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自然对流换热问题基于混合元法的差分格式及其数值模拟*

罗振东[1,2], 朱 江[2], 谢正辉[2], 张桂芳[1]

(1. 首都师范大学 数学系, 北京 100037;
2. 中国科学院 大气物理研究所, 北京 100029)

(许政范推荐)

摘要: 研究自然对流换热问题, 通过对于空间变量采用有限元离散而对于时间变量用差分离散, 导出一种基于混合有限元法的最低阶的差分格式, 这种格式可以同时求出流体的速度、温度和压力的数值解, 并给出了模拟方腔流的自然换热的数值例子•

关 键 词: 自然对流换热问题; 混合元法; 差分格式
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引 言

设 $\Omega\subset R^2$ 是有界区域, 考虑二维非定常的自然对流换热问题:

问题(Ⅰ) 求 $\boldsymbol{u}=(u_1,u_2),p,T$ 使得对于任意 $t_1>0$ 满足

$$\begin{cases}\boldsymbol{u}_t-\mu\Delta\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\nabla p=\lambda\boldsymbol{j}T & ((x,y,t)\in\Omega\times(0,t_1)),\\ \mathrm{div}\,\boldsymbol{u}=0 & ((x,y,t)\in\Omega\times(0,t_1)),\\ T_t-\Delta T+\lambda\boldsymbol{u}\cdot\nabla T=0 & ((x,y,t)\in\Omega\times(0,t_1)),\\ \boldsymbol{u}=0,\ T=0 & ((x,y,t)\in\partial\Omega\times(0,t_1)),\\ \boldsymbol{u}(x,y,0)=0,\ T(x,y,0)=f(x,y) & ((x,y)\in\Omega),\end{cases}$$

其中 $\boldsymbol{u}$ 是速度向量, p 为压力, T 为温度, $\mu=1/Ra>0$ 为粘性系数, Ra 是 Rayleigh 数, $\lambda>0$ 为 Groshoff 数, $\boldsymbol{j}=(0,1)$ 为向量, $f(x,y)$ 为初值函数•

非定常的自然对流换热问题(Ⅰ)是大气动力学中的一个重要方程组, 是强迫耗散的非线性系统方程• 由于该方程组除了含速度和压力外, 还含有温度场, 所以数值方法不容易• 过去虽然提出了一些有限差分方法, 都是将温度场视为常量, 即简单的 N_S 方程求解, 而且往往用涡函数法去求解, 这样对压力须要引入外加的边界条件, 但效果也欠佳, 特别是对压力的计算往往是很困难的(参见[1~4])• 本文利用混合元方法, 导出的一种最低阶的稳定和收敛的差

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作者简介: 罗振东(1958—), 广西桂平人, 男, 教授, 博士, 博士生导师, 研究方向: 有限元法及其应用(E_mail: luozhd@mail.cnu.edu.cn)•

分格式，并给出了模拟方腔流的自然换热的数值例子，用以说明该格式的有效性• 与以往的处理N_S 问题的数值方法(如有限差分法，涡函数法等)不同之处是该方法能同时求出速度、压力和温度的近似解，并能计算粘性很小(即 Rayleigh 数很大)的情形•

1 广义解的存在性和混合有限元解的存在性、收敛性回顾

本文使用的 Sobolev 空间是熟知的(可参考[5])•

自然对流换热问题的混合广义解为

问题(Ⅰ)* 求 $(\boldsymbol{u}, p, T) \in [L^2(0, t_1; X) \cap H^1(0, t_1; V)] \times L^2(0, t_1; M) \times H^1(0, t_1; W)$ 满足

$$\begin{cases} (\boldsymbol{u}_t, \boldsymbol{v}) + \mu a(\boldsymbol{u}, \boldsymbol{v}) + a_1(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - b(p, \boldsymbol{v}) = \lambda(\boldsymbol{j} T, \boldsymbol{v}) & (\forall \boldsymbol{v} \in X), \\ b(\varphi, \boldsymbol{u}) = 0 & (\forall \varphi \in W), \\ (T_t, \psi) + d(T, \psi) + \lambda a_1(\boldsymbol{u}, T, \psi) = 0 & (\forall \psi \in W), \\ \boldsymbol{u}(x, y, 0) = 0,\ T(x, y, 0) = f(x, y) & ((x, y) \in \Omega), \end{cases} \tag{1}$$

其中

$$X = H_0^1(\Omega)^2,$$
$$W = H_0^1(\Omega),$$
$$V = \left\{ \boldsymbol{v} \in X;\ \operatorname{div} \boldsymbol{v} = 0 \right\},$$
$$M = L_0^2(\Omega) = \left\{ \varphi \in L^2(\Omega);\ \int_\Omega \varphi \,\mathrm{d}x\,\mathrm{d}y = 0 \right\},$$
$$a(\boldsymbol{u}, \boldsymbol{v}) = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) = \int_\Omega \left(\frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{\partial y}\frac{\partial v_1}{\partial y} + \frac{\partial u_2}{\partial x}\frac{\partial v_2}{\partial x} + \frac{\partial u_2}{\partial y}\frac{\partial v_2}{\partial y} \right) \mathrm{d}x\,\mathrm{d}y,$$
$$a_1(\boldsymbol{u}, \boldsymbol{v}, w) = \int_\Omega \left(u_1 \frac{\partial v_1}{\partial x} w_1 + u_1 \frac{\partial v_2}{\partial x} w_2 + u_2 \frac{\partial v_1}{\partial y} w_1 + u_2 \frac{\partial v_2}{\partial y} w_2 \right) \mathrm{d}x\,\mathrm{d}y,$$
$$a_1(\boldsymbol{u}, T, \psi) = \int_\Omega \left(u_1 \frac{\partial T}{\partial x} \psi + u_2 \frac{\partial T}{\partial y} \psi \right) \mathrm{d}x\,\mathrm{d}y,$$
$$b(\varphi, \boldsymbol{v}) = \int_\Omega \varphi \operatorname{div} \boldsymbol{v} \,\mathrm{d}x\,\mathrm{d}y,$$
$$d(T, \psi) = (\nabla T, \nabla \psi) = \int_\Omega \left(\frac{\partial T}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial T}{\partial y}\frac{\partial \psi}{\partial y} \right) \mathrm{d}x\,\mathrm{d}y\ •$$

我们在[6]中已经证明了问题(Ⅰ)* 存在唯一的解•

为了导出问题(Ⅰ)* 基于混合有限元法的差分格式，对于空间变量我们采用有限元离散而对于时间变量差分离散• 设 I_h 是 Ω 的拟一致的直角三角形剖分，混合有限元空间取为：

$$X_h = \left\{ (v_{1h}, v_{2h}) \in L^2(\Omega)^2;\ v_{jh}|_K \in P_1(K) \cap H^1(K)\ \forall K \in \mathrm{I}_h,\ v_{jh} n_i^1 |_e = v_{jh} n_i^2 |_e,\ e = \partial K_1 \cap \partial K_2 \subset \Omega,\ v_{jh} n_j |_e = 0,\ e \subset \partial K \cap \partial \Omega,\ i, j = 1, 2 \right\}, \tag{2}$$
$$M_h = \left\{ \varphi \in M;\ \varphi|_K \in P_0(K),\ \forall K \in \mathrm{I}_h \right\}, \tag{3}$$
$$W_h = \left\{ \phi \in W;\ \phi|_K \in P_1(K),\ \forall K \in \mathrm{I}_h \right\}, \tag{4}$$

其中 $P_m(K)$ 表示次数 $\leqslant m$ 的多项式空间，$\boldsymbol{n}_l = (n_l^1, n_l^2)$ 表示 $K_l \in \mathrm{I}_h$ 上的单位外法向量• 容易证明 $X_h \subset X$(参见[6])• 利用[7]中的方法，可以证明 X_h 和 M_h 满足 inf_sup 条件：

$$\sup_{\boldsymbol{v}_h \in X_h} \frac{b(\varphi_h, \boldsymbol{v}_h)}{\| \nabla \boldsymbol{v}_h \|_0} \geqslant \beta \| \varphi_h \|_0 \qquad (\forall \varphi_h \in M_h), \tag{5}$$

其中 $\beta > 0$ 是与 h 和 k 都无关的正常数• 设 L 为正整数，$k = t_1/L$ 是时间步长，$t^{(n)} = nk$, 0

$\leqslant n \leqslant L$; $(\boldsymbol{u}_h^n, p_h^n, T_h^n) \in X_h \times M_h \times W_h$ 对应于 $(\boldsymbol{u}(t^{(n)}), p(t^{(n)}), T(t^{(n)})) \equiv (\boldsymbol{u}^n, p^n, T^n)$ 的混合有限元解. 则问题(I)* 的混合有限元解为:

问题 (I_h^n) 求 $(\boldsymbol{u}_h^n, p_h^n, T_h^n) \in X_h \times M_h \times W_h$, $1 \leqslant n \leqslant L$, 满足

$$\begin{cases} (\boldsymbol{u}_h^n, \boldsymbol{v}_h) + k\mu a(p_h^n, \boldsymbol{v}_h) - kb(p_h^n, \boldsymbol{v}_h) = k\lambda(\boldsymbol{j}T_h^n, \boldsymbol{v}_h) + (\boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h) - \\ \qquad ka_1(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h) \qquad (\forall \boldsymbol{v}_h \in X_h), \\ b(\varphi_h, \boldsymbol{u}_h^n) = 0 \qquad (\forall \varphi_h \in M_h), \\ (T_h^n, \psi_h) + kd(T_h^n, \psi_h) = (T_h^{n-1}, \psi_h) - \lambda ka_1(\boldsymbol{u}_h^{n-1}, T_h^{n-1}, \psi_h) \qquad (\forall \psi_h \in W_h), \\ T_h^0 = P_h f, \quad \boldsymbol{u}_h^0 = 0, \end{cases} \tag{6}$$

其中 $P_h f$ 是 f 在 W_h 中 L^2_投影.

在[4]中已经证明了问题 (I_h^n) 存在唯一的稳定解满足

$$k^{1/2}\left(\sum_{i=1}^{n}\| \nabla(T^i - T_h^i)\|_0 + \sum_{i=1}^{n}\|\nabla(\boldsymbol{u}^i - \boldsymbol{u}_h^i)\|_0 + \|p^n - p_h^n\|_0\right) + \|\boldsymbol{u}^n - \boldsymbol{u}_h^n\|_0 + \|T^n - T_h^n\|_0 \leqslant C(h + k), \tag{7}$$

其中 C 是与 h 和 k 都无关的常数.

2 一种基于混合有限元法的最低阶差分格式

为了导出问题 (I_h^n) 的基于混合有限元法的最低阶差分格式, 我们将 Ω 剖分为拟一致的直角三角形单元 I_h, 直角三角形 K 的顶角点记为 1、2、3 并按逆时针排列, 直角点记为 2. 如图 1 所示.

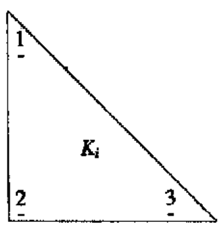


图 1 单元示意图

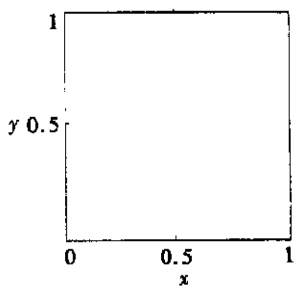


图 2 物理模型示意图

于是, 问题 (I_h^n) 在 K_i 上可以写成

$$\begin{aligned} &\int_{K_i} u_{1h}^n v_h \,\mathrm{d}x\,\mathrm{d}y + k\mu\int_{K_i}\nabla u_{1h}^n \cdot \nabla v_h \,\mathrm{d}x\,\mathrm{d}y - k\int_{K_i} p_h^n \frac{\partial v_h}{\partial x}\mathrm{d}x\,\mathrm{d}y = \\ &\quad \int_{K_i} u_{1h}^{n-1} v_h \,\mathrm{d}x\,\mathrm{d}y - k\int_{K_i} u_{1h}^{n-1}\frac{\partial u_{1h}^{n-1}}{\partial x} v_h \,\mathrm{d}x\,\mathrm{d}y - \\ &\quad k\int_{K_i} u_{2h}^{n-1}\frac{\partial u_{1h}^{n-1}}{\partial y} v_h \,\mathrm{d}x\,\mathrm{d}y \qquad (\forall v_h \in P_1(K_i)), \end{aligned} \tag{8}$$

$$\int_{K_i} u_{2h}^n v_h \,\mathrm{d}x\,\mathrm{d}y + \mu k\int_{K_i}\nabla u_{2h}^n \cdot \nabla v_h \,\mathrm{d}x\,\mathrm{d}y - k\int_{K_i} p_h^n \frac{\partial v_h}{\partial y}\mathrm{d}x\,\mathrm{d}y =$$

$$\int_{K_i} u_{2h}^{n-1} v_h \mathrm{d}x\mathrm{d}y + k\lambda\int_{K_i} T_h^n v_h \mathrm{d}x\mathrm{d}y - k\int_{K_i} u_{1h}^{n-1}\frac{\partial u_{2h}^{n-1}}{\partial x} v_h \mathrm{d}x\mathrm{d}y -$$

$$k\int_{K_i} u_{2h}^{n-1}\frac{\partial u_{2h}^{n-1}}{\partial y} v_h \mathrm{d}x\mathrm{d}y \qquad (\forall v_h \in P_1(K_i)), \tag{9}$$

$$\int_{K_i} \varphi_h \frac{\partial u_{1h}}{\partial x}\mathrm{d}x\mathrm{d}y + \int_{K_i} \varphi_h \frac{\partial u_{2h}}{\partial y}\mathrm{d}x\mathrm{d}y = 0 \qquad (\forall \varphi_h \in P_0(K_i)), \tag{10}$$

$$\int_{K_i} T_h^n \psi_h \mathrm{d}x\mathrm{d}y + k\int_{K_i} \nabla T_h^n \cdot \nabla \psi_h \mathrm{d}x\mathrm{d}y =$$

$$\int_{K_i} T_h^{n-1} \psi_h \mathrm{d}x\mathrm{d}y - k\lambda\int_{K_i} u_{1h}^{n-1}\frac{\partial T_h^{n-1}}{\partial x}\psi_h \mathrm{d}x\mathrm{d}y -$$

$$k\lambda\int_{K_i} u_{2h}^{n-1}\frac{\partial T_h^{n-1}}{\partial y}\psi_h \mathrm{d}x\mathrm{d}y \qquad (\forall \psi \in P_1(K_i)). \tag{11}$$

注意 (8)~(11)是问题(I_h^n)成立的充分条件. 在K_i上, u_{jh}^n, p_h^n和T_h^n可以表示为

$$u_{jh}^n = u_{ij1}^n\lambda_1 + u_{ij2}^n\lambda_2 + u_{ij3}^n\lambda_3 \qquad (j = 1, 2), \tag{12}$$

$$T_h^n = T_{i1}^n\lambda_1 + T_{i2}^n\lambda_2 + T_{i3}^n\lambda_3, \tag{13}$$

$$p_h^n = \frac{1}{\Delta_{K_i}}\int_{K_i} p_i^n \mathrm{d}x\mathrm{d}y, \tag{14}$$

其中 λ_1, λ_2和λ_3表示面积坐标(参见[8]或[9]), $\Delta_{K_i} = h^2/2$是三角形K_i的面积. 则利用面积坐标的性质(参见[8]或[9])可得

$$\frac{\partial}{\partial x} = \frac{-2}{h}\frac{\partial}{\partial\lambda_2}, \quad \frac{\partial}{\partial y} = \frac{2}{h}\left(\frac{\partial}{\partial\lambda_2} - \frac{\partial}{\partial\lambda_1}\right) \tag{15}$$

注意到 $\lambda_1 + \lambda_2 + \lambda_3 = 1$可有

$$\begin{cases} \dfrac{\partial\lambda_1}{\partial x} = 0, & \dfrac{\partial\lambda_2}{\partial x} = -\dfrac{2}{h}, & \dfrac{\partial\lambda_3}{\partial x} = \dfrac{2}{h}, \\ \dfrac{\partial\lambda_1}{\partial y} = -\dfrac{2}{h}, & \dfrac{\partial\lambda_2}{\partial y} = \dfrac{2}{h}, & \dfrac{\partial\lambda_3}{\partial x} = 0, \end{cases} \tag{16}$$

$$\frac{\partial T_h^n}{\partial x} = \frac{\partial T_h^n}{\partial\lambda_1}\frac{\partial\lambda_1}{\partial x} + \frac{\partial T_h^n}{\partial\lambda_2}\frac{\partial\lambda_2}{\partial x} = \frac{2}{h}(T_{i3}^n - T_{i2}^n), \tag{17}$$

$$\frac{\partial T_h^n}{\partial y} = \frac{\partial T_h^n}{\partial\lambda_1}\frac{\partial\lambda_1}{\partial y} + \frac{\partial T_h^n}{\partial\lambda_2}\frac{\partial\lambda_2}{\partial y} = \frac{2}{h}(T_{i2}^n - T_{i1}^n), \tag{18}$$

$$\nabla T_h^n = \frac{2}{h}(T_{i3}^n - T_{i2}^n, T_{i2}^n - T_{i1}^n). \tag{19}$$

这样(11)可写成

$$\int_{K_i}(T_{i1}^n\lambda_1 + T_{i2}^n\lambda_2 + T_{i3}^n\lambda_3)\lambda_m \mathrm{d}x\mathrm{d}y +$$

$$\frac{2k}{h}\int_{K_i}\left[(T_{i3}^n - T_{i2}^n)\frac{\partial\lambda_m}{\partial x} + (T_{i2}^n - T_{i1}^n)\frac{\partial\lambda_m}{\partial y}\right]\mathrm{d}x\mathrm{d}y =$$

$$\int_{K_i}(T_{i1}^{n-1}\lambda_1 + T_{i2}^{n-1}\lambda_2 + T_{i3}^{n-1}\lambda_3)\lambda_m \mathrm{d}x\mathrm{d}y -$$

$$k\lambda\int_{K_i}(u_{i11}^{n-1}\lambda_1 + u_{i12}^{n-1}\lambda_2 + u_{i13}^{n-1}\lambda_3)\frac{2}{h}(T_{i3}^{n-1} - T_{i2}^{n-1})\lambda_m \mathrm{d}x\mathrm{d}y -$$

$$k\lambda\int_{K_i}\left(u_{i21}^{n-1}\lambda_1+u_{i22}^{n-1}\lambda_2+u_{i23}^{n-1}\lambda_3\right)\frac{2}{h}\left(T_{i2}^{n-1}-T_{i1}^{n-1}\right)\lambda_m\,\mathrm{d}x\,\mathrm{d}y$$

$$(m=1,2,3)\cdot \tag{20}$$

再利用面积坐标的性质可得

$$\begin{pmatrix} 26 & -23 & 1 \\ -23 & 50 & -23 \\ 1 & -23 & 26 \end{pmatrix}\begin{pmatrix} T_{i1}^{n} \\ T_{i2}^{n} \\ T_{i3}^{n} \end{pmatrix}=\begin{pmatrix} F_{i1} \\ F_{i2} \\ F_{i3} \end{pmatrix}\cdot \tag{21}$$

其中

$$F_{i1}=2T_{i1}^{n-1}+T_{i2}^{n-1}+T_{i3}^{n-1}+2\lambda h\left(2u_{i21}^{n-1}+u_{i22}^{n-1}+u_{i23}^{n-1}\right)\left(T_{i1}^{n-1}-T_{i2}^{n-1}\right)+$$
$$2\lambda h\left(2u_{i11}^{n-1}+u_{i12}^{n-1}+u_{i13}^{n-1}\right)\left(T_{i2}^{n-1}-T_{i3}^{n-1}\right), \tag{22}$$

$$F_{i2}=T_{i1}^{n-1}+2T_{i2}^{n-1}+T_{i3}^{n-1}+2\lambda h\left(u_{i21}^{n-1}+2u_{i22}^{n-1}+u_{i23}^{n-1}\right)\left(T_{i1}^{n-1}-T_{i2}^{n-1}\right)+$$
$$2\lambda h\left(u_{i11}^{n-1}+2u_{i12}^{n-1}+u_{i13}^{n-1}\right)\left(T_{i2}^{n-1}-T_{i3}^{n-1}\right), \tag{23}$$

$$F_{i3}=T_{i1}^{n-1}+T_{i2}^{n-1}+2T_{i3}^{n-1}+2\lambda h\left(u_{i21}^{n-1}+u_{i22}^{n-1}+2u_{i23}^{n-1}\right)\left(T_{i1}^{n-1}-T_{i2}^{n-1}\right)+$$
$$2\lambda h\left(2u_{i11}^{n-1}+u_{i12}^{n-1}+2u_{i13}^{n-1}\right)\left(T_{i2}^{n-1}-T_{i3}^{n-1}\right)\cdot \tag{24}$$

解(21)可得

$$T_{i1}^{n}=\Big[T_{i1}^{n-1}+T_{i2}^{n-1}+T_{i3}^{n-1}+2\lambda h\left(u_{i21}^{n-1}+u_{i22}^{n-1}+u_{i23}^{n-1}\right)\left(T_{i1}^{n-1}-T_{i2}^{n-1}\right)+$$
$$2\lambda h\left(u_{i11}^{n-1}+u_{i12}^{n-1}+u_{i13}^{n-1}\right)\left(T_{i2}^{n-1}-T_{i3}^{n-1}\right)\Big]/4+$$
$$\Big[771T_{i1}^{n-1}+575T_{i2}^{n-1}+479T_{i3}^{n-1}+$$
$$2\lambda h\left(771u_{i21}^{n-1}+575u_{i22}^{n-1}+479u_{i23}^{n-1}\right)\left(T_{i1}^{n-1}-T_{i2}^{n-1}\right)+$$
$$2\lambda h\left(771u_{i11}^{n-1}+575u_{i12}^{n-1}+479u_{i13}^{n-1}\right)\left(T_{i2}^{n-1}-T_{i3}^{n-1}\right)\Big]/7300, \tag{25}$$

$$T_{i2}^{n}=\Big[T_{i1}^{n-1}+T_{i2}^{n-1}+T_{i3}^{n-1}+2\lambda h\left(u_{i21}^{n-1}+u_{i22}^{n-1}+u_{i23}^{n-1}\right)\left(T_{i1}^{n-1}-T_{i2}^{n-1}\right)+$$
$$2\lambda h\left(u_{i11}^{n-1}+u_{i12}^{n-1}+u_{i13}^{n-1}\right)\left(T_{i2}^{n-1}-T_{i3}^{n-1}\right)\Big]/4+$$
$$\Big[575T_{i1}^{n-1}+675T_{i2}^{n-1}+575T_{i3}^{n-1}+$$
$$2\lambda h\left(575u_{i21}^{n-1}+675u_{i22}^{n-1}+575u_{i23}^{n-1}\right)\left(T_{i1}^{n-1}-T_{i2}^{n-1}\right)+$$
$$2\lambda h\left(575u_{i11}^{n-1}+675u_{i12}^{n-1}+575u_{i13}^{n-1}\right)\left(T_{i2}^{n-1}-T_{i3}^{n-1}\right)\Big]/7300, \tag{26}$$

$$T_{i3}^{n}=\Big[T_{i1}^{n-1}+T_{i2}^{n-1}+T_{i3}^{n-1}+2\lambda h\left(u_{i21}^{n-1}+u_{i22}^{n-1}+u_{i23}^{n-1}\right)\left(T_{i1}^{n-1}-T_{i2}^{n-1}\right)+$$
$$2\lambda h\left(u_{i11}^{n-1}+u_{i12}^{n-1}+u_{i13}^{n-1}\right)\left(T_{i2}^{n-1}-T_{i3}^{n-1}\right)\Big]/4+$$
$$\Big[479T_{i1}^{n-1}+575T_{i2}^{n-1}+771T_{i3}^{n-1}+$$
$$2\lambda h\left(479u_{i11}^{n-1}+575u_{i12}^{n-1}+771u_{i13}^{n-1}\right)\left(T_{i1}^{n-1}-T_{i2}^{n-1}\right)+$$
$$2\lambda h\left(479u_{i21}^{n-1}+575u_{i22}^{n-1}+771u_{i23}^{n-1}\right)\left(T_{i2}^{n-1}-T_{i3}^{n-1}\right)\Big]/7300, \tag{27}$$

与(21)同理,从(8)～(10)可得

$$\begin{pmatrix} 2+24\mu & 1-24\mu & 1 & 0 & 0 & 0 & 0 \\ 1-24\mu & 2+48\mu & 1-24\mu & 0 & 0 & 0 & 12h \\ 1 & 1-24\mu & 2+24\mu & 0 & 0 & 0 & -12h \\ 0 & 0 & 0 & 2+24\mu & 1-24\mu & 1 & 12h \\ 0 & 0 & 0 & 1-24\mu & 2+48\mu & 1-24\mu & -12h \\ 0 & 0 & 0 & 1 & 1-24\mu & 2+24\mu & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 & 0 \end{pmatrix}\times$$

$$\begin{pmatrix} u_{i11}^n \\ u_{i12}^n \\ u_{i13}^n \\ u_{i21}^n \\ u_{i22}^n \\ u_{i23}^n \\ p_i^n \end{pmatrix} = \begin{pmatrix} f_{i11} \\ f_{i12} \\ f_{i13} \\ f_{i21} \\ f_{i22} \\ f_{i23} \\ 0 \end{pmatrix} . \tag{28}$$

其中

$$\begin{cases} f_{i11} = 2h(2u_{i21}^{n-1} + u_{i22}^{n-1} + u_{i23}^{n-1})(u_{i11}^{n-1} - u_{i12}^{n-1}) + 2h(2u_{i11}^{n-1} + u_{i12}^{n-1} + u_{i13}^{n-1}) \cdot \\ \qquad (u_{i12}^{n-1} - u_{i13}^{n-1}) + (2u_{i11}^{n-1} + u_{i12}^{n-1} + u_{i13}^{n-1}), \\ f_{i12} = 2h(u_{i21}^{n-1} + 2u_{i22}^{n-1} + u_{i23}^{n-1})(u_{i11}^{n-1} - u_{i12}^{n-1}) + 2h(u_{i11}^{n-1} + 2u_{i12}^{n-1} + u_{i13}^{n-1}) \cdot \\ \qquad (u_{i12}^{n-1} - u_{i13}^{n-1}) + (u_{i11}^{n-1} + 2u_{i12}^{n-1} + u_{i13}^{n-1}), \\ f_{i13} = 2h(u_{i21}^{n-1} + u_{i22}^{n-1} + 2u_{i23}^{n-1})(u_{i11}^{n-1} - u_{i12}^{n-1}) + 2h(u_{i11}^{n-1} + u_{i12}^{n-1} + 2u_{i13}^{n-1}) \cdot \\ \qquad (u_{i12}^{n-1} - u_{i13}^{n-1}) + (u_{i11}^{n-1} + u_{i12}^{n-1} + 2u_{i13}^{n-1}), \\ f_{i21} = \lambda(2T_{i1}^n + T_{i2}^n + T_{i3}^n) + 2h(2u_{i21}^{n-1} + u_{i22}^{n-1} + u_{i23}^{n-1})(u_{i21}^{n-1} - u_{i22}^{n-1}) + \\ \qquad 2h(2u_{i11}^{n-1} + u_{i12}^{n-1} + u_{i13}^{n-1})(u_{i22}^{n-1} - u_{i23}^{n-1}) + (2u_{i21}^{n-1} + u_{i22}^{n-1} + u_{i23}^{n-1}), \\ f_{i22} = \lambda(T_{i1}^n + 2T_{i2}^n + T_{i3}^n) + 2h(u_{i21}^{n-1} + 2u_{i22}^{n-1} + u_{i23}^{n-1})(u_{i21}^{n-1} - u_{i22}^{n-1}) + \\ \qquad 2h(u_{i11}^{n-1} + 2u_{i12}^{n-1} + u_{i13}^{n-1})(u_{i22}^{n-1} - u_{i23}^{n-1}) + (u_{i21}^{n-1} + 2u_{i22}^{n-1} + u_{i23}^{n-1}), \\ f_{i23} = \lambda(T_{i1}^n + T_{i2}^n + 2T_{i3}^n) + 2h(u_{i21}^{n-1} + u_{i22}^{n-1} + 2u_{i23}^{n-1})(u_{i21}^{n-1} - u_{i22}^{n-1}) + \\ \qquad 2h(u_{i11}^{n-1} + u_{i12}^{n-1} + 2u_{i13}^{n-1})(u_{i22}^{n-1} - u_{i23}^{n-1}) + (u_{i21}^{n-1} + u_{i22}^{n-1} + 2u_{i23}^{n-1}) . \end{cases}$$

解(28)可得

$$\begin{aligned} p_i^n = {} & \frac{1}{18(1+36\mu)}\Big[2(u_{i11}^{n-1} - u_{i12}^{n-1})(u_{i22}^{n-1} - u_{i23}^{n-1}) + (u_{i12}^{n-1} - u_{i13}^{n-1})^2 + \\ & (u_{i21}^{n-1} - u_{i22}^{n-1})^2 + \mu(u_{i21}^{n-1} - 24u_{i22}^{n-1} + 23u_{i23}^{n-1})(u_{i12}^{n-1} - u_{i11}^{n-1}) + \\ & \mu(u_{i11}^{n-1} - 24u_{i12}^{n-1} + 23u_{i13}^{n-1})(u_{i13}^{n-1} - u_{i12}^{n-1}) + \\ & \mu(73u_{i21}^{n-1} - 24u_{i22}^{n-1} - 49u_{i23}^{n-1})(u_{i21}^{n-1} - u_{i22}^{n-1}) + \\ & \mu(73u_{i11}^{n-1} - 24u_{i12}^{n-1} - 49u_{i13}^{n-1})(u_{i22}^{n-1} - u_{i23}^{n-1})\Big] + \\ & \frac{1}{36h(1+36\mu)}\Big[(u_{i12}^{n-1} - u_{i13}^{n-1}) + (u_{i21}^{n-1} - u_{i22}^{n-1}) + \lambda(T_{i1}^n - T_{i2}^n) - \\ & \mu(u_{i11}^{n-1} - 24u_{i12}^{n-1} + 23u_{i13}^{n-1}) + \mu\lambda(73T_{i1}^n - 24T_{i2}^n - 49T_{i3}^n) + \\ & \mu(73u_{i21}^{n-1} - 24u_{i22}^{n-1} - 49u_{i23}^{n-1})\Big], \end{aligned} \tag{29}$$

$$\begin{aligned} u_{i13}^n = {} & \frac{2h(1+48\mu)}{3(1+24\mu)(1+36\mu)(1+72\mu)}\Big[2(u_{i11}^{n-1} - u_{i12}^{n-1})(u_{i22}^{n-1} - u_{i23}^{n-1}) + \\ & (u_{i21}^{n-1} - u_{i22}^{n-1})^2 + \mu(u_{i21}^{n-1} - 24u_{i22}^{n-1} + 23u_{i23}^{n-1})(u_{i12}^{n-1} - u_{i11}^{n-1}) + \\ & (u_{i12}^{n-1} - u_{i13}^{n-1})^2 + \mu(u_{i11}^{n-1} - 24u_{i12}^{n-1} + 23u_{i13}^{n-1})(u_{i13}^{n-1} - u_{i12}^{n-1}) + \\ & \mu(73u_{i21}^{n-1} - 24u_{i22}^{n-1} - 49u_{i23}^{n-1})(u_{i21}^{n-1} - u_{i22}^{n-1}) + \\ & \mu(73u_{i11}^{n-1} - 24u_{i12}^{n-1} - 49u_{i13}^{n-1})(u_{i22}^{n-1} - u_{i23}^{n-1})\Big] + \end{aligned}$$

$$\frac{1+48\mu}{3(1+24\mu)(1+36\mu)(1+72\mu)}\Big[(u_{i12}^{n-1}-u_{i13}^{n-1})+(u_{i21}^{n-1}-u_{i22}^{n-1})+$$
$$\lambda(T_{i1}^{n}-T_{i2}^{n})-\mu(u_{i11}^{n-1}-24u_{i12}^{n-1}+23u_{i13}^{n-1})+$$
$$\mu\lambda(73T_{i1}^{n}-24T_{i2}^{n}-49T_{i3}^{n})+\mu(73u_{i21}^{n-1}-24u_{i22}^{n-1}-49u_{i23}^{n-1})\Big]+$$
$$\frac{1}{1+72\mu}\Big[2hu_{i23}^{n-1}(u_{i11}^{n-1}-u_{i12}^{n-1})+2hu_{i13}^{n-1}(u_{i12}^{n-1}-u_{i13}^{n-1})+u_{i13}^{n-1}\Big]+$$
$$\frac{24\mu}{1+72\mu}\Big[2h(u_{i21}^{n-1}+u_{i22}^{n-1}+u_{i23}^{n-1})+(u_{i11}^{n-1}-u_{i12}^{n-1})+$$
$$2h(u_{i11}^{n-1}+u_{i12}^{n-1}+u_{i13}^{n-1})(u_{i12}^{n-1}-u_{i13}^{n-1})+(u_{i11}^{n-1}+u_{i12}^{n-1}+u_{i13}^{n-1})\Big]+$$
$$\frac{24\mu}{(1+24\mu)(1+72\mu)}\Big[2h(u_{i23}^{n-1}-u_{i21}^{n-1})(u_{i11}^{n-1}-u_{i12}^{n-1})+$$
$$2h(u_{i13}^{n-1}-u_{i11}^{n-1})(u_{i12}^{n-1}-u_{i13}^{n-1})+u_{i13}^{n-1}-u_{i11}^{n-1}\Big], \tag{30}$$

$$u_{i12}^{n}=-2u_{i13}^{n}+\frac{2h}{3(1+24\mu)(1+36\mu)}\Big[2(u_{i11}^{n-1}-u_{i12}^{n-1})(u_{i22}^{n-1}-u_{i23}^{n-1})+$$
$$(u_{i21}^{n-1}-u_{i22}^{n-1})^{2}+\mu(u_{i21}^{n-1}-24u_{i22}^{n-1}+23u_{i23}^{n-1})(u_{i12}^{n-1}-u_{i11}^{n-1})+$$
$$(u_{i12}^{n-1}-u_{i13}^{n-1})^{2}+\mu(u_{i11}^{n-1}-24u_{i12}^{n-1}+23u_{i13}^{n-1})(u_{i13}^{n-1}-u_{i12}^{n-1})+$$
$$\mu(73u_{i21}^{n-1}-24u_{i22}^{n-1}-49u_{i23}^{n-1})(u_{i21}^{n-1}-u_{i22}^{n-1})+$$
$$\mu(73u_{i11}^{n-1}-24u_{i12}^{n-1}-49u_{i13}^{n-1})(u_{i22}^{n-1}-u_{i23}^{n-1})\Big]+$$
$$\frac{1}{3(1+24\mu)(1+36\mu)}\Big[(u_{i12}^{n-1}-u_{i13}^{n-1})+(u_{i21}^{n-1}-u_{i22}^{n-1})+$$
$$\lambda(T_{i1}^{n}-T_{i2}^{n})-\mu(u_{i11}^{n-1}-24u_{i12}^{n-1}+23u_{i13}^{n-1})+$$
$$\mu\lambda(73T_{i1}^{n}-24T_{i2}^{n}-49T_{i3}^{n})+\mu(73u_{i21}^{n-1}-24u_{i22}^{n-1}-49u_{i23}^{n-1})\Big]+$$
$$2h(u_{i12}^{n-1}+u_{i12}^{n-1}+u_{i13}^{n-1})(u_{i11}^{n-1}-u_{i12}^{n-1})+u_{i11}^{n-1}+u_{i12}^{n-1}+u_{i13}^{n-1}+$$
$$2h(u_{i11}^{n-1}+u_{i12}^{n-1}+u_{i13}^{n-1})(u_{i12}^{n-1}-u_{i13}^{n-1})+$$
$$\frac{1}{1+24\mu}\Big[2h(u_{i23}^{n-1}-u_{i21}^{n-1})(u_{i11}^{n-1}-u_{i12}^{n-1})+$$
$$2h(u_{i13}^{n-1}-u_{i11}^{n-1})(u_{i12}^{n-1}-u_{i13}^{n-1})+u_{i13}^{n-1}-u_{i11}^{n-1}\Big], \tag{31}$$

$$u_{i11}^{n}=u_{i13}^{n}-\frac{2h}{3(1+24\mu)(1+36\mu)}\Big[2(u_{i11}^{n-1}-u_{i12}^{n-1})(u_{i22}^{n-1}-u_{i23}^{n-1})+$$
$$(u_{i21}^{n-1}-u_{i22}^{n-1})^{2}+\mu(u_{i21}^{n-1}-24u_{i22}^{n-1}+23u_{i23}^{n-1})(u_{i12}^{n-1}-u_{i11}^{n-1})+$$
$$(u_{i12}^{n-1}-u_{i13}^{n-1})^{2}+\mu(u_{i11}^{n-1}-24u_{i12}^{n-1}+23u_{i13}^{n-1})(u_{i13}^{n-1}-u_{i12}^{n-1})+$$
$$\mu(73u_{i21}^{n-1}-24u_{i22}^{n-1}-49u_{i23}^{n-1})(u_{i21}^{n-1}-u_{i22}^{n-1})+$$
$$\mu(73u_{i11}^{n-1}-24u_{i12}^{n-1}-49u_{i13}^{n-1})(u_{i22}^{n-1}-u_{i23}^{n-1})\Big]-$$
$$\frac{1}{3(1+24\mu)(1+36\mu)}\Big[(u_{i12}^{n-1}-u_{i13}^{n-1})+(u_{i21}^{n-1}-u_{i22}^{n-1})+\lambda(T_{i1}^{n}-T_{i2}^{n})-$$
$$\mu(u_{i11}^{n-1}-24u_{i12}^{n-1}+23u_{i13}^{n-1})+\mu\lambda(73T_{i1}^{n}-24T_{i2}^{n}-49T_{i3}^{n})+$$
$$\mu(73u_{i21}^{n-1}-24u_{i22}^{n-1}-49u_{i23}^{n-1})\Big]-$$
$$\frac{1}{1+24\mu}\Big[2h(u_{i23}^{n-1}-u_{i21}^{n-1})(u_{i11}^{n-1}-u_{i12}^{n-1})+$$
$$2h(u_{i13}^{n-1}-u_{i11}^{n-1})(u_{i12}^{n-1}-u_{i13}^{n-1})+u_{i13}^{n-1}-u_{i11}^{n-1}\Big], \tag{32}$$

$$u_{i23}^{n}=\frac{16h\mu}{(1+24\mu)(1+36\mu)(1+72\mu)}\Big[2(u_{i11}^{n-1}-u_{i12}^{n-1})(u_{i22}^{n-1}-u_{i23}^{n-1})+$$

$$(u^{n-1}_{i21} - u^{n-1}_{i22})^2 + \mu(u^{n-1}_{i21} - 24u^{n-1}_{i22} + 23u^{n-1}_{i23})(u^{n-1}_{i12} - u^{n-1}_{i11}) +$$
$$(u^{n-1}_{i12} - u^{n-1}_{i13})^2 + \mu(u^{n-1}_{i11} - 24u^{n-1}_{i12} + 23u^{n-1}_{i13})(u^{n-1}_{i13} - u^{n-1}_{i12}) +$$
$$\mu(73u^{n-1}_{i21} - 24u^{n-1}_{i22} - 49u^{n-1}_{i23})(u^{n-1}_{i21} - u^{n-1}_{i22}) +$$
$$\mu(73u^{n-1}_{i11} - 24u^{n-1}_{i12} - 49u^{n-1}_{i13})(u^{n-1}_{i22} - u^{n-1}_{i23})] +$$
$$\frac{8\mu}{(1+24\mu)(1+36\mu)(1+72\mu)}[(u^{n-1}_{i12} - u^{n-1}_{i13}) + (u^{n-1}_{i21} - u^{n-1}_{i22}) +$$
$$\lambda(T^n_{i1} - T^n_{i2}) - \mu(u^{n-1}_{i11} - 24u^{n-1}_{i12} + 23u^{n-1}_{i13}) +$$
$$\mu\lambda(73T^n_{i1} - 24T^n_{i2} - 49T^n_{i3}) + \mu(73u^{n-1}_{i21} - 24u^{n-1}_{i22} - 49u^{n-1}_{i23})] +$$
$$\frac{1}{1+72\mu}[\lambda T^n_{i3} + u^{n-1}_{i23} + 2hu^{n-1}_{i23}(u^{n-1}_{i21} - u^{n-1}_{i22}) + 2hu^{n-1}_{i13}(u^{n-1}_{i22} - u^{n-1}_{i23})] +$$
$$\frac{24\mu}{1+72\mu}[\lambda(T^n_{i1} + T^n_{i2} + T^n_{i3}) + 2h(u^{n-1}_{i21} + u^{n-1}_{i22} + u^{n-1}_{i23})(u^{n-1}_{i21} - u^{n-1}_{i22}) +$$
$$2h(u^{n-1}_{i11} + u^{n-1}_{i12} + u^{n-1}_{i13})(u^{n-1}_{i22} - u^{n-1}_{i23}) + u^{n-1}_{i21} + u^{n-1}_{i22} + u^{n-1}_{i23}] +$$
$$\frac{24\mu}{(1+24\mu)(1+72\mu)}[\lambda(T^n_{i3} - T^n_{i1}) + 2h(u^{n-1}_{i23} - u^{n-1}_{i21})(u^{n-1}_{i21} - u^{n-1}_{i22}) +$$
$$2h(u^{n-1}_{i13} - u^{n-1}_{i11})(u^{n-1}_{i22} - u^{n-1}_{i23}) + u^{n-1}_{i23} - u^{n-1}_{i21}], \tag{33}$$

$$u^n_{i22} = -2u^n_{i23} + \frac{2h}{3(1+24\mu)(1+36\mu)}[2(u^{n-1}_{i11} - u^{n-1}_{i12})(u^{n-1}_{i22} - u^{n-1}_{i23}) +$$
$$(u^{n-1}_{i21} - u^{n-1}_{i22})^2 + \mu(u^{n-1}_{i21} - 24u^{n-1}_{i22} + 23u^{n-1}_{i23})(u^{n-1}_{i12} - u^{n-1}_{i11}) +$$
$$(u^{n-1}_{i12} - u^{n-1}_{i13})^2 + \mu(u^{n-1}_{i11} - 24u^{n-1}_{i12} + 23u^{n-1}_{i13})(u^{n-1}_{i13} - u^{n-1}_{i12}) +$$
$$\mu(73u^{n-1}_{i21} - 24u^{n-1}_{i22} - 49u^{n-1}_{i23})(u^{n-1}_{i21} - u^{n-1}_{i22}) +$$
$$\mu(73u^{n-1}_{i11} - 24u^{n-1}_{i12} - 49u^{n-1}_{i13})(u^{n-1}_{i22} - u^{n-1}_{i23})] +$$
$$\frac{1}{3(1+24\mu)(1+36\mu)}[(u^{n-1}_{i12} - u^{n-1}_{i13}) + (u^{n-1}_{i21} - u^{n-1}_{i22}) + \lambda(T^n_{i1} - T^n_{i2}) -$$
$$\mu(u^{n-1}_{i11} - 24u^{n-1}_{i12} + 23u^{n-1}_{i13}) + \mu\lambda(73T^n_{i1} - 24T^n_{i2} - 49T^n_{i3}) +$$
$$\mu(73u^{n-1}_{i21} - 24u^{n-1}_{i22} - 49u^{n-1}_{i23})] + 2h(u^{n-1}_{i21} + u^{n-1}_{i22} + u^{n-1}_{i23})(u^{n-1}_{i21} - u^{n-1}_{i22}) +$$
$$2h(u^{n-1}_{i11} + u^{n-1}_{i12} + u^{n-1}_{i13})(u^{n-1}_{i22} - u^{n-1}_{i23}) + u^{n-1}_{i21} + u^{n-1}_{i22} + u^{n-1}_{i23} +$$
$$\lambda(T^n_{i1} + T^n_{i2} + T^n_{i3}) + \frac{1}{1+24\mu}[2h(u^{n-1}_{i23} - u^{n-1}_{i21})(u^{n-1}_{i21} - u^{n-1}_{i22}) +$$
$$\lambda(T^n_{i3} - T^n_{i1}) + 2h(u^{n-1}_{i13} - u^{n-1}_{i11})(u^{n-1}_{i22} - u^{n-1}_{i23}) + u^{n-1}_{i23} - u^{n-1}_{i21}], \tag{34}$$

$$u^n_{i21} = u^n_{i23} - \frac{2h}{3(1+24\mu)(1+36\mu)}[2(u^{n-1}_{i11} - u^{n-1}_{i12})(u^{n-1}_{i22} - u^{n-1}_{i23}) +$$
$$(u^{n-1}_{i21} - u^{n-1}_{i22})^2 + \mu(u^{n-1}_{i21} - 24u^{n-1}_{i22} + 23u^{n-1}_{i23})(u^{n-1}_{i12} - u^{n-1}_{i11}) +$$
$$(u^{n-1}_{i12} - u^{n-1}_{i13})^2 + \mu(u^{n-1}_{i11} - 24u^{n-1}_{i12} + 23u^{n-1}_{i13})(u^{n-1}_{i13} - u^{n-1}_{i12}) +$$
$$\mu(73u^{n-1}_{i21} - 24u^{n-1}_{i22} - 49u^{n-1}_{i23})(u^{n-1}_{i21} - u^{n-1}_{i22}) +$$
$$\mu(73u^{n-1}_{i11} - 24u^{n-1}_{i12} - 49u^{n-1}_{i13})(u^{n-1}_{i22} - u^{n-1}_{i23})] -$$
$$\frac{1}{3(1+24\mu)(1+36\mu)}[(u^{n-1}_{i12} - u^{n-1}_{i13}) + (u^{n-1}_{i21} - u^{n-1}_{i22}) +$$
$$\lambda(T^n_{i1} - T^2_{i2}) - \mu(u^{n-1}_{i11} - 24u^{n-1}_{i12} + 23u^{n-1}_{i13}) +$$
$$\mu\lambda(73T^n_{i1} - 24T^n_{i2} - 49T^n_{i3}) + \mu(73u^{n-1}_{i21} - 24u^{n-1}_{i22} - 49u^{n-1}_{i23})] -$$
$$\frac{1}{12+24\mu}[\lambda(T^n_{i3} - T^n_{i1}) + 2h(u^{n-1}_{i23} - u^{n-1}_{i21})(u^{n-1}_{i21} - u^{n-1}_{i22}) +$$

$$2h\left(u_{i13}^{n-1}-u_{i11}^{n-1}\right)\left(u_{i22}^{n-1}-u_{i23}^{n-1}\right)+u_{i23}^{n-1}-u_{i21}^{n-1}\Big]\cdot \tag{35}$$

结合(25)~(27)和(29)~(35)即得问题(Ⅰ)基于混合元法的差分格式•

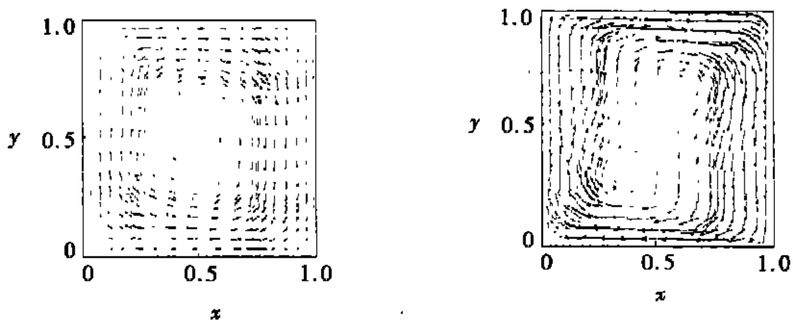
3 数值例子——封闭方腔自然对流的求解

本节我们以空气在封闭方腔自然对流的求解为数值例子,说明基于混合元法的差分格式的有效性• 边界的上$(u=v=0, T=1)$下$(u=v=0, T=0)$是两个恒温壁,边界的左$(u=v=0, T(0,y,t)=T(0))$右$(u=v=0, T(1,y,t)=T(1))$两边是两个绝热壁,如图2所示• 方腔为空气,由于上下是两个恒温壁差引起自然对流• 假定$\Omega=[0,1]\times[0,1]$,将其剖分成100×100个小正方形然后连接一对角成为直角三角形• 所以$h=k=0.01$,相应的差分格式如下:

$$\begin{cases}u_{1h}^{n}(0,l)=u_{1h}^{n}(100,l)=u_{2h}^{n}(0,l)=u_{2h}^{n}(100,l)=0 \qquad (0\leqslant l\leqslant 100);\\ u_{1h}^{n}(s,0)=u_{1h}^{n}(s,100)=u_{2h}^{n}(s,0)=u_{2h}^{n}(s,100)=0 \qquad (0\leqslant s\leqslant 100);\\ u_{jh}^{n}(s,l)=\left[u_{2l+100(s-1),j,2}^{n}+u_{2l+100(s-1)+1,j,3}^{n}+u_{2l+100(s-1)+2,j,1}^{n}+\right.\\ \qquad \left.u_{2l+100s-1,j,1}^{n}+u_{2l+100s,j,3}^{n}+u_{2l+100s+1,j,2}^{n}\right]/6\\ \qquad\qquad (1\leqslant s\leqslant 99, 1\leqslant l\leqslant 99, j=1,2);\end{cases}$$

$$\begin{cases}T_{h}^{n}(0,l)=0;\ T_{h}^{n}(100,l)=0 \qquad (0\leqslant l\leqslant 100);\\ T_{h}^{n}(s,0)=0;\ T_{h}^{n}(s,100)=1 \qquad (0\leqslant s\leqslant 100);\\ T_{h}^{n}(s,l)=\left[T_{2l+100(s-1),2}^{n}+T_{2l+100(s-1)+1,3}^{n}+T_{2l+100(s-1)+2,1}^{n}+\right.\\ \qquad \left.T_{2l+100s-1,1}^{n}+T_{2l+100s}^{n}+T_{2l+100s+1,2}^{n}\right]/6\\ \qquad\qquad (1\leqslant s\leqslant 99, 1\leqslant l\leqslant 99);\end{cases}$$

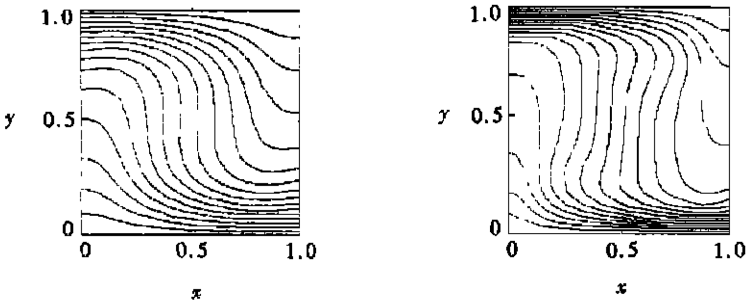
利用上式计算得出不同 Rayleigh 数的结果如图 3 至图 5 所示•



$Ra=10\,000$ $Ra=100\,000$

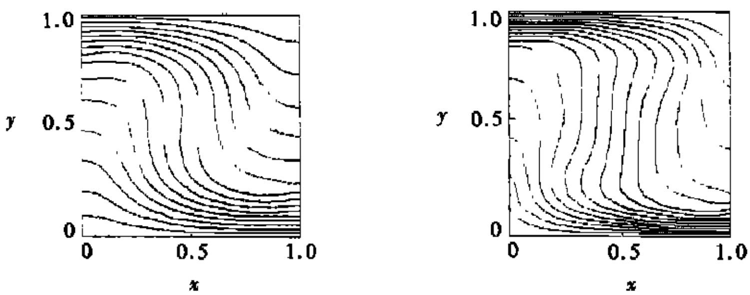
图 3 在不同的 Ra 值时速度向量的流线

结论 与现有的其他差分格式(如[1]和[2])比较可见,我们的数值结果对速度和温度只用一次元而压力用的零次元就很理想• 特别是我们的基于混合有限元法的差分格式能处理较小的粘性数即较大的 Ra 数(例如, $Ra=100\,000$)的情形,并且能同时求出速度、压力和温度的数值解•



$Ra=$ 10 000　　$Ra=$ 100 000

图4　在不同的 Ra 值时温度分布



$Ra=$ 10 000　　$Ra=$ 100 000

图5　在不同的 Ra 值时压力分布

[参 考 文 献]

[1] 傅德薰. 流体力学数值模拟[M]. 北京: 国防工业出版社, 1994.

[2] 忻孝康, 刘儒勋, 蒋伯诚. 计算流体动力学[M]. 长沙: 国防科技大学出版社, 1989.

[3] Paquet L. The boussinesq equations in presence of thermocapillarity at some part of the boundary [A]. In: Lumer G, Nicaise S, Shulze B W Eds. Partial Differential Equations, Mathematical Research[C]. **82**, Berlin: Akademie Verlag, 266—278.

[4] Girault V, Raviart P A. Finite Element Methods for Navier_Stokes Equations [M]. Berlin: Springer_Verlag, 1986.

[5] Adams R A. Sobolev Spaces[M]. New York: Academic Press, 1975.

[6] 罗振东. 非线性发展方程和热传导_对流方程的混合有限元分析与研究[D]. 合肥: 中国科学技术大学博士论文, 1997.

[7] Brezzi F, Douglas Jr J, Marrini L D. Two families of mixed finite elements for second order eliptic problems[J]. Numer Math, 1985, **47**(2): 217—235.

[8] Ciarlet P G. The Finite Element Method for Elliptic Problems[M]. Amsterdam: North_Holland, 1978.

[9] 罗振东. 有限元混合法理论基础及其应用, 发展与应用[M]. 济南: 山东教育出版社, 1996.

Difference Scheme and Numerical Simulation Based on Mixed Finite Element Method for Natural Convection Problem

LUO Zhen_dong[1,2], ZHU Jiang[2], XIE Zheng_hui[2], ZHANG Gui_fang[1]

(1. Department of Mathematics, Capital Normal University, Beijing 100037, P.R. China;
2. Institute of Academy Physics, Chinese Academy of Sciences, Beijing 100029, P.R. China)

Abstract: The non stationary natural convection problem is studied. A lowest order finite difference scheme based on mixed finite element method for non stationary natural convection problem, by the spatial variations discreted with finite element method and time with finite difference scheme was derived, where the numerical solution of velocity, pressure, and temperature can be found together, and a numerical example to simulate the close square cavity is given, which is of practical importance.

Key words: nutural convection equation; mixed element method; finite difference scheme