方向作用应力产生的 j 方向单位应变引起的 i 方向的横向应变. ε, σ_L, $\boldsymbol{S}$ 和 $\boldsymbol{C}$ 分别表示在材料主方向的应变, 应力向量, 材料柔度阵和材料刚度阵. E_1, E_2, E 表示在纤维主方向的弹性模量. G_{12}, G_2, G_1 表示相对于纤维主方向的剪切模量.

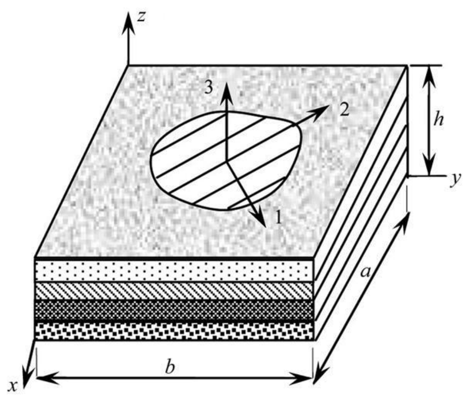


图 1 层合板的几何模型

对于角铺设层合板(见图 1, 材料主轴 与 z 轴一致), 任意层的应力应变关系可以表示为

$$\sigma = \boldsymbol{C} \cdot \varepsilon, \qquad (19)$$

这里

$$\sigma = (\sigma_x \quad \sigma_y \quad \sigma_z \quad \tau_{yz} \quad \tau_{xz} \quad \tau_{xy})^{\mathrm{T}}, \qquad (20)$$

$$\varepsilon = (\varepsilon_x \quad \varepsilon_y \quad \varepsilon_z \quad \gamma_{yz} \quad \gamma_{xz} \quad \gamma_{xy})^{\mathrm{T}}, \qquad (21)$$

$$\boldsymbol{C} = \begin{bmatrix} C_{11} & C_{12} & C_1 & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_2 & 0 & 0 & C_{26} \\ C_1 & C_2 & C & 0 & 0 & C_{\ 6} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{\ 6} & 0 & 0 & C_{66} \end{bmatrix} = \boldsymbol{A}^{-1} \cdot \boldsymbol{C} \cdot (\boldsymbol{A}^{-1})^{\mathrm{T}}, \qquad (22)$$

其中

$$\boldsymbol{A} = \begin{bmatrix} l_1^2 & m_1^2 & 0 & 0 & 0 & 2l_1 m_1 \\ l_2^2 & m_2^2 & 0 & 0 & 0 & 2l_2 m_2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 & l_2 & 0 \\ 0 & 0 & 0 & m_1 & l_1 & 0 \\ l_1 l_2 & m_1 m_2 & 0 & 0 & 0 & l_1 m_2 + l_2 m_1 \end{bmatrix}, \qquad (2\ )$$

$$l_1 = \cos\theta,\quad m_1 = \sin\theta,\quad l_2 = -\sin\theta,\quad m_2 = \cos\theta, \tag{24}$$

这里 θ 表示从总体坐标系 x 轴转向纤维方向的叠层的角度． 当 $\theta = 0°(90°)$ 时, 为正交铺设层合板． $\boldsymbol{A}$ 表示坐标转换矩阵． ε, σ 和 $\boldsymbol{C}$ 分别表示在总体坐标系中的应变, 应力向量, 和材料刚度阵.

对于三维结构包含非线性项的应变位移关系为

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2,\quad \varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2,\quad \varepsilon_z = \frac{\partial w}{\partial z}, \tag{25}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y},\quad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z},\quad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}. \tag{26}$$

与二维问题相似, 推导出角铺设层合板三维混合状态微分方程为

$$\frac{\partial}{\partial z}\boldsymbol{F} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{R} \\ \boldsymbol{Q} & \boldsymbol{0} \end{bmatrix}\boldsymbol{F}, \tag{27}$$

这里

$$\boldsymbol{F} = (u \quad v \quad \sigma_z \quad \tau_{zx} \quad \tau_{zy} \quad w)^{\mathrm{T}}, \tag{28}$$

$$\boldsymbol{R} = \begin{bmatrix} a_{10} & -a_{11} & -\dfrac{\partial}{\partial x} \\ -a_{11} & a_{12} & -\dfrac{\partial}{\partial y} \\ -\dfrac{\partial}{\partial x} & -\dfrac{\partial}{\partial y} & -\sigma_x^0\dfrac{\partial^2}{\partial x^2} - \sigma_y^0\dfrac{\partial^2}{\partial y^2} - 2\tau_{xy}^0\dfrac{\partial^2}{\partial x\partial y} + \rho\dfrac{\partial^2}{\partial t^2} \end{bmatrix}, \tag{29}$$

$$\boldsymbol{Q} = \begin{bmatrix} -a_2\dfrac{\partial^2}{\partial x^2} - a_1\dfrac{\partial^2}{\partial y^2} - 2a_{14}\dfrac{\partial^2}{\partial x\partial y} + \rho\dfrac{\partial^2}{\partial t^2} & -a_{14}\dfrac{\partial^2}{\partial x^2} - a_{15}\dfrac{\partial^2}{\partial y^2} - (a_1 + a)\dfrac{\partial^2}{\partial x\partial y} & a_1\dfrac{\partial}{\partial x} + a_{16}\dfrac{\partial}{\partial y} \\ -a_{14}\dfrac{\partial^2}{\partial x^2} - a_{15}\dfrac{\partial^2}{\partial y^2} - (a_1 + a)\dfrac{\partial^2}{\partial x\partial y} & -a_1\dfrac{\partial^2}{\partial x^2} - a_4\dfrac{\partial^2}{\partial y^2} - 2a_{15}\dfrac{\partial^2}{\partial x\partial y} + \rho\dfrac{\partial^2}{\partial t^2} & a_5\dfrac{\partial}{\partial y} + a_{16}\dfrac{\partial}{\partial x} \\ a_1\dfrac{\partial}{\partial x} + a_{16}\dfrac{\partial}{\partial y} & a_5\dfrac{\partial}{\partial y} + a_{16}\dfrac{\partial}{\partial x} & a_7 \end{bmatrix}, \tag{ 0}$$

$$\begin{cases} a_1 = -C_1/C,\quad a_2 = C_{11} - C_1^2/C,\quad a = C_{12} - C_1C_2/C, \\ a_4 = C_{22} - C_2^2/C,\quad a_5 = -C_2/C,\quad a_6 = C_{66},\quad a_7 = 1/C, \\ a_9 = 1/C_{44},\quad a_{10} = C_{44}/(C_{44}C_{55} - C_{45}^2),\quad a_{11} = C_{45}/(C_{44}C_{55} - C_{45}^2), \\ a_{12} = C_{55}/(C_{44}C_{55} - C_{45}^2),\quad a_1 = C_{66} - C_6^2/C, \\ a_{14} = C_{16} - C_1C_6/C,\quad a_{15} = C_{26} - C_2C_6/C,\quad a_{16} = -C_6/C. \end{cases} \tag{ 1}$$

方程(27) 是计算三维各向异性结构在初应力 σ_x^0, σ_y^0, τ_{xy}^0 下固有频率的控制方程． 该方程可用于计算对称层合板和非对称层合板(必须考虑耦合效应) .

2 方程的应用

对于四边简支板,这里 σ_x^0 是常量,其它初应力假定为 0. 令

$$u=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}u_z(z)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{i\omega t},\quad v=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}v_z(z)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{i\omega t},$$

$$w=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}w_z(z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{i\omega t},\quad \tau_{zx}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\tau_1(z)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{i\omega t},$$

$$\tau_{zy}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\tau_2(z)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{i\omega t},\quad \sigma_z=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sigma(z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{i\omega t},$$

其中 $i=\sqrt{-1}$, ω 是自振频率. 那么方程(27) 简化为

$$\frac{\partial}{\partial z}\boldsymbol{F}=\boldsymbol{H}\cdot\boldsymbol{F}, \tag{2}$$

其中

$$\boldsymbol{F}=(u_z\quad v_z\quad w_z\quad \sigma\quad \tau_1\quad \tau_2)^{\mathrm{T}}, \qquad (\quad)$$

$$\boldsymbol{H}=\begin{bmatrix} 0 & 0 & -p & 0 & a_{10} & -a_{11}\\ 0 & 0 & -q & 0 & -a_{11} & a_{12}\\ -a_1p+a_{16}q & -a_5q+a_{16}p & 0 & a_7 & 0 & 0\\ 0 & 0 & p^2\sigma_x^0-\rho\omega^2 & 0 & p & q\\ a_2p^2+a_1q^2+2a_{14}pq-\rho\omega^2 & a_{14}p^2+a_{15}q^2+(a_1+a\ )pq & 0 & a_1p+a_{16}q & 0 & 0\\ a_{14}p^2+a_{15}q^2+(a_1+a\ )pq & a_1p^2+a_4q^2+2a_{15}pq-\rho\omega^2 & 0 & a_5q+a_{16}p & 0 & 0 \end{bmatrix}, \tag{4}$$

这里 $p=m\pi/a$, $q=n\pi/b$.

应用精细积分法[1], 得到方程(2) 的解为

$$\boldsymbol{F}|_{z=h}=\boldsymbol{T}\cdot\boldsymbol{F}|_{z=0}. \tag{5}$$

令 $\boldsymbol{F}=(\boldsymbol{T},\ \sigma)^{\mathrm{T}}$, $\boldsymbol{T}=\begin{bmatrix}\boldsymbol{T}_{11} & \boldsymbol{T}_{12}\\ \boldsymbol{T}_{21} & \boldsymbol{T}_{22}\end{bmatrix}$, 方程(5) 表示为

$$\boldsymbol{u}_h=\boldsymbol{T}_{11}\boldsymbol{u}_0+\boldsymbol{T}_{12}\sigma_0,\quad \sigma_h=\boldsymbol{T}_{21}\boldsymbol{u}_0+\boldsymbol{T}_{22}\sigma_0. \tag{6}$$

将边界条件 $\sigma_h=\sigma_0=0$ 代入方程(6) 得

$$\boldsymbol{T}_{21}\boldsymbol{u}_0=0. \tag{7}$$

对于非零解 $\boldsymbol{u}_0\neq\boldsymbol{0}$, 系数矩阵 $\boldsymbol{T}_{21}$ 的行列式应该为 0.

结果与讨论

这里研究了弹性混合板在初应力下的固有频率和屈曲应力. 弹性混合板由 层组成(见

图 2) . 顶层和底层是铝. 中间层是 N 层叠加的玻璃纤维增强复合材料. 表 1 列出了组成弹性混合板各成分的材料特性.

表 1 弹性混合板各成分的材料特性

材料常数	混合板各成分		
	铝(Al)	玻璃纤维增强复合材料(GFRP1)	玻璃纤维增强复合材料(GFRP2)
E_x/GPa	72.0	55.897 9	5.205 9
E_y/GPa	72.0	1 .729	28.7 5
E_z/GPa	72.0	1 .729	17.946 2
ν_{yx}	0.	0.277	0.177
ν_{zx}	0.	0.068	0.157
ν_{zy}	0.	0.400	0. 71
G_{xy}/GPa	28.0	5.589 8	7.45 1
G_{xz}/GPa	28.0	5.589 8	6.4724
G_{yz}/GPa	28.0	4.90	6.1782
ρ/(kg/m)	2.7 E+	2.55 E+	2.55 E+

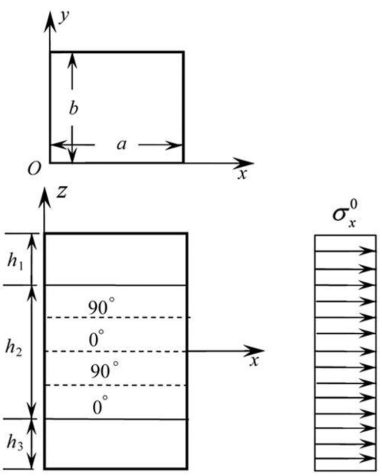


图 2 Al/GFRP/Al 矩形板和应力场

表 2 给出了几何参数 λ 和 $mh/a(nh/b)$ 对弹性混合板的固有频率的影响. 作为层厚比 $\lambda = h_2/h_1 = h_2/h$, 层厚比的增大意味着核心层厚度的增大. 对于 $\lambda = 0$ 的情况, 混合板成为纯铝板. 当 λ 趋于无穷大, 混合板成为玻璃纤维增强复合材料层合板. 可以看出固有频率随层厚比和厚度模式 $mh/a(nh/b)$ 的增大而减小. 在低层厚比时层合玻璃纤维增强复合材料的层数 N 对振动频率没有明显影响. 当层厚比增大的时候层数的影响是显著的. 混合板的低振动频率发生在大层厚比和小层数的时候. 表 说明本文结果与经典板理论[ ]结果相似, 但是与 Chen 的结果[0]存在很大差别. 在 Chen 的结果[0]中可能存在书写错误.

表 2 Al/(0°/90°)$_N$/Al 弹性混合方板固有频率的比较

($\lambda = h_2/h_1 = h_2/h$, $\omega = \omega h \sqrt{\rho_{Al}/E_{Al}}$, $a/b = 1$))

N	mh/a (nh/b)	λ						
		0	1	5	10	15	20	∞
	0.1	0.058 69	0.055 24	0.048 29	0.04 42	0.040 54	0.0 8 65	0.029 40
	0.2	0.215 6	0.182 61	0.157 96	0.145 76	0.1 8 28	0.1 16	0.105 48
2	0.	0.4 4	0. 7 7	0.287 29	0.270 17	0.259 72	0.252	0.207 64
	0.4	0.684 4	0.504 45	0.421 16	0.400 45	0. 88 5	0. 79 97	0. 22 2
	0.5	0.952 01	0.680 99	0.556 17	0.5 1 57	0.518 89	0.509 84	0.442 77

续表 2

N	mh/a (nh/b)	λ						
		0	1	5	10	15	20	∞
4	0.1	0.058 69	0.055 25	0.048 58	0.044 11	0.041 54	0.0 9 87	0.0 2 06
	0.2	0.215 6	0.182 62	0.158 51	0.147 1	0.140 66	0.1 6 19	0.11 12
	0.	0.4 4	0. 7 6	0.287 7	0.271 80	0.262 48	0.256 0	0.218 84
	0.4	0.684 4	0.504 4	0.421	0.401 62	0. 90 80	0. 8 22	0. 4 67
	0.5	0.952 01	0.681 00	0.556 09	0.5 2 1	0.520 2	0.512 10	0.454 1
6	0.1	0.058 69	0.055 25	0.048 6	0.044 25	0.041 7	0.040 11	0.0 2 57
	0.2	0.215 6	0.182 62	0.158 6	0.147 65	0.141 20	0.1 6 88	0.114 90
	0.	0.4 4	0. 7 6	0.287 84	0.272 2	0.26 24	0.257 06	0.222 18
	0.4	0.684 4	0.504 4	0.421 8	0.402 02	0. 91 61	0. 84 9	0. 9 48
	0.5	0.952 01	0.681 00	0.556 08	0.5 2 42	0.521 02	0.51 22	0.460 26
8	0.1	0.058 69	0.055 25	0.048 65	0.044 0	0.041 80	0.040 19	0.0 2 75
	0.2	0.215 6	0.182 62	0.158 67	0.147 77	0.141 9	0.1 7 14	0.115 55
	0.	0.4 4	0. 7 6	0.287 88	0.272 40	0.26 5	0.257 47	0.22 50
	0.4	0.684 4	0.504 4	0.421 40	0.402 19	0. 91 94	0. 84 88	0. 41 55
	0.5	0.952 01	0.681 00	0.556 08	0.5 2 55	0.521 4	0.51 7	0.46 07

表 3 Al/(0°/90°)$_N$/Al 弹性混合方板固有频率的比较

($\lambda = 0, \omega = \omega h \sqrt{\rho_{Al}/E_{Al}}, a/b = 1, mh/a = nh/b = 0.1, N = 2$)

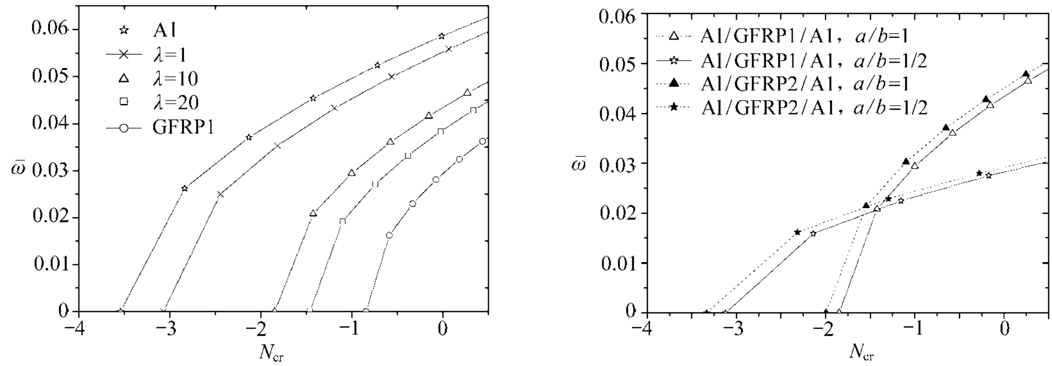
本文结果	经典板理论[ ]	Chen 的结果[0]
0.058 69	0.060 6	0.164 6

表 4 给出了屈曲行为的结果. 无量纲屈曲系数随板厚比 a/h 的增大而增大, 但是随层厚比 λ 的增大而减小. 在小层厚比的时候, 板的振动频率和屈曲系数随层数 N 的变化很小.

表 4 Al/(0°/90°)$_N$/Al 弹性混合方板屈曲系数的比较

($\lambda = h_2/h_1 = h_2/h, a/b = 1, mh/a = nh/b = 0.1, N_{cr} = \sigma_x^0 b^2/(E_2 h^2)$)

N	a/h	λ						
		0	1	5	10	15	20	∞
2	10	.541 85	.070 76	2.297 02	1.848 2	1.608 61	1.460 71	0.844 20
	20	14.167 42	12.28 04	9.188 08	7. 92 94	6.4 4 4	5.842 85	. 76 79
4	10	.541 85	.071 67	2. 2 06	1.904 87	1.685 2	1.550 97	0.999 55
	20	14.167 42	12.286 69	9.292 24	7.619 50	6.741 29	6.20 87	.998 19
6	10	.541 85	.071 85	2. 28 05	1.915 94	1.700 40	1.568 79	1.0 0 56
	20	14.167 42	12.287 8	9. 12 21	7.66 77	6.801 62	6.275 14	4.122 2
8	10	.541 85	.071 89	2. 29 8	1.919 87	1.705 76	1.575 10	1.041 58
	20	14.167 42	12.287 56	9. 19	7.679 48	6.82 06	6. 00 40	4.166



(a) Al/ GFRP1/ Al, $a/b = 1$, $mh/a = nh/b = 0.1$ (b) $a/h = 10$, $a/b = 1$, $m = 1$, $n = 1$, $\lambda = 10$

图 3 初应力对 Al/ (0°/90°) / Al 弹性混合板振动频率的影响

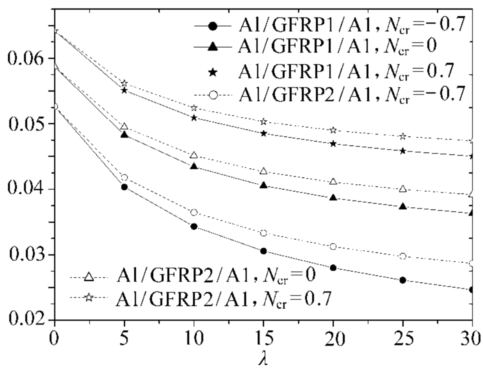


图 4 层厚比和初应力对 Al/ (0°/90°) / Al 弹性混合板振动频率的影响 ($a/b = 1$, $mh/a = nh/b = 0.1$)

图 (a) 显示了初应力对振动频率的影响. 可以看出压载荷对振动行为产生软化影响, 而拉载荷产生相反的影响. 当频率到达 0 时得到屈曲荷载. 小层厚比混合板得到大的屈曲系数, 在纯铝板时屈曲系数达到最大. 图 (b) 显示了不同玻璃纤维增强复合材料初应力对固有频率的影响. Al/ GFRP1/ Al 板的振动频率小于 Al/ GFRP2/ Al 板的振动频率. 图 4 显示了层厚比 λ 和初应力对振动频率的影响. 可以看出混合板的频率随层厚比的增大而减小, 在小层厚比时这种减小是显著的. 初应力的影响说明拉载荷增大固有频率, 而压载荷减小固有频率. 当 $\lambda = 0$ 时, 混合板成为纯铝板, Al/ GFRP1/ Al 板的振动频率等于 Al/ GFRP2/ Al 板的振动频率. 在 Chen 的文献[0] 中的图 4 可能是不正确的.

4 结 论

本文用精细积分法给出了混合板在初应力下的固有振动和屈曲的近似解析解, 在此之前还没有报导过. 精细积分法与传统的有限差分法相比可以给出计算机精度允许的很精确的数值结果. 这些结果可以作为衡量各种板理论准确性的一个标准. 而且, 非对称层合板中出现的各种耦合影响, 比如弯扭耦合, 拉弯耦合等在同一组控制方程里被考虑了.

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Vibration and Stability of Hybrid Plate Based on Elasticity Theory

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Abstract: The governing equations of elasticity theory for natural vibration and buckling of anisotropic plate were derived from Hellinger-Reissner's variational principle with nonlinear strain-displacement relations. Simply supported rectangular hybrid plates were studied by precise integration method. This method, in contrast to the traditional finite difference approximation, gives highly precise numerical results which approach the full computer precision. So the results for natural vibration and stability of hybrid plates presented can be regarded as approximate analytical solutions. Furthermore, the several types of coupling effects, such as coupling between bending and twisting, coupling between extension and bending, etc. when the layer stacking sequence is not symmetric, are considered by only one set of governing equations.

Key words: vibration; buckling; precise integration method; hybrid plate; state space