应用数学和力学编委会编
重 庆 出 版 社 出 版

文章编号: 1000_0887(2003) 02_0146_17

E_2 类二阶椭圆组一般形式的非线性边值问题*

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(江福汝推荐)

摘要: 研究了 E_2 类二阶椭圆型方程组相当广泛的一类非线性边值问题• 通过引进一种代换把它化为一类非线性广义 Riemann_Hilbert 边值问题, 再引进奇异积分算子, 建立与该问题等价的非线性奇异积分方程• 应用奇异积分算子性质和泛函分析与函数论方法, 在一定的假设条件下, 证得了该问题的可解性•

关 键 词: 椭圆型方程组; 边值问题; 奇异积分算子; 奇异积分方程; 存在性
中图分类号: O175.25; O175.8 文献标识码: A

引 言

本文研究 E_2 类二阶椭圆型方程组相当广泛的一类非线性边值问题, 这类非线性边值问题的研究在工程和力学中有重要的应用, 并越来越为国内许多学者所重视• 俄罗斯学者 L. V. Wolfersdorf 在研究全纯函数的边值问题时最先提出并讨论了这类非线性边值问题[1,2], 得到了有益的结果, 而后 Sigrid Kencht 和 Ali Seif Mashimba[3], R. P. Gilbert 和本文作者[4,5] 又对一阶和二阶方程组的这类问题做了更深入的研究, 本文在此基础上进一步探讨一般形式的二阶强椭圆型方程组这一类非线性边值问题• 通过引进一种代换把它化为一类非线性广义 Riemann_Hilbert 边值问题• 再通过引进奇异积分算子和计算积分, 建立与该问题等价的非线性奇异积分方程• 进而应用奇异积分算子性质和泛函分析与函数论方法, 在一定的假设条件下, 证得了该问题的可解性•

该文的方法, 为深入进行椭圆型方程组一般形式的非线性边值问题的研究, 开辟了新的思路• 有关的积分算子的建立也为深入开展这方面的研究打下了基础•

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* 收稿日期: 2001_07_24; 修订日期: 2002_10_17
基金项目: 国家自然科学基金资助项目 (19671056); 上海市自然科学基金资助项目(99ZA14030, 01ZA14023); 江西省自然科学基金资助项目(981102, 0211014)
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1 问题的提出与问题的转化

考察以下二阶拟线性椭圆型方程组

$$\frac{\partial^2 w}{\partial z\partial z} - q_1(z)\frac{\partial^2 w}{\partial z^2} - q_2(z)\frac{\partial^2 w}{\partial z^2} - q_3(z)\frac{\partial^2 w}{\partial z^2} - q_4(z)\frac{\partial^2 w}{\partial z^2} = F\left(z, w, \frac{\partial w}{\partial z}, \frac{\partial w}{\partial z}\right) \qquad z \in G: |z| < 1, \tag{1}$$

其中 F 可写为

$$F \equiv A_1(z, w)\frac{\partial w}{\partial z} + A_2(z, w)\frac{\partial w}{\partial z} + A_3(z, w)\frac{\partial w}{\partial z} + A_4(z, w)\frac{\partial w}{\partial z} + A_0(z, w),$$

$$w = u + \mathrm{i}v.$$

方程(1)是一致椭圆型的,即要求 $q_i(z)$ 满足不等式

$$\sum_{i=1}^{4}|q_i(z)| \leqslant q_0 < 1, \tag{2}$$

同时满足非线性边界条件

$$\begin{cases}\phi_1(s, u(\mathrm{e}^{\mathrm{i}s}), v(\mathrm{e}^{\mathrm{i}s})) = f_1(s) \\ \phi_2(s, u(\mathrm{e}^{\mathrm{i}s}), v(\mathrm{e}^{\mathrm{i}s})) = f_2(s)\end{cases} \qquad t = \mathrm{e}^{\mathrm{i}s} \in \Gamma: |t| = 1, \tag{3}$$

称问题(1)~(3)为非线性边值问题 **P**.

问题 P 是复杂非线性边值问题,为研究它,我们将该问题进行有效转化.

第一步:问题的转化与问题的等价性

引进参数 τ: $0 \leqslant \tau \leqslant 1$, 考察以下带参数 τ 的边值问题

$$\begin{cases}\dfrac{\partial^2 w}{\partial z\partial z} - q_1(z)\dfrac{\partial^2 w}{\partial z^2} - q_2(z)\dfrac{\partial^2 w}{\partial z^2} - q_3(z)\dfrac{\partial^2 w}{\partial z^2} - q_4(z)\dfrac{\partial^2 w}{\partial z^2} = \\ \qquad \tau F\left(z, w, \dfrac{\partial w}{\partial z}, \dfrac{\partial w}{\partial z}\right) \qquad z \in G, & (4) \\ \left.\begin{aligned}&\tau\phi_1(s, u, v) + (1-\tau)(u\phi_{1u}(s, u, v) + v\phi_{1v}(s, u, v)) = \tau f_1(s) \\ &\tau\phi_2(s, u, v) + (1-\tau)(u\phi_{2u}(s, u, v) + v\phi_{2v}(s, u, v)) = \tau f_2(s)\end{aligned}\right\} \\ \qquad\qquad t = \mathrm{e}^{\mathrm{i}s} \in \Gamma. & (5)\end{cases}$$

显见边界条件(3)是条件(5)中 $\tau = 1$ 的情形.

在方程(4)和边界条件(5)中对 τ 求导,即得到如下形式的广义 Riemann_Hilbert 边值问题

$$\begin{cases}\dfrac{\partial^2 \dot{w}}{\partial z\partial z} - q_1(z)\dfrac{\partial^2 \dot{w}}{\partial z^2} - q_2(z)\dfrac{\partial^2 \dot{w}}{\partial z^2} - q_3(z)\dfrac{\partial^2 \dot{w}}{\partial z^2} - q_4(z)\dfrac{\partial^2 \dot{w}}{\partial z^2} = \\ \qquad \tau F_*\left(z, \dot{w}, \dfrac{\partial \dot{w}}{\partial z}, \dfrac{\partial \dot{w}}{\partial z}\right) \qquad z \in G, & (6) \\ F_* \equiv F + \tau F_w \dot{w} + \tau F_{\bar{w}}\dot{\bar{w}} + \tau F_{w^{(1)}}\dfrac{\partial \dot{w}}{\partial z} + \tau F_{w^{(2)}}\dfrac{\partial \dot{w}}{\partial z} + \\ \qquad \tau F_{\overline{w^{(1)}}}\dfrac{\partial \dot{\bar{w}}}{\partial z} + \tau F_{\overline{w^{(2)}}}\dfrac{\partial \dot{\bar{w}}}{\partial z}, \\ w^{(1)} = \dfrac{\partial w}{\partial z},\ w^{(2)} = \dfrac{\partial w}{\partial z}, \\ \left.\begin{aligned}&\mathrm{Re}\left[\overline{\lambda_1(t, \tau, w)}\dot{w}\right] = g_1(t, \tau, w) \\ &\mathrm{Re}\left[\overline{\lambda_2(t, \tau, w)}\dot{w}\right] = g_2(t, \tau, w)\end{aligned}\right\} \qquad t \in \Gamma, & (7)\end{cases}$$

这里

$$\begin{cases} \dot w = \dfrac{\mathrm{d}w}{\mathrm{d}\tau} = \dot u + \mathrm{i}\dot v,\ \lambda_j = a_j + \mathrm{i}b_j,\ |\lambda_j| = 1 \qquad (j = 1, 2), \\ a_j = a_j \backslash |a_j + \mathrm{i}b_j|,\ a_j = \varphi_{ju} + (1-\tau)\varphi_{juu}u + (1-\tau)\varphi_{juv}v, \\ b_j = b_j \backslash |a_j + \mathrm{i}b_j|,\ b_j = \varphi_{jv} + (1-\tau)\varphi_{jvu}u + (1-\tau)\varphi_{jvv}v, \\ g_j = g_j \backslash |a_j + \mathrm{i}b_j|,\ g_j = u\varphi_{ju} + v\varphi_{jv} - \varphi_j. \end{cases} \tag{8}$$

易见问题(4)～(5)的解必为(6)～(7)的解. 反之, 由以后的假设和定理 2 的证明可知当 $\tau = 0$ 时, (4)～(5) 仅有零解, 即 $w(z, 0) = 0$, 于是对问题(6)～(7)的解 $\dot w(z, \tau)$ 关于 τ 在区间[0, 1] 上积分立即得到问题(4)～(5) 在 $\tau = 1$ 时的解, 即原问题(1)～(3)的解.

假设 A

i) $q_i(z) \in D_{1,p}(G)$ 且适合不等式(2), $A_i(z, w)$ $(i = 0, 1, \ldots, 4)$ 关于 $z \in G$, $w \in E$(E 为复平面) 按 Hölder_Lipschitz 连续有界, 且当 $w(z) \in D_{1,p}(G)$ 时, $A_i(z, w(z)) \in D_{1,p}(G)$ 并按 $D_{1,p}(G)$ 的范数均匀有界;

ii) $\varphi_j(s, u, v)$ 和它的关于 u, v 的直到三阶偏导数按Hölder_Lipschitz 连续有界, $\varphi_{ju}, \varphi_{jv} > 0$, $j = 1, 2$;

iii) $f_j(s)$, $s \in [0, 2\pi]$ 是以 2π 为周期的实连续函数, 且 $f'_j(s) \in L_p[-\pi, \pi]$, $p > 2$, $j = 1, 2$;

iv) 对给定的 w, $\lambda_j(t, \tau, w) \neq 0$, 且 $\dfrac{1}{2\pi}\displaystyle\int_\Gamma \mathrm{d}\arg\lambda_j(t, \tau, 0) = 0$.

第二步: 引进代换, 把问题(6)～(7)转化为标准非线性边值问题.

现在引进代换

$$\dot w = w^* \mathrm{e}^{\mathrm{i}p(z) + \mathrm{i}\overline{p^*(z)}}, \tag{9}$$

其中

$$p(z) = p_1(z) + \mathrm{i}p_2(z) = \frac{1}{2\pi}\int_0^{2\pi} \arg\lambda_1(t, \tau, 0)\, \frac{\mathrm{e}^{\mathrm{i}t} + z}{\mathrm{e}^{\mathrm{i}t} - z}\mathrm{d}t,$$

$$p^*(z) = p_1^*(z) + \mathrm{i}p_2^*(z) = \frac{1}{2\pi}\int_0^{2\pi} \arg\lambda_2(t, \tau, 0)\, \frac{\mathrm{e}^{\mathrm{i}t} + z}{\mathrm{e}^{\mathrm{i}t} - z}\mathrm{d}t.$$

显然在边界 Γ 上, 有

$$p_1(t) = \arg\lambda_1(t, \tau, 0),\ p_1^*(t) = \arg\lambda_2(t, \tau, 0), \tag{10}$$

所以我们有

$$\lambda_1\dot w = \mathrm{e}^{-\mathrm{i}\arg\lambda_1} w^* \mathrm{e}^{\mathrm{i}p(z) + \mathrm{i}\overline{p^*(z)}} = \mathrm{e}^{-\mathrm{i}\arg\lambda_1}\mathrm{e}^{\mathrm{i}p_1 - p_2 + \mathrm{i}p_1^* + p_2^*} w^*,$$

$$\lambda_1(z, \tau, 0)\dot w = \mathrm{e}^{-\mathrm{i}\arg\lambda_1(z, \tau, 0)}\mathrm{e}^{\mathrm{i}p_1 - p_2 + \mathrm{i}p_1^* + p_2^*} w^* + \lambda_1(z, \tau, 0)\dot w - \lambda_1(z, \tau, w)\dot w,$$

$$\lambda_2\dot w = \mathrm{e}^{-\mathrm{i}\arg\lambda_2} w^* \mathrm{e}^{\mathrm{i}p(z) + \mathrm{i}\overline{p^*(z)}} = \mathrm{e}^{-\mathrm{i}\arg\lambda_2}\mathrm{e}^{\mathrm{i}p_1 - p_2 + \mathrm{i}p_1^* + p_2^*} w^*,$$

$$\lambda_2(z, \tau, 0)\dot w = \mathrm{e}^{-\mathrm{i}\arg\lambda_2(z, \tau, 0)}\mathrm{e}^{\mathrm{i}p_2 - p_1 + \mathrm{i}p_1^* + p_2^*} w^* + \lambda_2(z, \tau, 0)\dot w - \lambda_2(z, \tau, w)\dot w.$$

于是边界条件(7)化为

$$\begin{cases} \mathrm{Re}[\mathrm{e}^{\mathrm{i}p_1^*} w^*] = \mathrm{e}^{p_2 - p_2^*} g_1 + \mathrm{Re}[(\lambda_1(z, \tau, 0) - \lambda_1(z, \tau, w))\mathrm{e}^{\mathrm{i}p_1 + \mathrm{i}p_1^*} w^*], \\ \mathrm{Re}[\mathrm{e}^{\mathrm{i}p_1} w^*] = \mathrm{e}^{p_2 - p_2^*} g_2 + \mathrm{Re}[(\lambda_2(z, \tau, 0) - \lambda_2(z, \tau, w))\mathrm{e}^{\mathrm{i}p_1 + \mathrm{i}p_1^*} w^*], \end{cases} \tag{11}$$

将(11)写成

$$\begin{cases} e^{ip_1^*}w^* + e^{-ip_1^*}\overline{w^*} = 2e^{p_2-p_2^*}g_1 + \left[(\lambda_1(z,\tau,0)-\lambda_1(z,\tau,w))e^{ip_1+ip_1^*}w^* - \right. \\ \qquad \left.(\lambda_1(z,\tau,0)-\lambda_1(z,\tau,w))e^{-ip_1-ip_1^*}w^*\right], \\ e^{ip_1^*}w^* + e^{-ip_1^*}\overline{w^*} = 2e^{p_2-p_2^*}g_2 + \left[(\lambda_2(z,\tau,0)-\lambda_2(z,\tau,w))e^{ip_1+ip_1^*}w^* - \right. \\ \qquad \left.(\lambda_2(z,\tau,0)-\lambda_2(z,\tau,w))e^{-ip_1-ip_1^*}w^*\right]\cdot \end{cases}$$

记上述方程组的系数行列式为

$$\Delta \equiv \begin{vmatrix} e^{ip_1^*} + (\lambda_1(z,\tau,0)-\lambda_1(z,\tau,w))e^{ip_1+ip_1^*} & e^{-ip_1^*} + (\lambda_1(z,\tau,0)-\lambda_1(z,\tau,w))e^{-ip_1-ip_1^*} \\ e^{ip_1} + (\lambda_1(z,\tau,0)-\lambda_1(z,\tau,w))e^{ip_1+ip_1^*} & e^{-ip_1} + (\lambda_1(z,\tau,0)-\lambda_1(z,\tau,w))e^{-ip_1-ip_1^*} \end{vmatrix},$$

则有

$$\Delta - \Delta = 2\left[e^{i(p_1^*-p_1)} - e^{i(p_1-p_1^*)}\right],$$

如果 $\Delta = 0$, 则 $\Delta = 0$, 即 $e^{i(p_1^*-p_1)} = e^{i(p_1-p_1^*)}$, 从而有 $p_1^* = p_1$, $\arg\lambda_1 = \arg\lambda_2$, $\lambda_1 = \lambda_2$, 即边界条件(7) 变为一个, 这与所作假设 A 相矛盾 , 所以 $\Delta\neq 0$, 从而由(11), 边界条件又可归结为

$$\begin{cases} \mathrm{Re}[w^*] = g_1^*(z,\tau,w^*), \\ \mathrm{Re}[iw^*] = g_2^*(z,\tau,w^*)\cdot \end{cases} \tag{12}$$

其中

$$g_1^*(z,\tau,w^*) = e^{-p_2+p_2^*}\left\{\mathrm{Re}[e^{ip_1^*}w^*] - \right.$$
$$\left.\mathrm{Re}[(\lambda_1(z,\tau,0)-\lambda_1(z,\tau,w))e^{ip_1+ip_1^*}w^*]\right\},$$
$$g_2^*(z,\tau,w^*) = e^{p_2^*-p_2}\left\{\mathrm{Re}[e^{ip_1^*}w^*] - \right.$$
$$\left.\mathrm{Re}[(\lambda_2(z,\tau,0)-\lambda_2(z,\tau,w))e^{ip_1+ip_1^*}w^*]\right\}\cdot$$

现将代换(9) 代入方程(6), 可得

$$e^{ip(z)+\overline{ip^*(z)}}\frac{\partial^2 w^*}{\partial z\partial \bar z} + \frac{\partial}{\partial \bar z}\left(e^{ip(z)+\overline{ip^*(z)}}\right)\frac{\partial w^*}{\partial z} +$$
$$\frac{\partial w^*}{\partial \bar z}\frac{\partial}{\partial z}\left(e^{ip(z)+\overline{ip^*(z)}}\right) + w^*\frac{\partial^2}{\partial z\partial \bar z}\left(e^{ip(z)+\overline{ip^*(z)}}\right) -$$
$$q_1(z)\left[\left(e^{ip(z)+\overline{ip^*(z)}}\right)\frac{\partial^2 w^*}{\partial z^2} + 2\frac{\partial w^*}{\partial z}\frac{\partial}{\partial z}\left(e^{ip(z)+\overline{ip^*(z)}}\right) + w^*\frac{\partial^2}{\partial z^2}\left(e^{ip(z)+\overline{ip^*(z)}}\right)\right] -$$
$$q_2(z)\left[\overline{\left(e^{ip(z)+\overline{ip^*(z)}}\right)\frac{\partial^2 w^*}{\partial z^2} + 2\frac{\partial w^*}{\partial z}\frac{\partial}{\partial z}\left(e^{ip(z)+\overline{ip^*(z)}}\right) + w^*\frac{\partial^2}{\partial z^2}\left(e^{ip(z)+\overline{ip^*(z)}}\right)}\right] -$$
$$q_3(z)\left[\left(e^{ip(z)+\overline{ip^*(z)}}\right)\frac{\partial^2 w^*}{\partial z^2} + 2\frac{\partial w^*}{\partial z}\frac{\partial}{\partial z}\left(e^{ip(z)+\overline{ip^*(z)}}\right) + w^*\frac{\partial^2}{\partial z^2}\left(e^{ip(z)+\overline{ip^*(z)}}\right)\right] -$$
$$q_4(z)\left[\overline{\left(e^{ip(z)+\overline{ip^*(z)}}\right)\frac{\partial^2 w^*}{\partial z^2} + 2\frac{\partial w^*}{\partial z}\frac{\partial}{\partial z}\left(e^{ip(z)+\overline{ip^*(z)}}\right) + w^*\frac{\partial^2}{\partial z^2}\left(e^{ip(z)+\overline{ip^*(z)}}\right)}\right] =$$
$$\tau F_*\left(z, w, \frac{\partial w}{\partial z}, \frac{\partial w}{\partial \bar z}\right)\cdot \tag{13}$$

注意到 $p(z)$、$p^*(z)$ 为全纯函数, 并以 $e^{ip(z)+\overline{ip^*(z)}}$ 除以上式两端, 则(13) 又可简化为

$$\frac{\partial^2 w^*}{\partial z\partial \bar z} - q_1(z)\frac{\partial^2 w^*}{\partial z^2} - q_2(z)e^{-2i(p_1+p_1^*)}\frac{\partial^2 \overline{w^*}}{\partial \bar z^2} -$$

$$q_3(z)\frac{\partial^2 w^*}{\partial z^2}-q_4(z)e^{-2i(p_1+p_1^*)}\frac{\partial^2 \overline{w^*}}{\partial z^2}=$$
$$\tau F^*\left(z, w^*, \frac{\partial w^*}{\partial z}, \frac{\partial w^*}{\partial z}\right), \tag{14}$$

其中

$$F^* = e^{-ip-\overline{ip^*}}F_* +$$
$$q_1(z)e^{-ip-\overline{ip^*}}\left[2i\overline{p^{*\prime}(z)}e^{\overline{ip^*}}\frac{\partial w^*}{\partial z}+(i\overline{p^{*\prime\prime}(z)}-\overline{[p^{*\prime}(z)]^2})e^{\overline{ip^*}}w^*\right]+$$
$$q_2(z)e^{-ip-\overline{ip^*}}\overline{\left[2i\overline{p^{*\prime}(z)}e^{\overline{ip^*}}\frac{\partial w^*}{\partial z}+(i\overline{p^{*\prime\prime}(z)}-\overline{[p^{*}(z)]^2})e^{\overline{ip^*}}w^*\right]}+$$
$$q_3(z)e^{-ip-\overline{ip^*}}\left[2ip'(z)e^{ip(z)}\frac{\partial w^*}{\partial z}+(i\overline{p''(z)}-[p'(z)]^2)e^{ip(z)}w^*\right]+$$
$$q_4(z)e^{-ip-\overline{ip^*}}\overline{\left[2ip'(z)e^{ip(z)}\frac{\partial w^*}{\partial z}+(i\overline{p''(z)}-[p'(z)]^2)e^{ip(z)}w^*\right]}. \tag{15}$$

显见, 由于 $p_1(z)$、$p_1^*(z)$ 分别是全纯函数 $p(z)$、$p^*(z)$ 的实部, 故方程(14) 的第 2、4 项的系数的绝对值不变, 即

$$|q_j(z)e^{-2i[p_1(z)+p_1^*(z)]}|=|q_j(z)| \qquad (j=2,4),$$

从而得知, 经过形如(9) 的代换, 方程(6) 和(14) 的椭圆型性质不变.

换言之, 我们可把原非线性边值问题(6) ~ (7) 化为关于 w^* 的方程(14) 适合边界条件(12) 的非线性边值问题以求解.

为方便起见, 以后仍不妨记 $q_j(z)e^{-2i[p_1(z)+p_1^*(z)]}$ 为 $q_j(z)$, $j=2,4$, 以下建立与该问题(12)、(14) 等价的非线性奇异积分方程.

为方便计, 我们首先研究如下问题(1)*、(12)*, 又简称问题 P^*:

$$\frac{\partial^2 w}{\partial z\partial \bar z}-q_1(z)\frac{\partial^2 w}{\partial z^2}-q_2(z)\frac{\partial^2 \bar w}{\partial \bar z^2}-q_3(z)\frac{\partial^2 w}{\partial \bar z^2}-q_4(z)\frac{\partial^2 \bar w}{\partial z^2}=$$
$$F\left(z, w, \frac{\partial w}{\partial z}, \frac{\partial w}{\partial \bar z}\right) \qquad z\in G: |z|<1, \tag{1*}$$
$$\begin{cases}\mathrm{Re}[w]=g_1(z,w),\\ \mathrm{Re}[iw]=g_2(z,w).\end{cases} \tag{12*}$$

采用奇异积分算子性质和先验估计, 获得该问题(1)*、(12)* 的可解性结果. 然后在假设 A 下可通过通常的连续性方法获得问题(12)、(14) 的可解性.

2 建立与问题 P^* 等价的奇异积分方程

应用文献[6] 中所熟悉的基本定理和算子 T 和 Π 的表示, 由方程(1)*, 可得到它的等价积分表示式

$$w(z)=\varphi_1(z)+\overline{\varphi_2(z)}+T\bar T\left(q_1(z)\frac{\partial^2 w}{\partial z^2}+q_2(z)\frac{\partial^2 \bar w}{\partial \bar z^2}+\right.$$
$$\left.q_3(z)\frac{\partial^2 w}{\partial \bar z^2}+q_4(z)\frac{\partial^2 \bar w}{\partial z^2}\right)+T\bar TF, \tag{16}$$

其中 $\varphi_1(z)$、$\varphi_2(z)$ 为任意全纯函数. 以下先考虑 $q_i(z)\in D^0_\infty(G)$ 情形.

首先计算 $T\bar T(q_1(z)\partial^2 w/\partial z^2)$, 应用 Green 公式, 我们有

$$T\left(q_1(z)\frac{\partial^2 w}{\partial z^2}\right) = -\frac{1}{\pi}\iint_G \frac{q_1(\zeta)\frac{\partial^2 w}{\partial \zeta^2}}{\zeta - z}\mathrm{d}\sigma_\zeta =$$

$$-\frac{1}{\pi}\iint_G \frac{\partial}{\partial \zeta}\left(\frac{q_1(\zeta)\frac{\partial w}{\partial \zeta}}{\zeta - z}\right)\mathrm{d}\sigma_\zeta + \frac{1}{\pi}\iint_G \frac{\partial w}{\partial \zeta}\frac{\partial}{\partial \zeta}\left(\frac{q_1(\zeta)}{\zeta - z}\right)\mathrm{d}\sigma_\zeta =$$

$$-\frac{1}{2\pi i}\int_\Gamma \frac{q_1(\zeta)\frac{\partial w}{\partial \zeta}}{\zeta - z}\mathrm{d}\zeta + \lim_{\varepsilon \to 0}\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{q_1(\zeta)\frac{\partial w}{\partial \zeta}}{\zeta - z}\mathrm{d}\zeta +$$

$$\frac{1}{\pi}\iint_G \frac{\partial w}{\partial \zeta}\frac{\partial}{\partial \zeta}\left(\frac{q_1(\zeta)}{\zeta - z}\right)\mathrm{d}\sigma_\zeta \qquad \Gamma_\varepsilon: |\zeta - z| = \varepsilon,$$

而

$$\lim_{\varepsilon \to 0}\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{q_1(\zeta)\frac{\partial w}{\partial \zeta}}{\zeta - z}\mathrm{d}\zeta =$$

$$\lim_{\varepsilon \to 0}\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{q_1(\zeta)\frac{\partial w}{\partial \zeta} - q_1(\zeta)\frac{\partial w}{\partial z}}{\zeta - z}\mathrm{d}\zeta + \lim_{\varepsilon \to 0}\frac{q_1(\zeta)\frac{\partial w}{\partial z}}{2\pi i}\int_{\Gamma_\varepsilon}\frac{\mathrm{d}\zeta}{\zeta - z}.$$

由假设 $q_i(z)\in D^0_\infty(G)$, 所以 $q_i(z)$ 在 G 内连续, 按广义解的定义, $w(z)$ 及其一阶偏导数也连续, 所以上式右端的第一个极限为零, 而第 2 项积分也为零, 这是因为

$$\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{\mathrm{d}\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_0^{2\pi}\frac{\varepsilon e^{i\theta}}{\varepsilon e^{-i\theta}}\mathrm{d}\theta = \frac{1}{2\pi i}\int_0^{2\pi}e^{2i\theta}\mathrm{d}\theta = 0.$$

因此有

$$T\left(q_1(z)\frac{\partial^2 w}{\partial z^2}\right) = \frac{1}{\pi}\iint_G \frac{\partial w}{\partial \zeta}\frac{\partial}{\partial \zeta}\left(\frac{q_1(\zeta)}{\zeta - z}\right)\mathrm{d}\sigma_\zeta. \tag{17}$$

另外

$$T\varphi_0 = -\frac{1}{\pi}\iint_G \frac{\overline{\varphi_0(z_1)}}{z_1 - z}\mathrm{d}\sigma_{z_1} = -\overline{\frac{1}{\pi}\iint_G \varphi_0(z_1)\frac{\partial}{\partial z_1}\lg(z_1 - z)\mathrm{d}\sigma_{z_1}} =$$

$$\overline{-\frac{1}{2\pi i}\int_\Gamma \varphi_0(z_1)\lg(z_1 - z)\mathrm{d}z_1 + \lim_{\varepsilon \to 0}\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\varphi_0(z_1)\lg(z_1 - z)\mathrm{d}z_1} =$$

$$\overline{-\frac{1}{2\pi i}\int_\Gamma \varphi_0(z_1)[\lg(1 - z_1 z) - \lg z_1]\mathrm{d}z_1} + O(\varepsilon\lg\varepsilon) =$$

$$-\frac{1}{2\pi i}\int_\Gamma \overline{\varphi_0(z_1)}[\lg(1 - z_1 z) - \lg z_1]\frac{\mathrm{d}z_1}{z_1^2}.$$

显见, 它是关于 z 的全纯函数, 不妨把它拼入任意全纯函数 $\varphi_1(z)$. 于是 $TT(q_1(z)\partial^2 w/\partial z^2)$ 只含有以下项

$$\frac{1}{\pi^2}\iint_G \frac{\mathrm{d}\sigma_{z_1}}{z_1 - z}\iint_G \frac{\partial w}{\partial \zeta}\frac{\partial}{\partial \zeta}\left(\frac{q_1(\zeta)}{\zeta - z}\right)\mathrm{d}\sigma_\zeta =$$

$$-\frac{1}{\pi^2}\iint_G \frac{\partial w}{\partial \zeta}\mathrm{d}\sigma_\zeta\iint_G \frac{(\zeta - z_1)\frac{\partial q_1}{\partial \zeta} - q_1(\zeta)}{(\zeta - z_1)^2(z_1 - z)}\mathrm{d}\sigma_{z_1} =$$

$$-\frac{1}{\pi^2}\iint_G \frac{\partial q_1}{\partial \zeta}\frac{\partial w}{\partial \zeta}\mathrm{d}\sigma_\zeta\iint_G \frac{\mathrm{d}\sigma_{z_1}}{(\zeta - z_1)(z_1 - z)} +$$

$$\frac{1}{\pi^2}\iint_G q_1(\zeta)\frac{\partial w}{\partial \zeta}\mathrm{d}\sigma_\zeta\iint_G \frac{\mathrm{d}\sigma_{z_1}}{(\zeta - z_1)^2(z_1 - z)}. \tag{18}$$

在(18)式中, 利用 Green 公式我们可计算得

$$\frac{1}{\pi}\iint_G \frac{d\sigma_{z_1}}{(\zeta-z_1)^2(z_1-z)} = -\frac{z}{1-\bar{\zeta}z}+\frac{1}{z-\zeta}, \tag{19}$$

和

$$\frac{1}{\pi}\iint_G \frac{d\sigma_{z_1}}{(\zeta-z_1)(z_1-z)} = -\frac{\bar{\zeta}z}{1-\bar{\zeta}z}+\frac{\zeta}{z-\zeta}-\frac{1}{\pi}\iint_G \frac{z_1 d\sigma_{z_1}}{(z_1-z)(z_1-\zeta)^2}. \tag{20}$$

这样一来, 我们有

$$\begin{aligned}TT\left(q_1(z)\frac{\partial^2 w}{\partial z^2}\right) = &-\frac{1}{\pi}\iint_G \frac{\partial q_1}{\partial \zeta}\frac{\partial w}{\partial \zeta}\left(-\frac{\bar{\zeta}z}{1-\bar{\zeta}z}+\frac{\zeta}{z-\zeta}\right)d\sigma_\zeta+\\ &\frac{1}{\pi}\iint_G \frac{\partial q_1}{\partial \zeta}\frac{\partial w}{\partial \zeta}d\sigma_\zeta\frac{1}{\pi}\iint_G \frac{z_1 d\sigma_{z_1}}{(\zeta-z_1)^2(z_1-z)}+\\ &\frac{1}{\pi}\iint_G q_1(\zeta)\frac{\partial w}{\partial \zeta}\left(\frac{\zeta}{z-\zeta}-\frac{z}{1-\bar{\zeta}z}\right)d\sigma_\zeta+\varphi_0^*(z);\end{aligned} \tag{21}$$

同样可以计算得

$$\frac{1}{\pi}\iint_G \frac{z_1 d\sigma_{z_1}}{(\zeta-z_1)^2(z_1-z)} = -\frac{\bar{\zeta}z}{1-\bar{\zeta}z}+\frac{\zeta}{z-\zeta}+\frac{1}{\pi}\iint_G \frac{d\sigma_{z_1}}{(z_1-\zeta)(z_1-z)}. \tag{22}$$

将(22)式代入(21)式, 并把 $\varphi_0^*(z)$ 拼入 $\varphi_1(z)$, 则我们有

$$\begin{aligned}TT\left(q_1(z)\frac{\partial^2 w}{\partial z^2}\right) = &\frac{1}{\pi}\iint_G q_1(\zeta)\frac{\partial w}{\partial \zeta}\left(\frac{\zeta}{z-\zeta}-\frac{z}{1-\bar{\zeta}z}\right)d\sigma_\zeta+\\ &\frac{1}{\pi}\iint_G \frac{\partial q_1}{\partial \zeta}\frac{\partial w}{\partial \zeta}d\sigma_\zeta\frac{1}{\pi}\iint_G \frac{d\sigma_{z_1}}{(\zeta-z_1)(z_1-z)},\end{aligned} \tag{23}$$

而

$$\begin{aligned}\mathrm{I} = &\frac{1}{\pi}\iint_G q_1(\zeta)\frac{\partial w}{\partial \zeta}\left(\frac{\zeta}{z-\zeta}-\frac{z}{1-\bar{\zeta}z}\right)d\sigma_\zeta = \\ &-\frac{1}{\pi}\iint_G \frac{\partial}{\partial \zeta}\frac{q_1(\zeta)w(\zeta)}{z-\zeta}d\sigma_\zeta+\frac{1}{\pi}\iint_G w(\zeta)\frac{\partial}{\partial \zeta}\frac{q_1(\zeta)}{\zeta-z}d\sigma_\zeta-\\ &\frac{1}{\pi}\iint_G \frac{\partial}{\partial \zeta}\frac{zq_1(\zeta)w(\zeta)}{1-\bar{\zeta}z}d\sigma_\zeta+\frac{1}{\pi}\iint_G w(\zeta)\frac{\partial}{\partial \zeta}\frac{zq_1(\zeta)}{1-\bar{\zeta}z}d\sigma_\zeta = \\ &-\frac{1}{2\pi i}\int_\Gamma \frac{q_1(\zeta)w(\zeta)}{\zeta-z}d\zeta+\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{|\zeta-z|=\varepsilon}\frac{q_1(\zeta)w(\zeta)}{\zeta-z}d\zeta-\\ &\frac{1}{\pi}\iint_G w(\zeta)\frac{\partial q_1(\zeta)}{\partial \zeta}\frac{d\sigma_\zeta}{\zeta-z}+\frac{1}{\pi}\iint_G \frac{q_1(\zeta)w(\zeta)}{(\zeta-z)^2}d\sigma_\zeta-\\ &\frac{1}{2\pi i}\int_\Gamma \frac{zq_1(\zeta)w(\zeta)}{1-\bar{\zeta}z}d\zeta+\frac{1}{\pi}\iint_G w(\zeta)z\frac{\partial q_1(\zeta)}{\partial \zeta}\frac{d\sigma_\zeta}{1-\bar{\zeta}z}+\\ &\frac{z^2}{\pi}\iint_G \frac{q_1(\zeta)w(\zeta)}{(1-\bar{\zeta}z)^2}d\sigma_\zeta = \frac{1}{\pi}\iint_G \frac{q_1(\zeta)w(\zeta)}{(\zeta-z)^2}d\sigma_\zeta+\frac{z^2}{\pi}\iint_G \frac{q_1(\zeta)w(\zeta)}{(1-\bar{\zeta}z)^2}d\sigma_\zeta-\\ &\frac{1}{\pi}\iint_G w(\zeta)\frac{\partial q_1(\zeta)}{\partial \zeta}\frac{d\sigma_\zeta}{\zeta-z}+\frac{z}{\pi}\iint_G w(\zeta)\frac{\partial q_1(\zeta)}{\partial \zeta}\frac{d\sigma_\zeta}{1-\bar{\zeta}z}+\\ &\varphi_1(z)+\overline{\varphi_2(z)} \qquad (\varphi_i \text{ 为任意全纯函数}),\end{aligned}$$

即

$$\begin{aligned}\mathrm{I} = &\frac{1}{\pi}\iint_G \frac{q_1(\zeta)w(\zeta)}{(\zeta-z)^2}d\sigma_\zeta+\frac{z^2}{\pi}\iint_G \frac{q_1(\zeta)w(\zeta)}{(1-\bar{\zeta}z)^2}d\sigma_\zeta+\\ &T_0(w)+\varphi_1(z)+\overline{\varphi_2(z)} \qquad (T_0(w) \text{ 为弱奇性积分});\end{aligned} \tag{24}$$

而

$$\mathrm{II}=\frac{1}{\pi}\iint_G\frac{\partial q_1}{\partial\zeta}\frac{\partial w}{\partial\zeta}d\sigma_\zeta\frac{1}{\pi}\iint_G\frac{d\sigma_{z_1}}{(\zeta-z_1)(z_1-z)}=$$
$$\frac{1}{\pi}\iint_G\frac{\partial}{\partial\zeta}\left(w\frac{\partial q_1}{\partial\zeta}\frac{1}{\pi}\iint_G\frac{d\sigma_{z_1}}{(z_1-\zeta)(z_1-z)}\right)d\sigma_\zeta-$$
$$\frac{1}{\pi}\iint_G w(\zeta)\frac{\partial^2 q_1}{\partial\zeta^2}d\sigma_\zeta\frac{1}{\pi}\iint_G\frac{d\sigma_{z_1}}{(z_1-\zeta)(z_1-z)}-$$
$$\frac{1}{\pi}\iint_G w(\zeta)\frac{\partial q_1}{\partial\zeta}d\sigma_\zeta\frac{1}{\pi}\iint_G\frac{d\sigma_{z_1}}{(z_1-\zeta)^2(z_1-z)}.$$

由[7]中阿达玛不等式可知

$$\begin{cases}\dfrac{1}{\pi}\iint_G\dfrac{d\sigma_{z_1}}{|z_1-\zeta||z_1-z|}\leqslant 8\lg\dfrac{\rho_0}{|\zeta-z|},\\[2ex]\dfrac{1}{\pi}\iint_G\dfrac{d\sigma_{z_1}}{|z_1-\zeta|^2|z_1-z|}\leqslant 8|\zeta-z|^{-1},\end{cases}\tag{25}$$

所以II式中的右端第2、第3项都是弱奇性积分, 而第一项又可应用Green公式写成

$$\frac{1}{\pi}\iint_G\frac{\partial}{\partial\zeta}\left(w\frac{\partial q_1}{\partial\zeta}\frac{1}{\pi}\iint_G\frac{d\sigma_{z_1}}{(z_1-\zeta)(z_1-z)}\right)d\sigma_\zeta=$$
$$\frac{1}{2\pi i}\int_\Gamma\left[w\frac{\partial q_1}{\partial\zeta}\frac{1}{\pi}\iint_G\frac{d\sigma_{z_1}}{(z_1-\zeta)(z_1-z)}\right]d\zeta-$$
$$\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{|\zeta-z|=\varepsilon}\left[w\frac{\partial q_1}{\partial\zeta}\frac{1}{\pi}\iint_G\frac{d\sigma_{z_1}}{(z_1-\zeta)(z_1-z)}\right]d\zeta,$$

而

$$\left|\frac{1}{2\pi i}\int_{|\zeta-z|=\varepsilon}\left[w\frac{\partial q_1}{\partial\zeta}\frac{1}{\pi}\iint_G\frac{d\sigma_{z_1}}{(z_1-\zeta)(z_1-z)}\right]d\zeta\right|\leqslant$$
$$\frac{1}{2\pi}\max_{\zeta\in G}\left|w(\zeta)\frac{\partial q_1}{\partial\zeta}\right|\int_{|\zeta-z|=\varepsilon}\left[\frac{1}{\pi}\iint_G\frac{d\sigma_{z_1}}{|z_1-\zeta||z_1-z|}\right]|d\zeta|\leqslant$$
$$\frac{1}{2\pi}\max_{\zeta\in G}\left|w(\zeta)\frac{\partial q_1}{\partial\zeta}\right|\int_{|\zeta-z|=\varepsilon}8\lg\frac{\rho_0}{|\zeta-z|}|d\zeta|\leqslant M'\varepsilon\lg\varepsilon\to 0.$$

所以第一项的弱奇性积分等于零, 即得到

$$TT\left(q_1(z)\frac{\partial^2 w}{\partial z^2}\right)=-\frac{1}{\pi}\iint_G\frac{q_1(\zeta)w(\zeta)}{(\zeta-z)^2}d\sigma_\zeta+$$
$$\frac{z^2}{\pi}\iint_G\frac{q_1(\zeta)w(\zeta)}{(1-\bar{\zeta}z)^2}d\sigma_\zeta+T_1(w)+\varphi_1(z)+\overline{\varphi_2(z)},\tag{26}$$

其中 $T_1(w)$ 是弱奇性积分算子, φ_i 为 全纯函数. 同样有

$$TT\left(q_4(z)\frac{\partial^2 w}{\partial z^2}\right)=-\frac{1}{\pi}\iint_G\frac{q_4(\zeta)\overline{w(\zeta)}}{(\zeta-z)^2}d\sigma_\zeta+$$
$$\frac{z^2}{\pi}\iint_G\frac{q_4(\zeta)\overline{w(\zeta)}}{(1-\bar{\zeta}z)^2}d\sigma_\zeta+T_4(w)+\varphi_1(z)+\overline{\varphi_2(z)}.\tag{27}$$

类似地可以计算

$$T\left(q_3(z)\frac{\partial^2 w}{\partial z^2}\right)=-\frac{1}{\pi}\iint_G\frac{q_3(\zeta)\dfrac{\partial^2 w}{\partial\zeta^2}}{\zeta-z}d\sigma_\zeta=-\frac{1}{2\pi i}\int_\Gamma\frac{q_3(\zeta)\dfrac{\partial w}{\partial\zeta}}{\zeta(1-\bar{\zeta}z)}d\zeta+$$

$$\lim_{\varepsilon\to 0}\frac{q_3(z)\dfrac{\partial w}{\partial z}}{2\pi i}\int_{\Gamma_\varepsilon}\frac{\varepsilon e^{-i\theta}\,i\,d\theta}{\varepsilon e^{-i\theta}}+\frac{1}{\pi}\iint_G\frac{\partial w}{\partial\zeta}\frac{\partial}{\partial\zeta}\left(\frac{q_3(\zeta)}{\zeta-z}\right)d\sigma_\zeta=$$

$$\overline{\varphi_0(z)}+q_3(z)\frac{\partial w}{\partial z}+\frac{1}{\pi}\iint_G\frac{\partial w}{\partial\zeta}\frac{\partial}{\partial\zeta}\left(\frac{q_3(\zeta)}{\zeta-z}\right)d\sigma_\zeta,$$

$$TT\left(q_3(z)\frac{\partial^2 w}{\partial z^2}\right)=-\frac{1}{\pi}\iint_G\frac{\partial}{\partial\zeta}\left(\frac{q_3(\zeta)w(\zeta)}{\zeta-z}\right)d\sigma_\zeta+\frac{1}{\pi}\iint_G w(\zeta)\frac{\partial}{\partial\zeta}\frac{q_3(\zeta)}{\zeta-z}d\sigma_\zeta-$$

$$\frac{1}{\pi^2}\iint_G\frac{d\sigma_{z_1}}{z_1-z}\iint_G\frac{\dfrac{\partial w}{\partial\zeta}\dfrac{\partial q_3(\zeta)}{\partial\zeta}}{\zeta-z_1}d\sigma_\zeta-\frac{1}{\pi}\iint_G\frac{\overline{\varphi_0(\zeta)}}{\zeta-z}d\sigma_\zeta=$$

$$\frac{1}{2\pi i}\int_\Gamma\frac{q_3(\zeta)w(\zeta)}{\zeta^2(\zeta-z)}d\zeta-\lim_{\varepsilon\to 0}\frac{q_3(z)w(z)}{2\pi i}\int_0^{2\pi}\frac{-ie^{-i\theta}d\theta}{\varepsilon e^{i\theta}}+$$

$$\frac{1}{\pi}\iint_G\frac{w(\zeta)\dfrac{\partial q_3(\zeta)}{\partial\zeta}}{\zeta-z}d\sigma_\zeta-\frac{1}{\pi}\iint_G\frac{q_3(\zeta)w(\zeta)}{(\zeta-z)^2}d\sigma_\zeta-$$

$$\frac{1}{\pi^2}\int_G\frac{\partial w}{\partial\zeta}\frac{\partial q_3(\zeta)}{\partial\zeta}d\sigma_\zeta\iint_G\frac{d\sigma_{z_1}}{(\zeta-z_1)(z_1-z)}-\frac{1}{\pi}\iint_G\frac{\overline{\varphi_0(\zeta)}}{\zeta-z}d\sigma_\zeta, \tag{28}$$

而

$$\frac{\partial^2}{\partial z\partial z}\left(-\frac{1}{\pi}\iint_G\frac{\overline{\varphi_0(\zeta)}}{\zeta-z}d\sigma_\zeta\right)=\frac{\partial}{\partial z}\overline{\varphi_0(z)}=0,$$

所以

$$-\frac{1}{2\pi i}\int_\Gamma\frac{q_3(\zeta)w(\zeta)}{\zeta^2(\zeta-z)}d\zeta-\frac{1}{\pi}\iint_G\frac{\overline{\varphi_0(\zeta)}}{\zeta-z}d\sigma_\zeta=\varphi_1(z)+\overline{\varphi_2(z)}$$

$$(\varphi_i\text{ 为任意全纯函数},\ i=0,1,2,)\cdot \tag{29}$$

利用 Green 公式和阿达玛估计式, 有

$$-\frac{1}{\pi^2}\iint_G\frac{\partial w}{\partial\zeta}\frac{\partial q_3(\zeta)}{\partial\zeta}d\sigma_\zeta\iint_G\frac{d\sigma_{z_1}}{(\zeta-z_1)(z_1-z)}=$$

$$-\frac{1}{\pi^2}\iint_G\frac{\partial}{\partial\zeta}\left[w(\zeta)\frac{\partial q_3(\zeta)}{\partial\zeta}\iint_G\frac{d\sigma_{z_1}}{(\zeta-z_1)(z_1-z)}\right]d\sigma_\zeta+$$

$$\frac{1}{\pi^2}\iint_G w(\zeta)\frac{\partial^2 q_3(\zeta)}{\partial\zeta^2}d\sigma_\zeta\iint_G\frac{d\sigma_{z_1}}{(\zeta-z_1)(z_1-z)}-$$

$$\frac{1}{\pi^2}\iint_G\left[w(\zeta)\frac{\partial q_3(\zeta)}{\partial\zeta}\frac{\partial}{\partial\zeta}\iint_G\frac{d\sigma_{z_1}}{(\zeta-z_1)(z_1-z)}\right]d\sigma_\zeta=$$

$$\frac{1}{2\pi i}\int_\Gamma\left[w(\zeta)\frac{\partial q_3(\zeta)}{\partial\zeta}\iint_G\frac{d\sigma_{z_1}}{(\zeta-z_1)(z_1-z)}\right]d\zeta-$$

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{|\zeta-z|=\varepsilon}\left[w(\zeta)\frac{\partial q_3(\zeta)}{\partial\zeta}\iint_G\frac{d\sigma_{z_1}}{(\zeta-z_1)(z_1-z)}\right]d\zeta+$$

$$\frac{1}{\pi^2}\iint_G w(\zeta)\frac{\partial^2 q_3(\zeta)}{\partial\zeta^2}d\sigma_\zeta\iint_G\frac{d\sigma_{z_1}}{(\zeta-z_1)(z_1-z)}-$$

$$\frac{1}{\pi}\iint_G w(\zeta)\frac{\partial q_3(\zeta)}{\partial\zeta}\frac{d\sigma_\zeta}{\zeta-z}, \tag{30}$$

而

$$\left|\frac{1}{\pi}\iint_G\frac{d\sigma_{z_1}}{(\zeta-z_1)(z_1-z)}\right|\leqslant 8\lg\frac{\rho_0}{|\zeta-z|},$$

$$\left|\frac{1}{2\pi i}\int_{|\zeta-z|=\varepsilon}\left[w(\zeta)\frac{\partial q_3(\zeta)}{\partial\zeta}\frac{1}{\pi}\iint_G\frac{d\sigma_{z_1}}{(\zeta-z_1)(z_1-z)}\right]d\zeta\right|\leqslant$$

$$\max_{\zeta\in G}\left|w(\zeta)\frac{\partial q_3(\zeta)}{\partial\zeta}\right|\varepsilon\cdot 8\lg\frac{\rho_0}{\varepsilon}\to 0.$$

所以(30)中第1、2、4项都是弱奇性积分，第2项为零，由此回到(28)，我们可表示

$$TT\left(q_3(z)\frac{\partial^2 w}{\partial z^2}\right)=-\frac{1}{\pi}\iint_G\frac{q_3(\zeta)w(\zeta)}{(\zeta-z)^2}d\sigma_\zeta+T_3(w)+\varphi_1(z)+\overline{\varphi_2(z)},\tag{31}$$

其中 $T_3(w)$ 为弱奇性积分。

由此又易推得

$$TT\left(q_2(z)\frac{\partial^2 w}{\partial z^2}\right)=-\frac{1}{\pi}\iint_G\frac{q_2(\zeta)\overline{w(\zeta)}}{(\zeta-z)^2}d\sigma_\zeta+T_2(w)+\varphi_1(z)+\overline{\varphi_2(z)},\tag{32}$$

而 $T_2(w)$ 也是弱奇性积分。

综合以上讨论结果(26)、(27)、(31)、(32)，得到与(16)等价的奇异积分方程

$$\begin{aligned}w(z)=&\ \varphi_1(z)+\overline{\varphi_2(z)}-\frac{1}{\pi}\iint_G\frac{q_1(\zeta)w(\zeta)}{(\zeta-z)^2}d\sigma_\zeta+\\&\frac{z^2}{\pi}\iint_G\frac{q_1(\zeta)w(\zeta)}{(1-\zeta z)^2}d\sigma_\zeta-\frac{1}{\pi}\iint_G\frac{q_2(\zeta)\overline{w(\zeta)}}{(\zeta-z)^2}d\sigma_\zeta-\\&\frac{1}{\pi}\iint_G\frac{q_3(\zeta)w(\zeta)}{(\zeta-z)^2}d\sigma_\zeta+\frac{1}{\pi}\iint_G\frac{q_4(\zeta)\overline{w(\zeta)}}{(\zeta-z)^2}d\sigma_\zeta+\\&\frac{z^2}{\pi}\iint_G\frac{q_4(\zeta)\overline{w(\zeta)}}{(1-\zeta z)^2}d\sigma_\zeta+T(w)+TTF,\end{aligned}\tag{33}$$

其中 $T(w)$ 是弱奇性积分：

$$T(w)=\sum_{j=1}^{4}T_j(w),\tag{34}$$

$\varphi_i(z)\ (i=1,2)$ 为域 G 内的任意全纯函数。

由于在 $\Gamma:|z|=1$ 上，有

$$\begin{cases}-\dfrac{1}{\pi}\displaystyle\iint_G\frac{q_1(\zeta)w(\zeta)}{(\zeta-z)^2}d\sigma_\zeta+\frac{z^2}{\pi}\iint_G\frac{q_1(\zeta)w(\zeta)}{(1-\zeta z)^2}d\sigma_\zeta=0,\\-\dfrac{1}{\pi}\displaystyle\iint_G\frac{q_4(\zeta)\overline{w(\zeta)}}{(\zeta-z)^2}d\sigma_\zeta+\frac{z^2}{\pi}\iint_G\frac{q_4(\zeta)\overline{w(\zeta)}}{(1-\zeta z)^2}d\sigma_\zeta=0,\end{cases}\tag{35}$$

不妨可取任意全纯函数 $\varphi_2(z)$ 为

$$\varphi_2(z)=\frac{z^2}{\pi}\iint_G\frac{\overline{q_2(\zeta)}w(\zeta)}{(1-\zeta z)^2}d\sigma_\zeta+\frac{z^2}{\pi}\iint_G\frac{\overline{q_3(\zeta)}\ \overline{w(\zeta)}}{(1-\zeta z)^2}d\sigma_\zeta+\varphi_2^*(z),\tag{36}$$

则方程(33)又可化为

$$\begin{aligned}w(z)=&\ \varphi_1(z)+\overline{\varphi_2^*(z)}+\overline{\Pi^*}(q_1w)+\Pi^*(q_2w)+\\&\Pi^*(q_3w)+\overline{\Pi^*}(q_4w)+T(w)+TTF,\end{aligned}\tag{37}$$

其中 $\overline{\Pi^*}$、Π^* 为以下奇异积分算子

$$\begin{cases}\overline{\Pi^*}f=-\dfrac{1}{\pi}\displaystyle\iint_G\frac{f(\zeta)}{(\zeta-z)^2}d\sigma_\zeta+\frac{z^2}{\pi}\iint_G\frac{f(\zeta)}{(1-\zeta z)^2}d\sigma_\zeta,\\\Pi^*f=-\dfrac{1}{\pi}\displaystyle\iint_G\frac{\overline{f(\zeta)}}{(\zeta-z)^2}d\sigma_\zeta+\frac{z^2}{\pi}\iint_G\frac{\overline{f(\zeta)}}{(1-\zeta z)^2}d\sigma_\zeta.\end{cases}\tag{38}$$

将边界条件(12)代入(38)，注意到 Π^*f 对任意 $f\in L_p(G)$，$p>1$，在边界 $\Gamma:|z|=1$ 上都取值

为零,所以我们有

$$\begin{cases}\operatorname{Re}[\varphi_1(z)+\varphi_2^*(z)]=g_1+\operatorname{Re}[T(w)+TTF],\\ \operatorname{Re}[i(\varphi_1(z)+\varphi_2^*(z))]=g_2-\operatorname{Im}[T(w)+TTF]\cdot\end{cases}\tag{39}$$

应用 Schwartz 公式,可解得

$$\begin{cases}\varphi_1(z)+\varphi_2^*(z)=\dfrac{1}{2\pi i}\displaystyle\int_\Gamma\frac{[g_1+\operatorname{Re}[T(w)+TTF]](\zeta+z)}{(\zeta-z)\zeta}d\zeta,\\ \varphi_1(z)-\varphi_2^*(z)=-\dfrac{1}{2\pi}\displaystyle\int_\Gamma\frac{[g_2-\operatorname{Im}[T(w)+TTF]](\zeta+z)}{(\zeta-z)\zeta}d\zeta,\end{cases}\tag{40}$$

由此得

$$\begin{cases}\varphi_1(z)=\dfrac{1}{4\pi i}\displaystyle\int_\Gamma\frac{[g+[T(w)+TTF]](\zeta+z)}{(\zeta-z)\zeta}d\zeta,\\ \varphi_2^*(z)=\dfrac{1}{4\pi i}\displaystyle\int_\Gamma\frac{[g+[\overline{T(w)+TTF}]](\zeta+z)}{(\zeta-z)\zeta}d\zeta,\end{cases}\quad g=g_1+ig_2\cdot\tag{41}$$

这就是说,我们最后得了与非线性边值问题 P^* 等价的奇异积分方程(37),其中全纯函数 $\varphi_1(z)$ 和 $\varphi_2^*(z)$ 由Schwartz积分(41)给定•

3 与问题 P^* 等价的奇异积分方程的可解性

由(37)和(41),与问题 P^* 等价的奇异积分方程为

$$w(z)-\overline{\Pi_1^*}(q_1w)-\Pi_2^*(q_2w)-\Pi_2^*(q_3w)+\overline{\Pi_1^*}(q_4w)-\Gamma g=\Gamma(T(w)+TTF)+T(w)+TTF,\tag{42}$$

其中 Γf 表示下列积分

$$\Gamma f=\frac{1}{4\pi i}\int_\Gamma\frac{f(\zeta)(\zeta+z)}{(\zeta-z)\zeta}d\zeta+\frac{1}{4\pi i}\int_\Gamma\frac{f(\zeta)(1+\bar\zeta z)}{(1-\bar\zeta z)\zeta}d\zeta\cdot\tag{43}$$

参考文献[6](p.61, Riesz M.定理),当 $g(\zeta,w(\zeta))\in L_p(\Gamma)$ 时,$p>1$,则有 $\Gamma g\in L_p(G)$,且

$$\|\Gamma g\|_{p(G)}\leqslant M_p^*\left(\int_\Gamma|g(\zeta,w)|^p d\zeta\right)^{1/p};$$

若 $g(\zeta,w(\zeta))\in L_p(G)$,$p>1$,则又有

$$\|\Gamma g\|_{p(G)}\leqslant M_p^*\|g\|_{p(G)}\cdot$$

由假设A知,$g(z,w)$ 满足H_L连续条件,所以有

$$\|\Gamma g\|_{p(G)}\leqslant M_p^*H_1\|w\|_{p(G)}+M_p^*\|\overline{g(z,0)}\|_{p(G)}\leqslant M_p^*H_1\|w\|_{p(G)}+M_0\cdot\tag{44}$$

注意到算子

$$Sw=\overline{\Pi_1^*}(q_1w)+\Pi_2^*(q_2w)+\Pi_2^*(q_3w)+\overline{\Pi_1^*}(q_4w)\tag{45}$$

是空间 $L_p(G)$ 中的线性有界算子,它在 L_p 中的范数为 Λ_p^*;当 $p=2$ 时,$\Lambda_2^*\leqslant q_0<1$;当 $2<p<2+\varepsilon$ 时,$0<\Lambda_p^*<1$• 因此当 H_1 适当小时,方程(42)又可表示成

$$w(z)=(S-\Gamma)^{-1}[\Gamma(T(w)+TTF)+T(w)+TTF]\cdot\tag{46}$$

显然,由前面讨论知道 $T(w)$ 是弱奇性算子,它把 $w\in L_p(G)$ 映射到 $C_\alpha(G)$,同时也把 $w(z)\in C_\alpha(G)$ 映射到 $C_\alpha^1(G)$•

现在进一步考察 TTF(和 $\overline{TTF}$),不妨只考察其中一小项,即考察 $TTF_{(1)}$,$F_{(1)}=e^{-i\varphi-i\varphi^*}A_1(\zeta,w)\partial w/\partial\zeta$,因此

$$TF_{(1)} = -\frac{1}{\pi}\iint_G \frac{e^{ip-ip^*}A_1(\zeta,w)\dfrac{\partial w}{\partial\zeta}}{\zeta-z}d\sigma_\zeta =$$

$$-\frac{1}{\pi}\iint_G \frac{\partial}{\partial\zeta}\left(\frac{e^{-ip-ip^*}A_1(\zeta,w)w(\zeta)}{\zeta-z}\right)d\sigma_\zeta +$$

$$\frac{1}{\pi}\iint_G \frac{w(\zeta)e^{-ip(\zeta)-i\overline{p^*(\zeta)}}}{\zeta-z}\frac{\partial A_1(\zeta,w)}{\partial\zeta}d\sigma_\zeta +$$

$$\frac{1}{\pi}\iint_G \frac{w(\zeta)e^{-ip(\zeta)-i\overline{p^*(\zeta)}}(-i)\overline{p^{*\prime}(\zeta)}A_1(\zeta,w)}{\zeta-z}d\sigma_\zeta =$$

$$-\frac{1}{2\pi i}\int_\Gamma \frac{e^{-ip-ip^*}A_1(\zeta,w)w(\zeta)}{\zeta-z}d\zeta + e^{-ip(z)-ip^*(z)}A_1(\zeta,w)w(z) +$$

$$\frac{-i}{\pi}\iint_G \frac{w(\zeta)e^{-ip(\zeta)-i\overline{p^*(\zeta)}}\overline{p^{*\prime}(\zeta)}A_1(\zeta,w)}{\zeta-z}d\sigma_\zeta +$$

$$\frac{1}{\pi}\iint_G \frac{w(\zeta)e^{-ip(\zeta)-i\overline{p^*(\zeta)}}}{\zeta-z}\frac{\partial A_1(\zeta,w)}{\partial\zeta}d\sigma_\zeta, \tag{47}$$

于是我们有

$$TTF_{(1)} = \frac{1}{\pi}\iint_G \frac{d\sigma_{z_1}}{z_1-z}\frac{1}{2\pi i}\int_\Gamma \frac{e^{-ip-ip^*}A_1(\zeta,w)w(\zeta)}{\zeta-z_1}d\zeta -$$

$$\frac{1}{\pi}\iint_G \frac{e^{-ip(z_1)-ip^*(z_1)}A_1(z_1,w)w(z_1)}{z_1-z}d\sigma_{z_1} +$$

$$\frac{1}{\pi}\iint_G \frac{d\sigma_{z_1}}{z_1-z}\left[\frac{-i}{\pi}\iint_G \frac{w(\zeta)e^{-ip(\zeta)-i\overline{p^*(\zeta)}}\overline{p^{*\prime}(\zeta)}A_1(\zeta,w)}{\zeta-z_1}d\sigma_\zeta +\right.$$

$$\left.\frac{1}{\pi}\iint_G \frac{w(\zeta)e^{-ip(\zeta)-i\overline{p^*(\zeta)}}}{\zeta-z}\frac{\partial A_1(\zeta,w)}{\partial\zeta}d\sigma_\zeta\right] =$$

$$I_1 + I_2 + I_3. \tag{48}$$

易知在第一项的积分 I_1 中, 当 $w\in L_p$ 时, $A_1(\zeta,w)$ 也属于 $D_{1,p}(G)$, 从而围道积分属于 $L_p(G)$; 而当 $w\in C_\alpha$ 时, $A_1(\zeta,w)$ 也属于 C_α; 所以当 w 分别属于 L_p 和 C_α 时, I_1 就分别属于 $C_\beta(G)$ 和 $C^1_\alpha(G)$, 同样也有积分 I_2 分别属于 $C_\beta(G)$ 和 $C^1_\alpha(G)$, $\beta=(p-2)/p$.

对于 I_3, 记

$$h(\zeta) = w(\zeta)e^{-ip(\zeta)-i\overline{p^*(\zeta)}}\left[-i\overline{p^{*\prime}(\zeta)} + \frac{\partial A_1}{\partial\zeta}\right].$$

易知, 当 $w(\zeta)\in L_p(G)$ 时, $p^{*\prime}(\zeta)\in L_p(G)$, $A_1(\zeta,w)\in D_{1,p}(G)$, 所以 $h(\zeta)\in L_{p_0}(G)$, $1/p_0 = 2/p$; 而当 $w(\zeta)\in C_\alpha(G)$, $p^{*\prime}\in C_\alpha(G)$, $h(\zeta)\in L_p(G)$. 因此相应地分别有

$$-\frac{1}{\pi}\iint_G \frac{d\sigma_{z_1}}{z_1-z}\frac{1}{\pi}\iint_G \frac{h(\zeta)}{\zeta-z}d\sigma_\zeta \in C_{(\gamma-2)/\gamma}(G),$$

(γ 是大于 2 而小于 $2p_0/(2-p_0) = 2p/(4-p)$ 的任意正数)

和

$$-\frac{1}{\pi}\iint_G \frac{d\sigma_{z_1}}{z_1-z}\frac{1}{\pi}\iint_G \frac{h(\zeta)}{\zeta-z}d\sigma_\zeta \in C^1_{(p-2)/p}(G).$$

同理可证对 F 的其它 4 项也有同样的结果. 综上所述, 当 $w(\zeta)\in L_p(G)$ 和 $C_\alpha(G)$ 时, 分别有

$$\begin{cases} TTF\,(\text{同样 } TTF) \in C_\delta(G) \text{ 和 } C^1_\gamma(G), \\ \delta = \min\left(\dfrac{p-2}{p}, \dfrac{\gamma-2}{\gamma}\right), \quad \gamma = \min\left(\alpha, \dfrac{p-2}{p}\right). \end{cases} \tag{49}$$

由于 $2/\gamma > (4-p)/p$，所以 $1-2/\gamma < 2(p-2)/p$，所以可取

$$1-2/\gamma > (p-2)/p, \quad \delta = (p-2)/p.$$

其于以上结果, 我们回到方程(46), 对任意的 $w_i\,(i=1,2)$, 我们可得估计式

$$\begin{cases} \|\Gamma(T(w_1)) + T(w_1) - \Gamma(T(w_2)) - T(w_2)\|_p \leqslant A^* \|w_1 - w_2\|_p, \\ \|\Gamma(TTF(w_1)) + TTF(w_1) - \Gamma(TTF(w_2)) - TTF(w_2)\|_p \leqslant \\ \qquad B^* \|w_1 - w_2\|_p. \end{cases} \tag{50}$$

于是由(42) 或(46) 可得

$$\|w_1 - w_2\|_p \leqslant (q_0\Lambda_p^* + HM_p^* + A^* + B^*)\|w_1 - w_2\|_p = \\ \alpha\|w_1 - w_2\|_p, \tag{51}$$

其中 α 与 τ 无关, 且令它取得适当小, 使得

$$0 < \alpha = q_0\Lambda_p^* + HM_p^* + A^* + B^* < 1, \tag{52}$$

则奇异积分方程(42) 或(46) 有唯一解, 且解 $w(z) \in L_p(G)$, 它适合模不等式

$$\|w(z)\|_p \leqslant \frac{M_0^*}{1 - q_0\Lambda_p^* - HM_p^* - A^* - B^*}, \quad M_0^* = \text{const}. \tag{53}$$

由本节前面的讨论可知, $w(z)$ 又可表示成(46) 的形式, 这时 $\Gamma(T(w) + TTF) + T(w) + TTF$ 是把$w \in L_p(G)$ 映射到空间 $C_\delta(G)$ 中的全连续算子, 所以属于 $L_p(G)$ 中的解必属于 $C_\delta(G)$, $0 < \delta < 1$. 同时由前面讨论又不难推得 $w(z) \in C^1_\gamma(G) \cap D_{2,p}(G)$.

综上所述, 我们即得

定理 1 在假设条件 A 下, 当 $q_i(z) \in D^0_\infty(G)$ 且 α 适合不等式(52) 时, 非线性边值问题 P^* 是可解的且解唯一, 解 $w(z) \in C_\gamma(G) \cap C^1_\gamma(G) \cap D_{1,p}(G)$, 其中 $\gamma = \min(\alpha, (p-2)/p)$, $1/2 < \alpha < 1$, $p > 2$.

以下进一步考察满足椭圆型条件的一般的 $q_i(z) \in D_{1,p}(G)$, 这时存在 $q_i^{(n)}(z) \in D^0_\infty(G)$, 使得

$$\begin{cases} L_\infty(q_i^{(n)} - q_i, G) \to 0, \quad L_p\left(\dfrac{\partial q_i^{(n)}}{\partial \zeta} - \dfrac{\partial q_i}{\partial \zeta}, G\right) \to 0, \\ L_p\left(\dfrac{\partial q_i^{(n)}}{\partial \bar\zeta} - \dfrac{\partial q_i}{\partial \bar\zeta}, G\right) \to 0 \qquad (n \to \infty). \end{cases} \tag{54}$$

考察边值问题 P^*:

$$\frac{\partial^2 w}{\partial z \partial \bar z} - q_1(z)\frac{\partial^2 w}{\partial z^2} - q_2(z)\frac{\partial^2 w}{\partial \bar z^2} - q_3(z)\frac{\partial^2 \bar w}{\partial z^2} - q_4(z)\frac{\partial^2 \bar w}{\partial \bar z^2} = \\ F\left(z, w, \frac{\partial w}{\partial z}, \frac{\partial w}{\partial \bar z}\right) \qquad z \in G, \tag{55}$$

$$w(z) = g(z, w), \quad g(z, w) = g_1 - \mathrm{i}g_2. \tag{56}$$

由[6] 可知, (55) 的一般解可表示成

$$w(z) = \varphi_1(z) + \overline{\varphi_2^*(z)} + \frac{2}{\pi}\iint_G f(\zeta)\lg\left|\frac{\zeta - z}{1 - \bar\zeta z}\right| \mathrm{d}\sigma_\zeta, \tag{57}$$

其中$f(\zeta)$ 是属于 $L_p(G)$ 的 待定函数. 将(57) 代入(56) 即有

$$\begin{cases} \operatorname{Re}[\varphi_1(z)+\varphi_2^*(z)]=g_1, \\ \operatorname{Rg}[\mathrm{i}(\varphi_1(z)+\varphi_2^*(z))]=g_2\cdot \end{cases} \tag{58}$$

由此可得

$$\begin{cases} \varphi_1(z)=\dfrac{1}{4\pi\mathrm{i}}\displaystyle\int_\Gamma \frac{g(\zeta,w(\zeta))(\zeta+z)}{(\zeta-z)\zeta}\mathrm{d}\zeta, \\ \varphi_2^*(z)=\dfrac{1}{4\pi\mathrm{i}}\displaystyle\int_\Gamma \frac{\overline{g(\zeta,w(\zeta))(\zeta+z)}}{(\zeta-z)\zeta}\mathrm{d}\zeta, \end{cases} \tag{59}$$

从而可得边值问题的解式

$$w(z)=\frac{1}{4\pi\mathrm{i}}\int_\Gamma \frac{g(\zeta,w(\zeta))(\zeta+z)}{(\zeta-z)\zeta}\mathrm{d}\zeta+\frac{1}{4\pi\mathrm{i}}\int_\Gamma \frac{g(\zeta,w(\zeta))(1+\zeta z)}{(1-\zeta z)\zeta}\mathrm{d}\zeta+ \\ \frac{2}{\pi}\iint_G f(\zeta)\lg\left|\frac{\zeta-z}{1-\zeta z}\right|\mathrm{d}\sigma_\zeta\cdot \tag{60}$$

相应地, 对于 $q_i^{(n)}$, 也有解式

$$w_n(z)=\frac{1}{4\pi\mathrm{i}}\int_\Gamma \frac{g(\zeta,w_n(\zeta))(\zeta+z)}{(\zeta-z)\zeta}\mathrm{d}\zeta+\frac{1}{4\pi\mathrm{i}}\int_\Gamma \frac{g(\zeta,w_n(\zeta))(1+\zeta z)}{(1-\zeta z)\zeta}\mathrm{d}\zeta+ \\ \frac{2}{\pi}\iint_G f_n(\zeta)\lg\left|\frac{\zeta-z}{1-\zeta z}\right|\mathrm{d}\sigma_\zeta\cdot \tag{61}$$

将(61) 代入方程(55) , 可得 $f_n(z)$ 所适合的积分方程

$$f_n(z)-q_1^{(n)}(z)S_1f_n-q_2^{(n)}(z)\overline{S_1f_n}-q_3^{(n)}(z)S_2f_n- \\ q_4^{(n)}(z)\overline{S_2f_n}-q_1(z)\frac{1}{\pi\mathrm{i}}\int_\Gamma\frac{\zeta^2g(\zeta,w_n(\zeta))}{(1-\zeta z)^3}\mathrm{d}\zeta- \\ q_2(z)\frac{1}{\pi\mathrm{i}}\int_\Gamma\frac{\zeta^2g(\zeta,w_n(\zeta))}{(1-\zeta z)^3}\mathrm{d}\zeta-q_3(z)\frac{1}{\pi\mathrm{i}}\int_\Gamma\frac{\zeta g(\zeta,w_n(\zeta))}{(\zeta-z)^3}\mathrm{d}\zeta- \\ q_4(z)\frac{1}{\pi\mathrm{i}}\int_\Gamma\frac{\zeta g(\zeta,w_n(\zeta))}{(\zeta-z)^3}\mathrm{d}\zeta= \\ F\left(\zeta,w_n,\frac{\partial w_n}{\partial z},\frac{\partial w_n}{\partial z}\right)\cdot \tag{62}$$

按假设 A, 可得

$$\frac{1}{\pi\mathrm{i}}\int_\Gamma\frac{\zeta g}{(\zeta-z)^3}\mathrm{d}\zeta=-\frac{1}{2\pi\mathrm{i}}\int_\Gamma\frac{1}{(\zeta-z)^2}\frac{\partial}{\partial\zeta}\zeta g\,\mathrm{d}\zeta= \\ -\frac{1}{2\pi\mathrm{i}}\int_\Gamma\frac{1}{(\zeta-z)^2}\left[g+\zeta\frac{\partial}{\partial\zeta}g\right]\mathrm{d}\zeta\in L_{p_1}(G),\quad p_1=\frac{1}{1-\alpha},\ \frac{1}{2}<\alpha<1,$$

所以

$$\left\|\frac{1}{\pi\mathrm{i}}\int_\Gamma\frac{\zeta g}{(\zeta-z)^3}\mathrm{d}\zeta\right\|_{p_1(G)}\leqslant M_{p_1}(L_{p_1}(g(\zeta,w_n(\zeta))))L_{p_1}\left(\frac{\partial}{\partial\zeta}g(\zeta,w_n(\zeta))\right)\leqslant \\ M_{p_1}H_1\|w_n(\zeta)\|_{p_1}+M'_{p_1}\left[L_p(g(\zeta,0))+L_p\left(\frac{\partial}{\partial\zeta}(g)(\zeta,0)\right)\right]\leqslant \\ M_{p_1}H_1\|w_n(\zeta)\|_{p_1}+M_0\qquad M_0=\text{const}\cdot \tag{63}$$

类似地, 也有

$$\left\|\frac{1}{\pi\mathrm{i}}\int_\Gamma\frac{\zeta^2 g}{(1-\zeta z)^3}\mathrm{d}\zeta\right\|_{p_1(G)}\leqslant M_{p_1}H_1\|w_n(\zeta)\|_{p_1}+M_0\cdot \tag{64}$$

于是再回到(62) , 又得估计式

$$\|f_n(z)\|_{p_1}\leqslant q_0\Lambda_{p_1}\|f_n(z)\|_{p_1}+q_0M_{p_1}H_1\|w_n(\zeta)\|_{p_1}+$$

$$\left\|F\left(z, w_n, \frac{\partial w_n}{\partial z}, \frac{\partial w_n}{\partial z}\right)\right\|_{p_1} + M_0. \tag{65}$$

另外,由(61),又有

$$\| w_n(\zeta) \|_{p_1} \leqslant M_{p_1} H_1 \| w_n(\zeta) \|_{p_1} + L_0 \| f_n(z) \|_{p_1}. \tag{66}$$

按假设A, Lipschitz 系数 H_1 可取适当小,所以有

$$\| w_n(\zeta) \|_{p_1} \leqslant \frac{L_0 \| f_n(z) \|_{p_1}}{1 - M_{p_1} H_1}. \tag{67}$$

再由 F 的表示式和(57)或(60)可得

$$\left\|F\left(z, w_n, \frac{\partial w_n}{\partial z}, \frac{\partial w_n}{\partial z}\right)\right\|_{p_1} \leqslant$$

$$M_{p_1} H_1 \| w_n(\zeta) \|_{p_1} + \sum_{i=1}^{4} A_i M \| f_n \|_{p_1} + A_0 M + M_0, \tag{68}$$

其中

$$A_i = \| A_i(z, w) \|_p.$$

不妨记 $A' = \sum_{i=1}^{4} A_i$, $M_0' = A_0 M + M_0$, 并将(67)和(68)代入(65)得到

$$\| f_n(z) \|_{p_1} \leqslant q_0 \Lambda_{p_1} \| f_n(z) \|_{p_1} + \frac{q_0 M_{p_1} H_1 L_0}{1 - M_{p_1} H_1} \| f_n \|_{p_1} +$$

$$L_0 \| f_n \|_{p_1} \frac{M_{p_1} H_1}{1 - M_{p_1} H_1} + A' \| f_n \|_{p_1} + M_0' =$$

$$\left(q_0 \Lambda_{p_1} + \frac{(1 + q_0) M_{p_1} H_1 L_0}{1 - M_{p_1} H_1} + A' \right) \| f_n \|_{p_1} + M_0' \leqslant$$

$$\alpha' \| f_n \|_{p_1} + M_0' \qquad \alpha' = q_0 \Lambda_p^* + \frac{(1 + q_0) M_{p_1} H_1 L_0}{1 - M_{p_1} H_1} + A' < 1. \tag{69}$$

回到(67)又有

$$\| w_n(\zeta) \|_{p_1} \leqslant \frac{L_0}{1 - M_{p_1} H_1} \| f_n(z) \|_{p_1} \leqslant \frac{L_0}{1 - M_{p_1} H_1} \frac{1}{1 - \alpha'} M_0'. \tag{70}$$

于是由(69)、(70)知 $f_n(z)$、$w_n(z)$ 按 $L_{p_1}(G)$ 范数均匀有界,因此存在子序列 $f_{n_k}(z)$、$w_{n_k}(z)$ 分别弱收敛也概收敛于 $f(z)$、$w(z) \in L_{p_1}(G)$;因为 $w_n(z) \in C_\delta(G)$,所以 $w_{n_k}(z)$ 也按 C 的范数一致收敛于 $w(z)$,且 $w(z) \in C_\delta(G)$.

由于积分

$$\frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta, w_{n_k}(\zeta))(\zeta + z)}{(\zeta - z)\zeta} d\zeta, \quad \frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta, w_{n_k}(\zeta))(1 + \zeta z)}{(1 - \zeta z)\zeta} d\zeta$$

也在 G 内分别内闭匀敛于

$$\frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta, w(\zeta))(\zeta + z)}{(\zeta - z)\zeta} d\zeta, \quad \frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta, w(\zeta))(1 + \zeta z)}{(1 - \zeta z)\zeta} d\zeta,$$

所以 $f_{n_k}(z)$ 也内闭匀敛,从而对 $\forall \varepsilon > 0$ 和 $\eta > 0$,存在 $G_\eta \subset G$,当 $\mathrm{mes}(G_\eta) < \eta$ 时,在 $G - G_\eta$ 上,$f_{n_k}(z)$ 按 C 的范数一致收敛于 $f(z)$,即有

$$\lim_{n_k \to \infty} \frac{1}{\pi}\iint_{G - G_\eta} f_{n_k}(\zeta) \lg \left| \frac{\zeta - z}{1 - \zeta z} \right| d\sigma_\zeta = \frac{1}{\pi}\iint_{G - G_\eta} f(\zeta) \lg \left| \frac{\zeta - z}{1 - \zeta z} \right| d\sigma_\zeta, \tag{71}$$

且

$$\left(\iint_{G_\eta}\left(\lg\left|\frac{\zeta-z}{1-\bar{\zeta}z}\right|\right)^q \mathrm{d}\sigma_\zeta\right)^{1/q} < \varepsilon. \tag{72}$$

进而也有

$$\begin{aligned}&\left|\iint_{G_\eta} f_{n_k}(\zeta)\lg\left|\frac{\zeta-z}{1-\bar{\zeta}z}\right|\mathrm{d}\sigma_\zeta\right| \leqslant \\ &\quad\left(\iint_{G_\eta}(f_{n_k}(\zeta))^p \mathrm{d}\sigma_\zeta\right)^{1/p}\left(\iint_{G_\eta}\left(\lg\left|\frac{\zeta-z}{1-\bar{\zeta}z}\right|\right)^q \mathrm{d}\sigma_\zeta\right)^{1/q} \leqslant \\ &\quad\left(\iint_{G_\eta}(f_{n_k}(\zeta))^p \mathrm{d}\sigma_\zeta\right)^{1/p}\cdot\varepsilon \leqslant \frac{M_0}{1-\alpha'}\varepsilon.\end{aligned} \tag{73}$$

再令 $\eta\to 0$,(同时 $\varepsilon\to 0$),即得

$$\lim_{n_k\to\infty}\frac{1}{\pi}\iint_G f_{n_k}(\zeta)\lg\left|\frac{\zeta-z}{1-\bar{\zeta}z}\right|\mathrm{d}\sigma_\zeta = \frac{1}{\pi}\iint_G f(\zeta)\lg\left|\frac{\zeta-z}{1-\bar{\zeta}z}\right|\mathrm{d}\sigma_\zeta. \tag{74}$$

这样就最后得到

$$\begin{aligned}w(z) = {}&\frac{1}{4\pi i}\int_\Gamma \frac{g(\zeta, w(\zeta))(\zeta+z)}{(\zeta-z)\zeta}\mathrm{d}\zeta + \frac{1}{4\pi i}\int_\Gamma \frac{\overline{g(\zeta, w(\zeta))}(1+\bar{\zeta}z)}{(1-\bar{\zeta}z)\bar{\zeta}}\mathrm{d}\bar{\zeta} + \\ &\frac{2}{\pi}\iint_G f(\zeta)\lg\left|\frac{\zeta-z}{1-\bar{\zeta}z}\right|\mathrm{d}\sigma_\zeta,\end{aligned} \tag{75}$$

它给出了边值问题 P^* 的解,$w(z)\in C_\gamma(G)$.

即我们又证明了

定理 2 在假设 A 和椭圆型条件 $\sum_{i=1}^4 |q_i(z)| \leqslant q_0 < 1$ 下,当成立不等式

$$\alpha' = q_0\Lambda_p^* + \frac{(1+q_0)M_{p_1}H_1L_0}{1-M_{p_1}H_1} + A' < 1$$

时,非线性边值问题 P^* 一定可解,且解 $w(z)\in C_\delta(G)\cap C^1_{(p_1-2)/p_1}(G)\cap D^2_{p_1}(G)$.

基于以上结果,采用文献[6, 7]中的连续性方法类似可证问题 P 的可解性.

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A Class of Nonlinear Boundary Value Problems for the Sceond_Order E_2 Class Elliptic Systems in General Form

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Abstract: A class of nonlinear boundary value problems(BVP) for the second_order E_2 class elliptic systems in general form is discussed. By introducing a kind of transformation, this kind of BVP is reduced to a class of generalized nonlinear Riemann_Hilbert BVP. And then some singular integral operators are introduced to establish the equivalent nonlinear singular integral equations. The solvability is proved under some suitable hypotheses by means of the properties of singular integral operators and the function theoretic methods.

Key words: elliptic systems; boundary value problems; singular integral equations; singular integral operators; existence