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一族 Liouville 可积系及其约束流的 Lax 表示、Darboux 变换*

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(我刊编委张鸿庆来稿)

摘要: 利用屠规彰格式求出了一族 Liouville 可积系, 通过高阶位势特征函数约束将可积系分解成 x 部分和 t_n 部分可积 Hamilton 系统, 求出了该系统的 Lax 表示及三类 Darboux 变换.

关 键 词: 可积系; 位势特征函数约束; Lax 表示; Darboux 变换
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引 言

取 Loop 代数 $\tilde{A}_1$ 的基为:

$$\begin{cases} \boldsymbol{h}(n) = \begin{pmatrix} \lambda^n & 0 \\ 0 & -\lambda^n \end{pmatrix}, \quad \boldsymbol{e}(n) = \begin{pmatrix} 0 & \lambda^n \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{f}(n) = \begin{pmatrix} 0 & 0 \\ \lambda^n & 0 \end{pmatrix}, \\ [\boldsymbol{h}(m), \boldsymbol{e}(n)] = 2\boldsymbol{e}(m+n), [\boldsymbol{h}(m), \boldsymbol{f}(n)] = \\ \qquad -2\boldsymbol{f}(m+n), [\boldsymbol{e}(m), \boldsymbol{f}(n)] = \boldsymbol{h}(m+n). \end{cases} \tag{1}$$

利用(1)建立线性等谱问题已经得到若干方程族, 如 AKNS 族, KN 族与 WKI 族[1,2]. 对(1)作线性组合可产生 Loop 代数 $\tilde{A}_1$ 的新基, 其中有[2]

$$\begin{cases} \boldsymbol{h}(n) = \frac{1}{2}\begin{pmatrix} \lambda^n & 0 \\ 0 & -\lambda^n \end{pmatrix}, \quad \boldsymbol{e}_\pm(n) = \frac{1}{2}\begin{pmatrix} 0 & \lambda^{n-1} \\ \pm\lambda^n & 0 \end{pmatrix}, \\ [\boldsymbol{h}(m), \boldsymbol{e}_\pm(n)] = \boldsymbol{e}_\mp(m+n), [\boldsymbol{e}_-(m), \boldsymbol{e}_+(n)] = \boldsymbol{h}(m+n-1), \\ \deg \boldsymbol{h}(n) = 2n, \quad \deg \boldsymbol{e}_\pm(n) = 2n-1. \end{cases} \tag{2}$$

与

$$\begin{cases} \boldsymbol{e}_1(n) = \begin{pmatrix} 0 & \lambda^n \\ \lambda^n & 0 \end{pmatrix}, \quad \boldsymbol{e}_2(n) = \begin{pmatrix} 0 & \lambda^n \\ -\lambda^n & 0 \end{pmatrix}, \quad \boldsymbol{e}_3(n) = \begin{pmatrix} \lambda^n & 0 \\ 0 & -\lambda^n \end{pmatrix}, \\ [\boldsymbol{e}_1(m), \boldsymbol{e}_2(n)] = -2\boldsymbol{e}_3(m+n), \quad [\boldsymbol{e}_1(m), \boldsymbol{e}_3(n)] = -2\boldsymbol{e}_2(m+n), \\ [\boldsymbol{e}_2(m), \boldsymbol{e}_3(n)] = -2\boldsymbol{e}_1(m+n), \\ \deg \boldsymbol{e}_i(n) = n \qquad (i = 1, 2, 3). \end{cases} \tag{3}$$

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下面我们利用基(3)构造一个等谱问题,利用屠规彰格式[1]导出 Liouville 可积的广义 Unnamed 耦合反应扩散方程族;再利用文献[3]~[5]的方法,将该方程族中的每一个方程分解成可换的 x 和 t_n 两部分有限可积可积 Hamilton 系统,求出该系统的 Lax 表示• 众所周知, Darboux 变换是求孤立子解的强有力工具,在获得 Lax 表示之后,我们又构造了所得方程族的三类 Darboux 变换,并给出了利用 Darboux 变换求方程行波解的一般机制•

1 可积 Hamilton 方程族

考虑等谱问题

$$\begin{cases} \phi_x = U\phi, \quad \lambda_t = 0, \quad \phi = (\phi_1, \phi_2)^{\mathrm{T}}, \\ \boldsymbol{U} = \boldsymbol{e}_1(1) + q\boldsymbol{e}_2(0) + r\boldsymbol{e}_3(0) \bullet \end{cases} \tag{4}$$

设 $\boldsymbol{V} = \sum_{m=0}^{\infty}(a_m\boldsymbol{e}_1(-m) + b_m\boldsymbol{e}_2(-m) + c_m\boldsymbol{e}_3(-m))\bullet$

解辅助方程 $V_x = [\boldsymbol{U}, \boldsymbol{V}],$ (5)

得递推关系

$$a_{mx} = -2qc_m + 2rb_m, \quad b_{mx} = -2c_{m+1} + 2ra_m, \quad c_{mx} = -2b_{m+1} + 2qa_m, \tag{6}$$

$$b_0 = c_0 = 0, \quad a_0 = \beta, \quad c_1 = \beta r, \quad b_1 = \beta q, \quad a_1 = 0,$$

记 $\boldsymbol{V}_+^{(n)} = (\lambda^n\boldsymbol{V})_+ = \sum_{m=0}^{n}(a_m\boldsymbol{e}_1(n-m) + b_m\boldsymbol{e}_2(n-m) + c_m\boldsymbol{e}_3(n-m)),$

$\boldsymbol{V}_-^{(n)} = \lambda^n\boldsymbol{V} - \boldsymbol{V}_+^{(n)}$

则(5)可写为

$$-\boldsymbol{V}_{+x}^{(n)} + [\boldsymbol{U}, \boldsymbol{V}_+^{(n)}] = \boldsymbol{V}_{-x}^{(n)} - [\boldsymbol{U}, \boldsymbol{V}_-^{(n)}] \tag{7}$$

(7)左端所含基元阶数 ⩾0,右端阶数 ⩽0,写出(7)右端阶数为 0 的基元得:

$$-\boldsymbol{V}_{+x}^{(n)} + [\boldsymbol{U}, \boldsymbol{V}_+^{(n)}] = 2c_{n+1}\boldsymbol{e}_2(0) + 2b_{n+1}\boldsymbol{e}_3(0),$$

取 $\boldsymbol{V}^{(n)} = \boldsymbol{V}_+^{(n)}$, $\Delta_n = 0$,则由零曲率方程 $\boldsymbol{U}_t - \boldsymbol{V}_x^{(n)} + [\boldsymbol{U}, \boldsymbol{V}^{(n)}] = \boldsymbol{0}$ (8)

确定可积系 $\boldsymbol{u}_t = \begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} -b_{n+1} \\ c_{n+1} \end{pmatrix} = J\begin{pmatrix} -b_{n+1} \\ c_{n+1} \end{pmatrix}$ (9)

由(6)易见

$$\begin{pmatrix} -b_{m+1} \\ c_{m+1} \end{pmatrix} = \begin{pmatrix} 2q\partial^{-1}r & \frac{1}{2}\partial + 2q\partial^{-1}q \\ \frac{1}{2}\partial - 2r\partial^{-1}r & -2r\partial^{-1}q \end{pmatrix}\begin{pmatrix} -b_m \\ c_m \end{pmatrix} = L\begin{pmatrix} -b_m \\ c_m \end{pmatrix} \tag{10}$$

由(10)知,(9)可写为 $\boldsymbol{u}_t = \begin{pmatrix} q \\ r \end{pmatrix}_t = JL^n\begin{pmatrix} -\beta q \\ \beta r \end{pmatrix}$ (11)

(11)与 AKNS 族非常类似,但递推算子不同[1]•

当 $n = 1$ 时,(11)约化为平凡方程: $q_t = -\beta q_x$, $r_t = -\beta r_x$,

当 $n = 2$, $\beta = -2$ 时,(11)约化为广义 Unnamed 耦合反应扩散方程

$$\begin{cases} q_t = -r_{xx} + 2r^3 - 2q^2r, \\ r_t = -q_{xx} + 2r^2q - 2q^3 \bullet \end{cases}$$

下面考虑方程族(11)的 Hamilton 系统•

记 $\boldsymbol{V} = a\boldsymbol{e}_1(0) + b\boldsymbol{e}_2(0) + c\boldsymbol{e}_3(0) = \begin{pmatrix} c & a+b \\ a-b & -c \end{pmatrix}$,易见

$$\langle V, \frac{\partial U}{\partial \lambda}\rangle = 2a,\ \langle V, \frac{\partial U}{\partial q}\rangle = -2b,\ \langle V, \frac{\partial U}{\partial r}\rangle = 2c,$$

将其代入迹恒等式

$$\frac{\delta}{\delta u}\langle V, \frac{\partial U}{\partial \lambda}\rangle = \lambda^{-r}\frac{\partial}{\partial \lambda}\left(\lambda^{r}\langle V, \frac{\partial U}{\partial u}\rangle\right) \text{ 得:}$$

$$\frac{\delta}{\delta u}(2a) = \lambda^{-r}\frac{\partial}{\partial \lambda}\begin{pmatrix} -2b\lambda^{r} \\ 2c\lambda \end{pmatrix},\text{ 比较 } \lambda^{-n-1} \text{ 的系数得:} \begin{pmatrix} \delta/\delta q \\ \delta/\delta r \end{pmatrix}(a_{n+1}) = (-n+r)\begin{pmatrix} -b_{n+1} \\ c_n \end{pmatrix},$$

取 $n=1$ 得: $r=0$, 于是得到(11)的Hamilton形式

$$u_t = \begin{pmatrix} q \\ r \end{pmatrix}_t = JL\begin{pmatrix} -b_n \\ c_n \end{pmatrix} = J\frac{\delta H_{n+1}}{\delta u}, \tag{12}$$

其中 $H_u = -\frac{a_{n+1}}{n}$, 易验证: $JL = L^{*}J = \begin{pmatrix} \partial - 4r\partial^{-1}r & -4r\partial^{-1}q \\ -4q\partial^{-1}r & -\partial - 4q\partial^{-1}q \end{pmatrix}$, 因此方程族(11)在Liounivlle意义下可积•

2 方程族(11)的约束流的Lax表示

对于 N 个不同特征值 λ_j, 下列约束系统

$$\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \end{pmatrix}_x = U(u, \lambda_j)\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \end{pmatrix} = \begin{pmatrix} r & q+\lambda_j \\ \lambda_j - q & -r \end{pmatrix}\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \end{pmatrix}, \tag{13}$$

$$\frac{\delta H_{k+1}}{\delta u} - c_{k+1}\sum_{j=1}^{N}\frac{\delta \lambda_j}{\delta u} = 0$$

是流(11)的不变子空间[3,4], 并称(13)为(11)的 x 约束流•

考虑(4)的共轭谱问题

$$\psi_x = U^{*}\psi = -U^{T}\psi = \begin{pmatrix} -r & q-\lambda_j \\ -q-\lambda_j & r \end{pmatrix}\psi,\quad \psi = (\psi_1, \psi_2)^{T} \tag{14}$$

当 $\lim_{|x|\to\infty}\psi_i = \lim_{|x|\to\infty}\varphi_i = 0$ 时, 由(4)和(14)直接计算知:

$$\frac{\delta \lambda_j}{\delta u} = \frac{1}{E}\begin{pmatrix} \varphi_{1j}^2 + \varphi_{2j}^2 \\ 2\varphi_{1j}^2\varphi_{2j}^2 \end{pmatrix},\quad E = \int_{-\infty}^{+\infty}(\varphi_1^2 - \varphi_2^2)\,dx,\quad L\frac{\delta \lambda_j}{\delta u} = \lambda_j\frac{\delta \lambda_j}{\delta u}\quad (j = 1, 2, \cdots, N) \tag{15}$$

设 $\varphi_i = (\varphi_{i1}, \cdots, \varphi_{in})^{T}$, $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$, $\langle\cdot, \cdot\rangle$ 表示 $\mathbf{R}^N$ 的内积($i = 1, 2,$), 则系统(13)可浓缩为

$$\begin{cases} \Phi_{1x} = r\Phi_1 + (q+\Lambda)\Phi_2,\quad \Phi_{2x} = (\Lambda - q)\Phi_1 - r\Phi_2, \\ \frac{\delta H_{k+1}}{\delta u} = \begin{pmatrix} b_{k+1} \\ c_{k+1} \end{pmatrix} = \begin{pmatrix} \langle\Phi_2, \Phi_2\rangle + \langle\Phi_1, \Phi_1\rangle \\ 2\langle\Phi_1, \Phi_2\rangle \end{pmatrix} \end{cases} \tag{16}$$

这里取 $c_{k+1} = E$•

易知: $\begin{pmatrix} -b_{m+1} \\ c_{m+1} \end{pmatrix} = L\begin{pmatrix} -b_m \\ c_m \end{pmatrix} = L^{m-k}\begin{pmatrix} -b_{k+1} \\ c_{k+1} \end{pmatrix} = \begin{pmatrix} \langle\Lambda^{m-k}\Phi_2, \Phi_2\rangle + \langle\Lambda^{m-k}\Phi_1, \Phi_1\rangle \\ 2\langle\Lambda^{m-k}\Phi_1, \Phi_2\rangle \end{pmatrix}$

所以

$$a_{m+1} = \langle\Lambda^{m-k}\Phi_2, \Phi_2\rangle - \langle\Lambda^{m-k}\Phi_1, \Phi_1\rangle,$$

$$b_{m+1} = -\langle\Lambda^{m-k}\Phi_1, \Phi_1\rangle - \langle\Lambda^{m-k}\Phi_2, \Phi_2\rangle,$$

$$c_{m+1} = 2\langle \Lambda^{m-k}\Phi_2, \Phi_2\rangle$$

记 $N^{(k)} \equiv \lambda^k V + V^{(k)} + N_0$, 则

$$N_0 = \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j}\begin{pmatrix} \phi_{2j}^2 - \phi_{1j}^2 & \phi_{1j}^2 - \phi_{2j}^2 \\ 2\phi_{1j}\phi_{2j} & \phi_{1j}^2 - \phi_{2j}^2 \end{pmatrix}.$$

由文献[4, 5] 知, 在约束流(16) 条件下, $N^{(k)}$ 满足 $N_x^{(k)} = [U, N^{(k)}]$; 反过来, 仍由[4, 5] 知, $N^{(k)}$ 的构造保证了上面公式给出了约束系统(16) 的Lax 表示. 于是有

定理 1 在伴随表示 $V_x = [U, V]$ 中, 将 V 换成$N^{(k)}$, 即可得到(16) 的Lax 表示: $N_x^{(k)} = [U, N^{(k)}]$, 其 Lax 对为 $\psi_x = U(u, \lambda)\psi$, $N^{(k)}\psi = \mu\psi$.

对于方程族(11) 的 t_n 约束流

$$\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \end{pmatrix}_{t_n} = V^{(n)}(u, \lambda_j)\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \end{pmatrix}, \quad \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = J\begin{pmatrix} -b_{n+1} \\ c_{n+1} \end{pmatrix} \tag{17}$$

有类似结果.

定理 2 在 $V_t = [V^{(n)}, V]$ 中, 将 V 换成$N^{(k)}$, 即可得到 t_n 约束系统

(17) 的Lax 表示: $N_{t_n}^{(k)} = [V^{(n)}, N^{(k)}]$, 其Lax 对为 $\psi_{t_n} = V^{(n)}(u, \lambda)\psi$, $N^{(k)}\psi = \mu\psi$.

定理 1 和 2 的证明仿照文献[3] 直接计算即可.

根据定理 1, 我们易得到关于含附加外项的约束流的Lax 表示[4], 即

定理 3 下列系统

$$\begin{cases} \phi_{1x} = r\phi_1 + (q + \Lambda)\phi_2, \quad \phi_{2x} = (\Lambda - q)\phi_1 - r\phi_2, & (18a) \\ \dfrac{\delta H_{k+1}}{\delta u} = \begin{pmatrix} -b_{k+1} \\ c_{k+1} \end{pmatrix} = \begin{pmatrix} \langle\Phi_2, \Phi_2\rangle + \langle\Phi_1, \Phi_1\rangle \\ 2\langle\Phi_1, \Phi_2\rangle \end{pmatrix} & (18b) \end{cases}$$

的Lax 表示为: $U_{t_k}^{(k)} - N_x^{(k)} + [U, N^{(k)}] = 0$.

其Lax 对是 $\varphi_x = U\varphi$, $\varphi_{t_k} = N^{(k)}\varphi$, 其中 $\psi_i = (\psi_{i1}, \cdots, \psi_{iN})^{\mathrm{T}}$, $i = 1, 2$.

约束系统(16) ~ (18) 均可化为有限维可积 Hamilton 系统. 下面以约束系统(16) 和(17) 为例说明上面的结果.

1) 当 $k = 0$, $\beta = 1$ 时, (16) 约化为

$$\begin{cases} b_1 = q = -\langle\Phi_1, \Phi_2\rangle - \langle\Phi_2, \Phi_2\rangle, \\ c_1 = r = 2 - \langle\Phi_1, \Phi_2\rangle. \end{cases} \tag{19}$$

此时的 $U = \begin{pmatrix} 2\langle\Phi_1, \Phi_2\rangle & \lambda - \langle\Phi_1, \Phi_1\rangle - \langle\Phi_2, \Phi_2\rangle \\ \lambda + \langle\Phi_1, \Phi_1\rangle + \langle\Phi_2, \Phi_2\rangle & -2\langle\Phi_1, \Phi_2\rangle \end{pmatrix}$.

于是(16) 约化为有限维可积Hamilton 系统

$$\begin{cases} \Phi_{1x} = \dfrac{\partial H_0}{\partial \Phi_2}, \quad \Phi_{2x} = -\dfrac{\partial H_0}{\partial \Phi_1}, \\ H_0 = \langle\Phi_1, \Phi_2\rangle^2 + \dfrac{1}{2}\langle\Lambda\Phi_2, \Phi_2\rangle - \dfrac{1}{2}\langle\Lambda\Phi_1, \Phi_1\rangle - \dfrac{1}{4}\langle\Phi_1, \Phi_1\rangle^2, \\ -\dfrac{1}{4}\langle\Phi_2, \Phi_2\rangle^2 - \dfrac{1}{2}\langle\Phi_1, \Phi_1\rangle\langle\Phi_2, \Phi_2\rangle, \end{cases} \tag{20}$$

其Lax 表示为: $N_x^{(0)} = [U, N^{(0)}]$, $N^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + N_0$,

对于 $n=1$, 在(19)和(20)成立条件下, 有

$$V^{(1)}=\begin{pmatrix}\lambda & -\langle\phi_1,\phi_1\rangle-\langle\phi_2,\phi_2\rangle\\ 2\langle\phi_1,\phi_2\rangle & -\lambda\end{pmatrix},$$

此时时间部分约束系统(20)可化为有限维可积Hamilton系统

$$\begin{cases}\phi_{1x}=\dfrac{\partial H_1}{\partial\phi_2},\quad \phi_{2x}=-\dfrac{\partial H_1}{\partial\phi_1},\\ H_1=\langle\Lambda\phi_1,\phi_2\rangle-\langle\phi_1,\phi_2\rangle^2-\dfrac{1}{4}\langle\phi_2,\phi_2\rangle^2-\dfrac{1}{2}\langle\phi_1,\phi_1\rangle\langle\phi_2,\phi_2\rangle.\end{cases}\tag{21}$$

(21)的Lax表示为: $N^{(0)}_{t_1}=[V^{(1)},N^{(0)}]$,

2) 当 $k=2$, $\beta=2$ 时, 由(16)得

$$\begin{cases}\dfrac{1}{4}q_{xx}=\dfrac{1}{2}(qr^2-q^3)-\dfrac{1}{2}(\langle\phi_1,\phi_1\rangle+\langle\phi_2,\phi_2\rangle),\\ \dfrac{1}{4}r_{xx}=\dfrac{1}{2}(-rq^2+r^3)+\langle\phi_1,\phi_2\rangle.\end{cases}\tag{22}$$

引入Jacobi_Ostrogradsky坐标

$$\begin{cases}Q=(\varphi_{11},\cdots,\varphi_{1N},q_1,q_2)^{\mathrm T},\quad P=(\varphi_{21},\cdots,\varphi_{2N},p_1,p_2)^{\mathrm T},\\ q_1=q,\quad q_2=r,\quad p_1=\dfrac{1}{4}q_x,\quad p_2=-\dfrac{1}{4}r_x.\end{cases}\tag{23}$$

则(16)可化为一个有限维可积Hamilton系统

$$\begin{cases}Q_x=\dfrac{\partial H_0}{\partial P},\quad P_x=\dfrac{\partial H_0}{\partial Q},\\ H_0=q_2\langle\phi_1,\phi_2\rangle+\dfrac{1}{2}q_1(\langle\phi_2,\phi_2\rangle+\langle\phi_1,\phi_1\rangle)-\dfrac{1}{2}(\langle\Lambda\phi_2,\phi_2\rangle-\\ \qquad\langle\Lambda\phi_1,\phi_1\rangle)-2p_1^2-2p_2^2+\dfrac{1}{8}(q_1^4+q_2^4)-\dfrac{1}{4}q_1^3q_2^2.\end{cases}\tag{24}$$

(24)的Lax表示为:

$$N^{(2)}_x=[U,N^{(2)}],\quad N^{(2)}=V^{(2)}+N_0,\quad V^{(2)}=\begin{pmatrix}2\lambda^2-q_2^2+q_1^2 & 2q_1\lambda+4p_2\\ 2q_2\lambda-4p_1 & -2\lambda^2+q_2^2-q_1^2\lambda\end{pmatrix},$$

$$U=\begin{pmatrix}q_2 & q_1+\lambda\\ \lambda-q_1 & -q_2\end{pmatrix}.$$

当 $n=2$, 在(23)和(24)成立条件下, (17)可化为一个有限维可积Hamilton系统,

$$\begin{cases}Q_{t_2}=\dfrac{\partial H_2}{\partial P},\quad P_{t_2}=-\dfrac{\partial H_2}{\partial Q},\\ H_2=2\langle\Lambda^2\phi_1,\phi_2\rangle+(q_1^2-q_2^2)\langle\phi_1,\phi_1\rangle+q_1\langle\Lambda\phi_2,\phi_2\rangle+\\ \qquad 2p_2\langle\phi_2,\phi_2\rangle-q_2\langle\Lambda\phi_1,\phi_1\rangle+2p_1\langle\phi_1,\phi_1\rangle,\end{cases}\tag{25}$$

其中Lax表示为: $N^{(2)}_{t_2}=[V^{(2)},N^{(2)}]$.

3 可积Hamilton系统(24)的Darboux变换

易知, (24)的Lax对为

$$\psi_x=U(u,\lambda)\psi\tag{26a}$$

$$N^{(2)}\psi=\mu\psi,\quad \psi=(\psi_1,\psi_2)^{\mathrm T}\tag{26b}$$

由文献[4, 5]知, 通过规范变换

$$\phi = T\phi, \quad T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}. \tag{27}$$

(26)可化为

$$\phi_x = U\phi \tag{28}$$
$$N^{(2)}\phi = \mu\phi,$$

$U, N^{(2)}$ 满足方程

$$T_x = UT - TU, \tag{29}$$
$$N^{(2)}T = TN^{(2)}. \tag{30}$$

其中 $U, N^{(2)}$ 的表达式与 $U, N^{(2)}$ 类同, 只需将 $U, N^{(2)}$ 中的 q, r 分别用q, r 代替.

设 $\phi(x, t, \lambda) = (\phi_{ij})$ 和 $\phi(x, t, \lambda) = (\phi_{ij})$ 分别为(26)和(27)的解矩阵, 因为$\mathrm{tr}\,U = \mathrm{tr}\,U = \mathrm{tr}N^{(2)} = \mathrm{tr}N^{(2)} = 0$, 所以$\det\phi_{ij}$ 与 $\det\phi_{ij}$ 是常数且与x, t 无关. 设 $\lambda = \eta_1(\eta \neq \lambda_j, j = 1, 2, \cdots, N)$ 为行列式$\det T$ 的一个零点, 则由(27)知$\det\varphi(x, \eta_1) = 0$, 故存在常数 $\mu_1, v_1, |v_1| + |\mu_1| \neq 0$, 满足 $\mu_1\overline{\varphi_{i1}}(x, \eta_1) + v_1\overline{\varphi_{i2}}(x, \eta_1) = 0, i = 1, 2.$

令 $\phi_1(x, \eta_1) = \mu_1\phi_{11}(x, \eta_1) + v_1\phi_1(x, \eta_1), \phi_2(x, \eta_1) = \mu_1\phi_{21}(x, \eta_1) + v_1\phi_{22}(x, \eta_1)$,

$$\delta_1 = \frac{\phi_2(x, \eta_1)}{\phi_1(x, \eta_1)}. \tag{31}$$

则由(27)知:

$$\begin{cases} T_1 = -T_2\delta_1, & (32\text{a}) \\ T_3 = -T_4\delta_1, & (32\text{b}) \end{cases}$$

则由(29)知:

$$\begin{cases} T_{1x} = (r - r)T_1 + (q + \lambda)T_3 - (\lambda - q)T_2, \\ T_{2x} = (r + r)T_2 + (q + \lambda)T_4 - (\lambda + q)T_1, \\ T_{3x} = -(r + r)T_3 + (\lambda - q)T_1 - (\lambda - q)T_4, \\ T_{4x} = -(r + r)T_4 + (\lambda - q)T_2 - (\lambda + q)T_3. \end{cases} \tag{33}$$

将(31)～(32)代入(33)(令 $\lambda = \eta_1$)得: $T_1 = \delta_1^2, T_2 = -\delta_1, T_3 = -\delta_1, T_4 = 1$

$$q = \frac{2\delta_{1x}}{\delta_1^2 - 1} - q, \quad r = r + \frac{2\delta_1\delta_{1x}}{\delta_1^2 - 1}, \quad T_I = \begin{pmatrix} \lambda - \eta_1 + \delta_1^2 & -\delta_1 \\ -\delta_1 & 1 \end{pmatrix}. \tag{34}$$

经直接计算知, 当

$$\phi_{1j} = \left(\sqrt{\lambda_j - \eta_1} + \frac{\delta_1^2}{\sqrt{\lambda_j - \eta_1}}\right)\phi_{1j} - \frac{\delta_1}{\sqrt{\lambda_j - \eta_1}}\phi_{2j},$$
$$\phi_{2j} = \frac{1}{\sqrt{\lambda_j - \eta_1}}(-\delta_1\phi_{1j} + \phi_{2j}) \qquad (j = 1, 2, \cdots, N). \tag{35}$$

时 $\phi(x, \eta_1)$ 同时满足(26a), (26b), 于是我们得到了约束流(24)的第 Ⅰ 类 Darboux 变换.

定理 4 设 $\phi = (\phi_{ij})$ 为约束流(24)的解矩阵, $\delta_1 = \dfrac{\mu_1\phi_{21}(x, \eta_1) + v_1\phi_{22}(x, \eta_1)}{\mu_1\phi_{11}(x, \eta_1) + v_1\phi_{12}(x, \eta_1)}$, 则当 T, $\phi_{ij}(i = 1, 2, ; ; j = 1, 2, \cdots, N)$ 分别满足(34)和(35)时, $\phi = T_I\phi$ 为(24)的 Darboux 变换.

同理, 取

$$\delta_2 = \frac{\phi_1(x, \eta_2)}{\phi_2(x, \eta_2)}(\eta_2 \neq \lambda_2),$$ 可求得(24)的第 Ⅱ 类 Darboux 变换:

定理 5 设 $\phi = (\phi_{ij})$ 为约束流(24)的基本解矩阵, 存在常数 $\mu_2, v_2, |v_2| + |\mu_2| \neq 0$, 令

$$\delta_2=\frac{\mu_2\phi_{11}(x,\eta_2)+\nu_2\phi_{12}(x,\eta_2)}{\mu_2\phi_{21}(x,\eta_2)+\nu_2\phi_{22}(x,\eta_2)}(\eta_2\neq\lambda_2)\cdot \tag{36}$$

$$\phi_{1j}=\frac{1}{\sqrt{\lambda_j-\eta_2}}\phi_{1j}-\frac{\delta_2}{\sqrt{\lambda_j-\eta_2}}\phi_{2j},\ \phi_{2j}=\left(\sqrt{\lambda_j-\eta_2}+\frac{\delta_2^2}{\sqrt{\lambda_j-\eta_1}}\right)\phi_{2j}-\frac{\delta_2}{\sqrt{\lambda_j-\eta_2}}\phi_{1j}\cdot$$

$q=-\frac{2\delta_{2x}}{\delta_2^2-1}-q,\ r=r-\frac{2\delta_{2x}}{\delta_1^2-1}$，取 $T_{\mathrm{II}}=\begin{pmatrix}1 & -\delta_2\\ -\delta_2 & \lambda-\eta_2+\delta_2^2\end{pmatrix}$，则变换 $\phi=T_{\mathrm{II}}\phi$ 为(24)的Darboux 变换•

为求约束流(24)的第 III类 Darboux 变换，对第 I 类 Darboux 变换和第 II类 Darboux 变换作复合运算• 先做第 I 类 Darboux 变换

$\phi=T_{\mathrm{I}}\phi,\ q=\frac{2\delta_{1x}}{\delta_1^2-1}-q,\ r=r+\frac{2\delta_1\delta_{1x}}{\delta_1^2-1}$，再做第 II 第 Darboux 变换

$$\phi=T\phi,\ T=\begin{pmatrix}1 & -\delta_2\\ -\delta_2 & \lambda-\eta_2+\delta_2^2\end{pmatrix},\quad q=-\frac{2\delta_{2x}}{\delta_2^2-1}-q,\quad r=r-\frac{2\delta_2\delta_{2x}}{\delta_2^2-1},$$

于是 $\phi=T\phi=TT_{\mathrm{I}}\phi$,

$$TT_{\mathrm{I}}=\begin{pmatrix}\lambda-\eta_1+\delta_1^2+\delta_1\delta_2 & -\delta_1-\delta_2\\ -(\lambda-\eta_1+\delta_1^2)\delta_2-\delta_1(\delta_2+\lambda-\eta_2) & \lambda-\eta_2+\delta_2^2+\delta_1\delta_2\end{pmatrix},$$

而 $\delta_2=\frac{\mu_2\phi_{11}(x,\eta_2)+\nu_2\phi_{12}(x,\eta_2)}{\mu_2\phi_{21}(x,\eta_2)+\nu_2\phi_{22}(x,\eta_2)}$(在 T_{I} 中取$(\eta_2=\lambda)$) =

$$\frac{(\eta_2-\eta_1+\delta_1^2)\delta_2-\delta_1}{1-\delta_1\delta_2}\cdot \tag{37}$$

由此得到(24)的第 III类 Darboux 变换•

定理 6 令

$$\delta_1=\frac{\mu_1\phi_{21}(x,\eta_1)+\nu_1\phi_{22}(x,\eta_1)}{\mu_1\phi_{11}(x,\eta_1)+\nu_1\phi_{12}(x,\eta_1)},\qquad \delta_2=\frac{\mu_2\phi_{21}(x,\eta_2)+\nu_2\phi_{22}(x,\eta_2)}{\mu_2\phi_{11}(x,\eta_2)+\nu_2\phi_{12}(x,\eta_2)},$$

$$q=q-\frac{2\delta_{1x}}{\delta_1^2-1}-\frac{2\delta_{2x}}{\delta_2^2-1},\qquad r=r+\frac{2\delta_1\delta_{1x}}{\delta_1^2-1}-\frac{2\delta_2\delta_{2x}}{\delta_2^2-1},$$

$$\phi_{1j}=\frac{1}{\sqrt{(\lambda_j-\eta_1)(\lambda_j-\eta_2)}}[(\lambda_j-\eta_1+\delta_1^2+\delta_1\delta_2)\phi_{1j}-(\delta_1+\delta_2)\phi_{2j}],$$

$$\phi_{2j}=\frac{1}{\sqrt{(\lambda_j-\eta_1)(\lambda_j-\eta_2)}}[(\eta_1-\lambda_j-\delta_1^2)\delta_2-\delta_1(\delta_2^2+\lambda_j-\eta_2))\phi_{1j}-$$
$$(\delta_1\delta_2+\delta_2^2+\lambda_j-\eta_2)\phi_{2j}]$$

$j=1,2,\cdots,N$• δ_2 取为(36)，则线性变换 $\phi=TT_{\mathrm{I}}\phi$ 是约束流(24) 的 Darboux 变换•

另外，定理 4~ 定理 6 还提供了求约束流(16) 的孤波解的一种有效方法• 事实上，令

$N^{(k)}=\begin{pmatrix}a_k(\lambda) & b_k(\lambda)\\ c_k(\lambda) & -a_k(\lambda)\end{pmatrix}$，则由(28) 知:

$$\mu(\lambda)=\pm\sqrt{a_k^2(\lambda)+b_k(\lambda)c_k(\lambda)}, \tag{38}$$

$$\delta_{1i}(\eta_i)=\frac{\phi_2(\eta_i)}{\phi_1(\eta_i)}=\frac{\mu(\eta_i)-a_k(\eta_i)}{b_k(\eta_i)}\qquad(i=1,2), \tag{39}$$

由此可由已知量(q,r,Φ_1,Φ_2) 直接求出 $\delta_{1i}(i=1,2)$，再将(39) 代入定理4~ 定理6中的 ϕ_{1j}，ϕ_{2j}，ϕ_{1j}，ϕ_{2j}，就可由(q,r,Φ_1,Φ_2) 求出新的孤波解(q,r,Φ_1,Φ_2)•

[参 考 文 献]

[1] TU Gui_zhang. The trace identity, a powerful tool for constructing the Hamiltonian structure of integrable systems[J]. J Math Phys, 1989, **30**(2): 330—338.

[2] 郭福奎. Loop 代数 A_1 的子代数与 Hamilton 方程族[J]. 数学物理学报, 1999, **19**(5): 507—512.

[3] ZENG Yun_bo. New factorization of the Kaup_Newell hierarchy[J]. Physica D, 1994, **73**(6): 171—188.

[4] ZENG Yun_bo, LI Yi_shen. The Lax representation and Darboux transfirmations for constrained flows of the AKNS hierarchy[J]. Acta Matjematics Sinica NeWeries, 1996, **12**(2): 217—224.

[5] ZENG Yun_bo. An approach to the deduction of the finite_dimensional integrabilty from infinite_dimensional integrability[J]. Phys Letters A, 1991, **160**(4): 541—547.

A Family of Integrable Systems of Liouville and Lax Representation, Darboux Transformations for its Constrained Flows

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Abstract: A family of integrable systems of Liouville are obtained by Tu pattern. Using higher_order potential_eigenfuction constraints, the integrable systems are factorized to two x_and t_n_ integrable Hamiltonian systems whose Lax representation and three kinds of Darboux transformations are presented.

Key words: integrable system; potential_eigenfuction constraint; Lax representation; Darboux transformation