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# Koiter 壳动力学方程解的存在性和唯一性*

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摘要: 用 Galerkin 方法研究了 Koiter 壳动力学方程, 得到了解的存在性与唯一性结果•

关 键 词: Galerkin 方法; Koiter 壳动力学方程; 存在性和唯一性
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## 1 问题提出

考虑一族具有相同中面 $S = \varphi(\omega) \subset \mathbf{R}^3$ 厚度为 $2\varepsilon$ 的线性弹性壳, 这里 $\omega$ 为 $\mathbf{R}^2$ 中有界的具有 Lipschitz 连续边界的连通子集, $\varphi \in C^3(\omega; \mathbf{R}^3)$, 壳沿着他们侧面一部份被夹紧( 位移为 0), 该部分中线为 $\varphi(\gamma_0)$, $\gamma_0$ 为 $\partial\omega$ 一部份, 其长度 $\gamma_0 > 0$( $\gamma_0$ 的长度大于 0)• $\forall \varepsilon > 0$, 让 $\zeta_i^\varepsilon$ 为解二维 W.T. Koiter 模型所得到的中面 $S$ 任一点 $\varphi(y)$ 位移 $\zeta_i^\varepsilon \boldsymbol{a}^i(y)$ 的共变分量• 即找 $\zeta^\varepsilon = (\zeta_i^\varepsilon) \in V_k(\omega)$, 满足:

$$\varepsilon\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{u}^\varepsilon)\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}\,\mathrm{d}y + \frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{u}^\varepsilon)\rho_{\alpha\beta}(\boldsymbol{v})\sqrt{a}\,\mathrm{d}y = \int_\omega p^{i,\varepsilon}v_i\sqrt{a}\,\mathrm{d}y$$

$$(\forall \boldsymbol{v} = (v_i) \in V_k(\omega))• \tag{1}$$

$$V_k(\omega) = \left\{\boldsymbol{v} = (v_i) \in H^1(\omega)\times H^1(\omega)\times H^2(\omega); v_i = \partial_\nu v_3 = 0 \text{ 在 } \gamma_0 \text{ 上}\right\},$$

$$a^{\alpha\beta\sigma\tau} = \frac{4\lambda\mu}{\lambda + 2\mu}a^{\alpha\beta}a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}),$$

为二维弹性张量的反变分量, 满足对任意对称张量 $(t_{\alpha\beta})$, $\exists$ 常数 $C > 0$,

$$Ct_{\alpha\beta}t_{\alpha\beta} \leqslant a^{\alpha\beta\sigma\tau}t_{\alpha\beta}t_{\sigma\tau},$$

$\lambda > 0$ 及 $\mu > 0$ 为与 $\varepsilon$ 无关, 构成弹性壳材料的 Lamé 常数•

$a = \det(a^{\alpha\beta})$, $a^{\alpha\beta}$ 为中面 $S$ 的第一基本形式

$$\gamma_{\alpha\beta}(\boldsymbol{v}) = \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma^\sigma_{\alpha\beta}v_\sigma - \boldsymbol{b}_{\alpha\beta}v_3,$$

为度量张量线性化改变量• $\Gamma^\sigma_{\alpha\beta}$ 为中面 $S$ 的 Christoffel 记号, $\boldsymbol{b}_{\alpha\beta}$ 为中面 $S$ 的曲率张量•

$$\rho_{\alpha\beta}(\boldsymbol{v}) = \partial_{\alpha\beta}v_3 - \Gamma^\sigma_{\alpha\beta}\partial_\sigma v_3 + b^\sigma_\beta(\partial_\alpha v_\sigma - \Gamma^\tau_{\alpha\sigma}v_\tau) + b^\sigma_\alpha(\partial_\beta v_\sigma - \Gamma^\tau_{\beta\sigma}v_\tau) + b^\sigma_{\alpha|\beta}v_\sigma - C_{\alpha\beta}v_3,$$

为中面 $S$ 曲率张量改变量• $a$ 满足:

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$$0 < C_1 \leqslant a(y) \leqslant C_2 \qquad (\forall y \in \omega),$$

$C_1, C_2$ 为与自变量 $y$ 无关的正常数•

$C_{\alpha\beta}$ 为中面 $S$ 的第三基本形式

$$p^{i,\varepsilon} = \frac{1}{2}\int_{-\varepsilon}^{\varepsilon} f^{i,\varepsilon}\mathrm{d}x_3^{\varepsilon},$$

$f^{i,\varepsilon}$ 为作用在弹性壳上体力密度•

## 2 Koiter 壳动力学方程近似解的先验估计

我们考虑如下 Koiter 壳动力学问题:

$$\forall T > 0, 0 \leqslant t \leqslant T,$$

$$\int_\omega \boldsymbol{u}_{tt}^{\varepsilon}(y,t)\cdot \boldsymbol{v}(y)\sqrt{a}\,\mathrm{d}y + \varepsilon\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{u}^{\varepsilon})\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}\,\mathrm{d}y +$$
$$\frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{u}^{\varepsilon})\rho_{\alpha\beta}(\boldsymbol{v})\sqrt{a}\,\mathrm{d}y = \int_\omega p^{i,\varepsilon}(y,t)v_i(y)\sqrt{a}\,\mathrm{d}y \quad (\forall \boldsymbol{v} \in V_k(\omega)), \tag{2}$$

$$\boldsymbol{u}^{\varepsilon}(y,0) = \varphi_0(y) \quad (y \in \omega), \tag{3}$$

$$\boldsymbol{u}_t^{\varepsilon}(y,0) = \phi_0(y) \quad (y \in \omega), \tag{4}$$

$$(p^{i,\varepsilon} \in L_\infty((0,T);V_k^*(\omega)),\ p_t^{i,\varepsilon} \in L_\infty((0,T);V_k^*(\omega)))\text{•}$$

$$\varphi_0(y) \in V_k(\omega),\ \phi_0(y) \in L_2(\omega)\text{•}$$

注:(3),(4) 也可选为

$$\boldsymbol{u}^{\varepsilon}(y,0) = \varphi_0(\varepsilon)(y),\ \boldsymbol{u}_t^{\varepsilon}(y,0) = \phi_0(\varepsilon)(y),$$

只需要求 $\varphi_0(\varepsilon)(y) \in V_k(\omega)$, $\phi_0(\varepsilon)(y) \in L_2(\omega)$•

因空间 $V_k(\varepsilon)$ 是可分的,我们可选 $V_k(\omega)$ 中一组基$\{\boldsymbol{w}_i\}_{i=1}^{+\infty}$,作近似解

$$\boldsymbol{u}^{m,\varepsilon}(y,t) = \sum_{i=1}^{m}\alpha_{im}(t)\boldsymbol{w}_i(y)$$

由 Galerkin 方法, $\alpha_{im}(t)(i = 1,2,\cdots,m)$ 应满足如下常微分方程组的初值问题:

$$\begin{cases} \int_\omega \boldsymbol{u}_{tt}^{m,\varepsilon}\cdot \boldsymbol{w}_i\sqrt{a}\,\mathrm{d}y + \varepsilon\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{u}^{m,\varepsilon})\gamma_{\alpha\beta}(\boldsymbol{w}_i)\sqrt{a}\,\mathrm{d}y + \\ \quad \frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{u}^{m,\varepsilon})\rho_{\alpha\beta}(\boldsymbol{w}_i)\sqrt{a}\,\mathrm{d}y = \\ \quad \int_\omega \boldsymbol{p}^{\varepsilon}(y,t)\cdot \boldsymbol{w}_i\sqrt{a}\,\mathrm{d}y \qquad (i = 1,2,\cdots,m), \\ \hfill (0 \leqslant t \leqslant T) \quad (5) \\ \boldsymbol{u}^{m,\varepsilon}(y,0) = \boldsymbol{u}_{0m} = \sum_{i=1}^{m} a_{im}\boldsymbol{w}_i, \hfill (6) \\ \boldsymbol{u}_t^{m,\varepsilon}(y,0) = \boldsymbol{u}_{1m} = \sum_{i=1}^{m} b_{im}\boldsymbol{w}_i, \hfill (7) \\ \quad (\boldsymbol{p}^{\varepsilon}(y,t) = (p^{1,\varepsilon}(y,t), p^{2,\varepsilon}(y,t), p^{3,\varepsilon}(y,t)))\text{•} \end{cases}$$

当 $m \to +\infty$,

$$\boldsymbol{u}_{0m} = \sum_{i=1}^{m} a_{im}\boldsymbol{w}_i \to \varphi_0(y) \text{ 在 } V_k(\omega) \text{ 中强收敛}, \tag{8}$$

$$\boldsymbol{u}_{1m} = \sum_{i=1}^{m} b_{im}\boldsymbol{w}_i \to \psi_0(y) \text{ 在 } L_2(\omega) \text{ 中强收敛}\cdot \tag{9}$$

由线性常微分方程理论知, 在 $[0, T]$ 上(5) ~ (7) 存在唯一解 $\alpha_{im}(t)(i = 1, 2, \cdots, m)$・(5) 的两端同乘 $\alpha'_{im}(t)(i = 1, 2, \cdots, m)$ 相加起来得:

$$\int_\omega \boldsymbol{u}_{tt}^{m,\varepsilon}(y, t)\cdot \boldsymbol{u}_t^{m,\varepsilon}(y, t)\sqrt{a}\,\mathrm{d}y + \varepsilon\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{u}^{m,\varepsilon})\gamma_{\alpha\beta}(\boldsymbol{u}_t^{m,\varepsilon})\sqrt{a}\,\mathrm{d}y + \frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{u}^{m,\varepsilon})\rho_{\alpha\beta}(\boldsymbol{u}_t^{m,\varepsilon})\sqrt{a}\,\mathrm{d}y = \int_\omega \boldsymbol{p}^\varepsilon\cdot \boldsymbol{u}_t^{m,\varepsilon}\sqrt{a}\,\mathrm{d}y,$$

即

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\omega [\boldsymbol{u}_t^{m,\varepsilon}(y, t)]^2\sqrt{a}\,\mathrm{d}y + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\varepsilon\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{u}^{m,\varepsilon})\gamma_{\alpha\beta}(\boldsymbol{u}^{m,\varepsilon})\sqrt{a}\,\mathrm{d}y + \frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{u}^{m,\varepsilon})\rho_{\alpha\beta}(\boldsymbol{u}^{m,\varepsilon})\sqrt{a}\,\mathrm{d}y\right] = \int_\omega \boldsymbol{p}^\varepsilon\cdot \boldsymbol{u}_t^{m,\varepsilon}\sqrt{a}\,\mathrm{d}y$$

$\forall 0 \leqslant t \leqslant T$, 从 0 到 $t$ 积分得:

$$\frac{1}{2}\int_\omega [\boldsymbol{u}_t^{m,\varepsilon}(y, t)]^2\sqrt{a}\,\mathrm{d}y + \frac{1}{2}\left[\varepsilon\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{u}^{m,\varepsilon})\gamma_{\alpha\beta}(\boldsymbol{u}^{m,\varepsilon})\sqrt{a}\,\mathrm{d}y + \frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{u}^{m,\varepsilon})\rho_{\alpha\beta}(\boldsymbol{u}^{m,\varepsilon})\sqrt{a}\,\mathrm{d}y\right] = \frac{1}{2}\int_\omega [\boldsymbol{u}_t^{m,\varepsilon}(y, 0)]^2\sqrt{a}\,\mathrm{d}y + \frac{1}{2}\left[\varepsilon\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{u}^{m,\varepsilon}(y, 0))\gamma_{\alpha\beta}(\boldsymbol{u}^{m,\varepsilon}(y, 0))\sqrt{a}\,\mathrm{d}y + \frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{u}^{m,\varepsilon}(y, 0))\rho_{\alpha\beta}(\boldsymbol{u}^{m,\varepsilon}(y, 0))\sqrt{a}\,\mathrm{d}y\right] + \int_0^t\int_\omega \boldsymbol{p}^\varepsilon\cdot \boldsymbol{u}_t^{m,\varepsilon}\sqrt{a}\,\mathrm{d}y\,\mathrm{d}t\cdot \tag{*}$$

从(8), (9)知

$$\frac{1}{2}\int_\omega [\boldsymbol{u}_t^{m,\varepsilon}(y, 0)]^2\sqrt{a}\,\mathrm{d}y + \frac{1}{2}\left[\varepsilon\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{u}^{m,\varepsilon}(y, 0))\gamma_{\alpha\beta}(\boldsymbol{u}^{m,\varepsilon}(y, 0))\sqrt{a}\,\mathrm{d}y + \frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{u}^{m,\varepsilon}(y, 0))\rho_{\alpha\beta}(\boldsymbol{u}^{m,\varepsilon}(y, 0))\sqrt{a}\,\mathrm{d}y\right] \leqslant C,$$

式中 $C$ 为与 $m$ 无关正常数, 以后 $C$ 在不同的位置代表不同的正常数・

让 $\langle\cdot, \cdot\rangle$ 表示 $V_k(\omega)$ 与 $V_k^*(\omega)$ 之间对偶内积,

$$\int_0^t\int_\omega \boldsymbol{p}^\varepsilon\cdot \boldsymbol{u}_t^{m,\varepsilon}\sqrt{a}\,\mathrm{d}y\,\mathrm{d}t = \int_0^t\langle \boldsymbol{p}^\varepsilon, \boldsymbol{u}_t^{m,\varepsilon}\rangle\mathrm{d}t = \int_0^t\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{p}^\varepsilon, \boldsymbol{u}^{m,\varepsilon}\rangle\mathrm{d}t - \int_0^t\langle \boldsymbol{p}_t^\varepsilon, \boldsymbol{u}^{m,\varepsilon}\rangle\mathrm{d}t =$$

$$\langle \boldsymbol{p}^{\varepsilon}(y,t), \boldsymbol{u}^{m,\varepsilon}(y,t)\rangle - \langle \boldsymbol{p}^{\varepsilon}(y,0), \boldsymbol{u}^{m,\varepsilon}(y,0)\rangle - \int_0^t \langle \boldsymbol{p}_t^{\varepsilon}, \boldsymbol{u}^{m,\varepsilon}\rangle \mathrm{d}t \cdot$$

从(6),(8)知,

$$-\langle \boldsymbol{p}^{\varepsilon}(y,0), \boldsymbol{u}^{m,\varepsilon}(y,0)\rangle \leqslant C,$$

$$\langle \boldsymbol{p}^{\varepsilon}(y,t), \boldsymbol{u}^{m,\varepsilon}(y,t)\rangle \leqslant \|\boldsymbol{p}^{\varepsilon}(y,t)\|_{V_k^*(\omega)} \|\boldsymbol{u}^{m,\varepsilon}(y,t)\|_{V_k(\omega)} \leqslant \frac{1}{2\delta}\|\boldsymbol{p}^{\varepsilon}(y,t)\|^2_{V_k^*(\omega)} + \frac{\delta}{2}\|\boldsymbol{u}^{m,\varepsilon}(y,t)\|^2_{V_k(\omega)},$$

式中 $\delta$ 为待定常数·

$$-\int_0^t \langle \boldsymbol{p}_t^{\varepsilon}, \boldsymbol{u}^{m,\varepsilon}\rangle \mathrm{d}t \leqslant \int_0^t \|\boldsymbol{p}_t^{\varepsilon}\|_{V_k^*(\omega)} \|\boldsymbol{u}^{m,\varepsilon}\|_{V_k(\omega)} \mathrm{d}t \leqslant \int_0^t \|\boldsymbol{p}_t^{\varepsilon}\|^2_{V_k^*(\omega)} \mathrm{d}t + \int_0^t \|\boldsymbol{u}^{m,\varepsilon}\|^2_{V_k(\omega)} \mathrm{d}t \cdot$$

对 $\boldsymbol{u}^{m,\varepsilon}(y,t) \in V_k(\omega)$, 由于

$$\left\{\sum_{\alpha,\beta}\|\gamma_{\alpha\beta}(\boldsymbol{u}^{m,\varepsilon})\|^2_{L_2(\omega)} + \sum_{\alpha,\beta}\|\rho_{\alpha\beta}(\boldsymbol{u}^{m,\varepsilon})\|^2_{L_2(\omega)}\right\}^{1/2}$$

与 $\|\boldsymbol{u}^{m,\varepsilon}\|_{V_k(\omega)}$ 等价(见[1],[2]),

故

$$\varepsilon\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{u}^{m,\varepsilon})\gamma_{\alpha\beta}(\boldsymbol{u}^{m,\varepsilon})\sqrt{a}\,\mathrm{d}y + \frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{u}^{m,\varepsilon})\rho_{\alpha\beta}(\boldsymbol{u}^{m,\varepsilon})\sqrt{a}\,\mathrm{d}y \geqslant C\left[\sum_{\alpha,\beta}\|\gamma_{\alpha\beta}(\boldsymbol{u}^{m,\varepsilon})\|^2_{L_2(\omega)} + \sum_{\alpha,\beta}\|\rho_{\alpha\beta}(\boldsymbol{u}^{m,\varepsilon})\|^2_{L_2(\omega)}\right] \geqslant C\|\boldsymbol{u}^{m,\varepsilon}\|^2_{V_k(\omega)} \cdot$$

取 $\delta$ 充分小使 $C-\delta/2 \geqslant C/2$, 将得到的所有不等式代入(*)式得:

$$\int_\omega [\boldsymbol{u}_t^{m,\varepsilon}(y,t)]^2\sqrt{a}\,\mathrm{d}y + \|\boldsymbol{u}^{m,\varepsilon}\|^2_{V_k(\omega)} \leqslant C\left(1+\int_0^t\int_\omega [\boldsymbol{u}_t^{m,\varepsilon}(y,t)]^2\sqrt{a}\,\mathrm{d}y\,\mathrm{d}t\right) \cdot$$

从 Gronwall 不等式得:

引理 2.1 若 $\boldsymbol{p}^{\varepsilon}(y,t), \boldsymbol{p}_t^{\varepsilon}(y,t) \in L_\infty((0,T);V_k^*(\omega))$, $\varphi(y) \in V_k(\omega)$, $\psi_0(y) \in L_2(\omega)$, 则(5) ~ (7) 的解 $\boldsymbol{u}^{m,\varepsilon}(y,t)$ 有如下估计:

$$\int_\omega [\boldsymbol{u}_t^{m,\varepsilon}(y,t)]^2\sqrt{a}\,\mathrm{d}y + \|\boldsymbol{u}^{m,\varepsilon}\|^2_{V_k(\omega)} \leqslant C \qquad (0 \leqslant t \leqslant T) \cdot$$

## 3 存在唯一性

由引理 2.1 知,

$\boldsymbol{u}^{m,\varepsilon}(y,t)$ 于 $L_\infty((0,T);V_k(\omega))$ 中关于 $m$ 一致有界,

$\boldsymbol{u}_t^{m,\varepsilon}(y,t)$ 于 $L_\infty((0,T);L_2(\omega))$ 中关于 $m$ 一致有界,

故可选一子序列(为方便仍记为 $\boldsymbol{u}^{m,\varepsilon}(y,t)$) 及存在函数 $\boldsymbol{u}^{\varepsilon}(y,t) \in L_\infty((0,T);V_k(\omega))$ 使得:

当 $m \to +\infty$,

$$\boldsymbol{u}^{m,\varepsilon}(y,t) \overset{*}{\rightharpoonup} \boldsymbol{u}^{\varepsilon}(y,t) \text{ 于 } L_\infty((0,T);V_k(\omega)) \text{ 中}, \tag{10}$$

$$\boldsymbol{u}_t^{m,\varepsilon}(y,t) \overset{*}{\rightharpoonup} \boldsymbol{u}_t^{\varepsilon}(y,t) \text{ 于 } L_\infty((0,T);L_2(\omega)) \text{ 中}, \tag{11}$$

现证 $\boldsymbol{u}^{\varepsilon}(y,t)$ 是问题的解· 引入 $\varphi$ 的空间 $E$·满足:

$$\left.\begin{aligned}&\varphi(t)=\sum_{j=1}^{\mu_0}\varphi_j(t)\boldsymbol{w}_j,\qquad \varphi_j(t)\in C^1[0,T],\\&\varphi_j(T)=0,\ \mu_0\ \text{任意有限}\cdot\end{aligned}\right\}\tag{12}$$

由(5)推出对 $m=\mu\geqslant\mu_0$,

$$\begin{aligned}&\int_\omega \boldsymbol{u}_{tt}^{\mu,\varepsilon}(y,t)\varphi(t)\sqrt{a}\,\mathrm{d}y+\varepsilon\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{u}^{\mu,\varepsilon})\gamma_{\alpha\beta}(\varphi(t))\sqrt{a}\,\mathrm{d}y+\\&\qquad\frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{u}^{\mu,\varepsilon})\rho_{\alpha\beta}(\varphi(t))\sqrt{a}\,\mathrm{d}y=\\&\qquad\int_\omega \boldsymbol{p}^{\varepsilon}\cdot\varphi(t)\sqrt{a}\,\mathrm{d}y\cdot\end{aligned}\tag{13}$$

引入算子 A: $V_k(\omega)\to V_k^*(\omega)$,

$\forall\,\boldsymbol{u},\boldsymbol{v}\in V_k(\omega)$,

$$\begin{aligned}\langle \mathrm{A}\boldsymbol{u},\boldsymbol{v}\rangle=&\ \varepsilon\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{u})\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}\,\mathrm{d}y+\\&\frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{u})\rho_{\alpha\beta}(\boldsymbol{v})\sqrt{a}\,\mathrm{d}y,\end{aligned}\tag{14}$$

(13) 可改写为

$$\langle \boldsymbol{u}_{tt}^{\mu,\varepsilon},\varphi\rangle+\langle A\boldsymbol{u}^{\mu,\varepsilon},\varphi\rangle=\langle \boldsymbol{p}^{\varepsilon},\varphi\rangle,$$

$\varphi$ 由(12) 给定, 因此

$$\int_0^T[-\langle \boldsymbol{u}_t^{\mu,\varepsilon},\varphi_t\rangle+\langle A\boldsymbol{u}^{\mu,\varepsilon},\varphi\rangle-\langle \boldsymbol{p}^{\varepsilon},\varphi\rangle]\,\mathrm{d}t=\langle\psi,\varphi(0)\rangle\cdot\tag{15}$$

在(15)中令 $\mu\to+\infty$, 对任意 $\varphi\in E$, 下式成立:

$$\int_0^T[-\langle \boldsymbol{u}_t,\varphi_t\rangle+\langle A\boldsymbol{u},\varphi\rangle-\langle \boldsymbol{p}^{\varepsilon},\varphi\rangle]\,\mathrm{d}t=\langle\psi,\varphi(0)\rangle\cdot\tag{16}$$

由于 $\boldsymbol{w}_j$ 的有限线性组合在 $V_k(\omega)$ 中稠密, (16) 对任意满足如下条件的 $\varphi$ 皆成立

$$\varphi\in C^1([0,T];V_k(\omega)),\ \varphi(T)=0\cdot$$

由此推出, 定义在 $(0,T)$ 上取值在 $V_k(\omega)$ 中广义函数意义下

$$\boldsymbol{u}_{tt}+A\boldsymbol{u}=\boldsymbol{p}^{\varepsilon},\tag{17}$$

于是 $\boldsymbol{u}_{tt}=\boldsymbol{p}^{\varepsilon}-A\boldsymbol{u}\in L_\infty(0,T);V_k^*(\omega))$·

在(17)的两端取与 $\varphi\in E$ 的内积并与(16) 比较得:

$$\langle\psi(y),\varphi(0)\rangle=\langle \boldsymbol{u}_t(y,0),\varphi(0)\rangle\quad(\forall\,\varphi\in E),$$

故 $\boldsymbol{u}_t(y,0)=\psi_0(y)$·

由(10),(11) 知

$$\boldsymbol{u}^{\mu,\varepsilon}(y,0)\overset{*}{\rightharpoonup}\boldsymbol{u}(y,0)\qquad 于\ L_2(\omega)\ 中,$$

而 $$\boldsymbol{u}^{\mu,\varepsilon}(y,0)\to\varphi_0(y)\qquad 于\ L_2(\omega)\ 中\cdot$$

因此 $\boldsymbol{u}(y,0)=\varphi_0(y)$·

现证解的唯一性, 设 $\boldsymbol{u}(y,t)$ 满足:

$$\boldsymbol{u}(y,t)\in L_\infty((0,T);V_k(\omega)),$$

$$\boldsymbol{u}_t(y,t)\in L_\infty((0,T);L_2(\omega)),$$

$$\boldsymbol{u}_{tt}(y,t) \in L_\infty((0,T); V_k^*(\omega))\cdot$$

和

$$\left.\begin{aligned} &\boldsymbol{u}_{tt} + A\boldsymbol{u} = 0, \\ &\boldsymbol{u}(0) = 0, \boldsymbol{u}_t(0) = 0\cdot \end{aligned}\right\} \tag{18}$$

对 $\varphi \in C^1([0,T]; V_k^*(\omega))$, 由证明的第一部分知, 存在函数 $\boldsymbol{w}$ 使

$$\begin{aligned} &\boldsymbol{w} \in L_\infty((0,T); V_k(\omega)), \boldsymbol{w}_t \in L_\infty((0,T); L_2(\omega)), \\ &\boldsymbol{w}_{tt} \in L_\infty((0,T); V_k^*(\omega)), \\ &\boldsymbol{w}_{tt} + A\boldsymbol{w} = \varphi, \\ &\boldsymbol{w}(T) = 0, \boldsymbol{w}_t(T) = 0\cdot \end{aligned}$$

下列分部积分是允许的

$$\int_0^T \langle \boldsymbol{u}_{tt}, \boldsymbol{w} \rangle \mathrm{d}t = \int_0^T \langle \boldsymbol{u}, \boldsymbol{w}_{tt} \rangle \mathrm{d}t\cdot \tag{19}$$

(18) 的两端取与 $\boldsymbol{w}$ 的内积并利用(19) 即得:

$$\int_0^T \langle \boldsymbol{u}, \boldsymbol{w}_{tt} + A\boldsymbol{w} \rangle \mathrm{d}t = 0,$$

即

$$\int_0^T \langle \boldsymbol{u}, \varphi \rangle \mathrm{d}t = 0, \forall \varphi \in C^1([0,T]; V_k^*(\omega)),$$

故 $\boldsymbol{u} = 0\cdot$

定理 3.1 如果 $\varphi_0(y) \in V_k(\omega)$, $\phi_0(y) \in L_2(\omega)$, $\boldsymbol{p}^\varepsilon(y,t)$, $\boldsymbol{p}_t^\varepsilon(y,t) \in L_\infty((0,T); V^*(\omega))$, 则问题(2) ~ (4) 存在唯一解 $\boldsymbol{u}(y,t)$ 满足:

$$\begin{aligned} &\boldsymbol{u}(y,t) \in L_\infty((0,T); V_k(\omega)), \\ &\boldsymbol{u}_t(y,t) \in L_\infty((0,T); L_2(\omega)), \\ &\boldsymbol{u}_{tt}(y,t) \in L_\infty((0,T); V_k^*(\omega))\cdot \end{aligned}$$

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[ 参 考 文 献 ]


[1] Ciarlet P G, Lods V. Asymptotic analysis of linearly elastic shells. Ⅲ, Justification of Koiter' s shell equations[J]. Arch Rational Mech Anal, 1995, **136**: 191~ 200.

[2] Bernadou M, Ciarlet P G, Miara B. Existence theorems for two_dimensional linear shell theories[J]. J Elasticity, 1994, **34**: 111~ 138.

[3] Ciarlet P G. Mathematical Elasticity, Vol Ⅱ: Plates and Shells [M]. Amsterdam: North_Holland, 1996.

[4] Lions J L, Magenes E, et al. 非齐次边值问题及其应用(第一卷) [M]. 陈怒行等译. 北京: 高等教育出版社, 1987.

# Existence and Uniqueness of Solutions to the Dynamic Equations for Koiter Shells

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**Abstract**: In this paper the dynamic equations for Koiter shells by Galerkin method have been studied, and the existence and uniqueness to the solutions are got.

**Key words**: Galerkin method; the dynamic equations for Koiter shells; existence and uniqueness