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压电介质内裂纹问题的精确解*

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摘要: 在压电介质断裂力学分析中, 人们常假定裂纹面上的电位移法向分量为零, 可是实验表明, 这一假设将导致错误的结果• 本文基于精确的电边界条件, 并应用 Stroh 公式的方法, 导出了含裂纹压电介质在无限远处均匀外载作用下二维问题的精确解• 结果表明: (ⅰ) 应力强度因子与各向同性材料相同, 而电位移强度因子取决于材料常数和机械载荷, 但与电载荷无关; (ⅱ) 能量释放率大于纯弹性各向异性材料内的值, 即总是正的, 且与电载荷无关; (iii) 裂纹内所含空气的介电常数对介质内的场强无影响•

关键词: 压电介质; 二维问题; 裂纹; 能量释放率; 精确解
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引　言

最近几年, 压电介质内的裂纹问题引起了人们的广泛关注, 人们在此领域已做了许多理论研究[1~16]• 但值得一提的是, 这些研究大都是基于一个所谓的非穿透裂纹假设, 即假设裂纹面上的电位移法向分量为零• 根据这一假设, 人们将获得下述结果[2,3,5,6,9,16]: 在电载荷单独作用下, 电位移在裂纹尖端呈现平方根的奇异性, 且能量释放率总是负的• 正如文献[9, 11, 12, 8, 6, 17]所指出的, 这一结论是与现有的实验结果相矛盾的•

本文基于精确的电边界条件, 给出了含裂纹压电介质二维问题的精确解, 并通过这些解证实介质内的电位移场强因子与电载荷无关, 且能量释放率总是非负的•

1　Stroh 公式

在一固定坐标系 $x_i\ (i = 1, 2, 3)$ 内, 压电介质内的场方程可表示为:

$$\sigma_{ij} = C_{ijkm}u_{k,m} + e_{mij}\varphi_{,m}, \quad D_i = e_{ikm}u_{k,m} - \varepsilon_{im}\varphi_{,m} \quad (i, j, k, m = 1, 2, 3) \tag{1.1}$$

其中, 重复下标表示求和, 逗号代表偏微分; σ_i, u_i, D_i 和 φ 分别是应力, 位移, 电位移分量和电势函数; C_{ijkl}, e_{mij}, ε_{im} 分别代表弹性常数, 压电常数和介电常数; 电场 E_i 与电势 φ 的关系为:

$$E_i = -\varphi_{,i} \tag{1.2}$$

对于二维变形问题, 设 u_i 仅与 x_1, x_2 有关, 并引入下列广义位移函数矢量 $\boldsymbol{u}$, 广义应力函数矢量 Φ 及场强表达式:

$$\boldsymbol{u} = [u_1, u_2, u_3, \varphi]^{\mathrm{T}}, \quad \Phi = [\Phi_1, \Phi_2, \Phi_3, \Phi_4]^{\mathrm{T}}, \tag{1.3a}$$

$$\sigma_{i1} = -\Phi_{i,2}, \quad \sigma_{i2} = \Phi_{i,1} \quad (i = 1, 2, 3), \quad D_1 = -\Phi_{4,2}, \quad D_2 = \Phi_{4,1}, \tag{1.3b}$$

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其中, T 表示转置, 那么 $\boldsymbol{u}$ 和 φ 的一般解可表示为[3]:

$$\left.\begin{aligned}&\boldsymbol{u} = \boldsymbol{A}\boldsymbol{f}(z) + \overline{\boldsymbol{A}\boldsymbol{f}(z)}, \quad \phi = \boldsymbol{B}\boldsymbol{f}(z) + \overline{\boldsymbol{B}\boldsymbol{f}(z)},\\ &\boldsymbol{A} = [a_1, a_2, a_3, a_4], \quad \boldsymbol{B} = [b_1, b_2, b_3, b_4],\\ &\boldsymbol{f}(z) = [f_1(z), f_2(z), f_3(z), f_4(z)]^{\mathrm{T}}, z = x_1 + p_a x_2,\end{aligned}\right\} \tag{1.4}$$

其中,"—"代表取共轭; $p_a(a = 1\sim 4)$ 为具有正的虚部的特征值, 由特征方程确定[3, 18], a_a, b_a 为相应于 p_a 的特征向量; $f(z)$ 为未知矢量函数• 应指出的是, 为了简化分析, Suo 的一元复变量分析法[19, 3] 在这里被采用, 一但根据边界条件确定 $f(z)$ 之后, 在计算场量时应将 $f(z)$ 中的分量函数的变量依次用 z_1, z_2, z_3, z_4 来替换• 同时, 还应注意到上述解是基于 $p_a(a = 1\sim 4)$ 互不相等的假设, 在此条件下, 可以证明矩阵 $\boldsymbol{A}, \boldsymbol{B}$ 是非奇异的, 并且存在下列正交关系[18]:

$$\begin{bmatrix}\boldsymbol{B}^{\mathrm{T}} & \boldsymbol{A}^{\mathrm{T}}\\ \overline{\boldsymbol{B}}^{\mathrm{T}} & \overline{\boldsymbol{A}}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\boldsymbol{A} & \overline{\boldsymbol{A}}\\ \boldsymbol{B} & \overline{\boldsymbol{B}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I}\end{bmatrix} \tag{1.5}$$

其中, $\boldsymbol{I}$ 是 4×4 的单位矩阵•

定义矩阵:

$$\boldsymbol{Y} = \mathrm{i}\boldsymbol{A}\boldsymbol{B}^{-1}, \tag{1.6}$$

这里, $\mathrm{i} = \sqrt{-1}$, $\boldsymbol{Y}$ 称为阻抗矩阵; Suo 已证明[3] $\boldsymbol{Y}$ 是 Hermitian 矩阵, 并能写成分块矩阵的形式:

$$\boldsymbol{Y} = \begin{bmatrix}\boldsymbol{Y}_{11} & \boldsymbol{Y}_{14}\\ \boldsymbol{Y}_{41} & \boldsymbol{Y}_{44}\end{bmatrix},$$

其中, $\boldsymbol{Y}_{11}$ 是 3×3 的块矩阵, $\boldsymbol{Y}_{44}$ 是实数, 它们有不同的量纲:

$$\boldsymbol{Y}_{11}\sim[\text{弹力}]^{-1}, \quad \boldsymbol{Y}_{44}\sim[\text{介电常数}]^{-1}, \quad \boldsymbol{Y}_{14} = (\boldsymbol{Y}_{41})^{\mathrm{T}}\sim[\text{压电}]^{-1} \tag{1.7}$$

可以证明[3, 20]:

$$\boldsymbol{Y}_{11}\text{ 是正定的Hermitian 矩阵, 且 }\boldsymbol{Y}_{44} < 0• \tag{1.8}$$

再进一步引入下列矩阵:

$$\boldsymbol{H}^{\mathrm{R}} = 2\mathrm{Re}\,\boldsymbol{Y}, \quad \boldsymbol{H}^{\mathrm{I}} = 2\mathrm{Im}\,\boldsymbol{Y}, \tag{1.9}$$

这里, Re 和 Im 分别表示取实部和虚部, 然后将 $H^{\mathrm{R}}, H^{\mathrm{I}}$ 也写成分块矩阵:

$$\boldsymbol{H}^{\mathrm{R}} = \begin{bmatrix}\boldsymbol{H}_{11}^{\mathrm{R}} & \boldsymbol{H}_{14}^{\mathrm{R}}\\ \boldsymbol{H}_{41}^{\mathrm{R}} & \boldsymbol{H}_{44}^{\mathrm{R}}\end{bmatrix}, \boldsymbol{H}^{\mathrm{I}} = \begin{bmatrix}\boldsymbol{H}_{11}^{\mathrm{I}} & \boldsymbol{H}_{14}^{\mathrm{I}}\\ \boldsymbol{H}_{41}^{\mathrm{I}} & \boldsymbol{H}_{44}^{\mathrm{I}}\end{bmatrix}, \tag{1.10a}$$

其中

$$\boldsymbol{H}_{11}^{\mathrm{R}} = \boldsymbol{Y}_{11} + \overline{\boldsymbol{Y}_{11}}, \quad \boldsymbol{H}_{44}^{\mathrm{R}} = \boldsymbol{Y}_{44} + \overline{\boldsymbol{Y}_{44}} = 2\boldsymbol{Y}_{44}, \quad \boldsymbol{H}_{41}^{\mathrm{R}} = (\boldsymbol{H}_{14}^{\mathrm{R}})^{\mathrm{T}} = \boldsymbol{Y}_{41} + \overline{\boldsymbol{Y}_{41}}, \tag{1.10b}$$

$$\boldsymbol{H}_{11}^{\mathrm{I}} = 2\mathrm{Im}\,\boldsymbol{Y}_{11}, \quad \boldsymbol{H}_{44}^{\mathrm{I}} = 2\mathrm{Im}\,\boldsymbol{Y}_{44} = 0, \quad \boldsymbol{H}_{41}^{\mathrm{I}} = (\boldsymbol{H}_{14}^{\mathrm{I}})^{\mathrm{T}} = 2\mathrm{Im}\,\boldsymbol{Y}_{41}, \tag{1.10c}$$

因 $\boldsymbol{Y}_{11}$ 是正定的 Hermite 矩阵, 则易证 $\overline{\boldsymbol{Y}_{11}}$ 也是正定的 Hermite 矩阵, 故由式(1.10b) 得:

$$\boldsymbol{H}_{11}^{\mathrm{R}}\text{ 是正定的实矩阵, 且 }\boldsymbol{H}_{44}^{\mathrm{R}} < 0, \tag{1.11}$$

另一方面, 利用关系 $\boldsymbol{A}\boldsymbol{B}^{-1} = (\boldsymbol{A}\boldsymbol{B}^{\mathrm{T}})(\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}})^{-1}$ 易得下列结果:

$$\boldsymbol{Y} = \boldsymbol{L}^{-1} - \mathrm{i}\boldsymbol{S}\boldsymbol{L}^{-1}, \quad \boldsymbol{H}^{\mathrm{R}} = 2\boldsymbol{L}^{-1}, \quad \boldsymbol{H}^{I} = -2\boldsymbol{S}\boldsymbol{L}^{-1}, \tag{1.12}$$

其中, $\boldsymbol{S}, \boldsymbol{H}, \boldsymbol{L}$ 为三个实矩阵, 可直接由材料常数确定[18], 它们与 $\boldsymbol{A}, \boldsymbol{B}$ 的关系为:

$$\boldsymbol{S} = \mathrm{i}(2\boldsymbol{A}\boldsymbol{B}^{\mathrm{T}} - \boldsymbol{I}), \quad \boldsymbol{H} = \mathrm{i}2\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}, \boldsymbol{L} = -\mathrm{i}2\boldsymbol{B}\boldsymbol{B}^{T}•$$

2 分　析

考虑一含裂纹 L 压电介质的二维问题• 设裂纹长为 $2a$, 定位于 x_1 轴上 $(-a, a)$ 处; 并假设介质在远处受均布的机械载荷和电载荷共同作用, 在裂纹表面外力和外加电荷为零; 同时认为裂纹内部被空气所充满•

2.1 裂纹内的场强

众所周知, 裂纹可以被看作是椭圆孔的极限情况• 最近文献[17]研究了含一椭圆孔压电介质的二维问题, 其结果表明当无限远处作用均布的外载时, 孔内的电位移场为常量• 因此, 在下面分析中, 我们假定裂纹内的电位移分量 (D_1^0, D_2^0) 是均匀的, 且与裂纹内的电场 (E_1^0, E_2^0) 有下列关系:

$$D_1^0 = \varepsilon_0 E_1^0, \qquad D_2^0 = \varepsilon_0 E_2^0, \tag{2.1}$$

其中, ε_0 为空气的电介常数•

2.2 介质内的场强

引入一个新的函数矢量 $\boldsymbol{F}(z)$:

$$\boldsymbol{F}(z) = \frac{\partial \boldsymbol{f}(z)}{\partial x_1} = \frac{\partial \boldsymbol{f}(z)}{\partial z}\frac{\partial z}{\partial x_1} = \frac{\partial \boldsymbol{f}(z)}{\partial z}•$$

根据式(1.4)得:

$$\boldsymbol{u}_{,1} = \boldsymbol{AF}(z) + \overline{\boldsymbol{A}}\,\overline{\boldsymbol{F}(z)}, \tag{2.2a}$$

$$\boldsymbol{\phi}_{,1} = \boldsymbol{BF}(z) + \overline{\boldsymbol{B}}\,\overline{\boldsymbol{F}(z)}, \tag{2.2b}$$

对于目前的问题, $\boldsymbol{F}(z)$ 可表示为:

$$\boldsymbol{F}(z) = \boldsymbol{c}^{\infty} + \boldsymbol{F}_0(z), \tag{2.3}$$

其中, $\boldsymbol{F}_0(z)$ 为在 z 平面上, 除无限远点 $(z \to \infty)$ 外, 处处全纯的函数矢量, 且 $\boldsymbol{F}_0(\infty) = 0$; $\boldsymbol{c}^{\infty}$ 为由无限远处载荷条件确定的常矢量•

将式(2.3)代入式(2.2b), 然后取 $z \to \infty$ 极限得:

$$\boldsymbol{Bc}^{\infty} + \overline{\boldsymbol{Bc}^{\infty}} = \boldsymbol{\phi}_{,1}^{\infty}, \tag{2.4}$$

其中

$$\boldsymbol{\phi}_{,1}^{\infty} = [\sigma_{21}^{\infty}, \sigma_{22}^{\infty}, \sigma_{23}^{\infty}, D_2^{\infty}]^{\mathrm{T}}$$

在裂纹表面, 力和电位移边界条件可表示为:

$$\boldsymbol{\phi}_{,1}^{0} = [0, 0, 0, D_2^0]^{\mathrm{T}} \tag{2.5}$$

把式(2.3)、(2.5)代入式(2.2b), 并利用式(2.4)得:

$$\boldsymbol{BF}_0(x_1) + \overline{\boldsymbol{BF}_0(x)} = \Delta\boldsymbol{\phi}_1 \quad (\text{在 } L \text{ 上}), \tag{2.6}$$

其中

$$\Delta\boldsymbol{\phi}_{,1} = -[\sigma_{21}^{\infty}, \sigma_{22}^{\infty}, \sigma_{23}^{\infty}, D_2^{\infty} - D_2^0]^{\mathrm{T}}• \tag{2.7}$$

再引入下列新的函数矢量:

$$\boldsymbol{\Omega}_0(z) = \boldsymbol{BF}_0(z), \qquad \boldsymbol{\Psi}_0(z) = -\overline{\boldsymbol{B}}\,\overline{\boldsymbol{F}}_0(z),$$

其中, $\overline{\boldsymbol{F}}_0(z) = \overline{\boldsymbol{F}_0(\bar{z})}$, 那么式(2.6)经整理可表示为:

$$[\boldsymbol{\Omega}_0(x_1) + \boldsymbol{\Psi}_0(x_1)]^+ - [\boldsymbol{\Omega}_0(x_1) + \boldsymbol{\Psi}_0(x_1)]^- = 0 \tag{2.8a}$$

$$[\boldsymbol{\Omega}_0(x_1) - \boldsymbol{\Psi}_0(x_1)]^+ + [\boldsymbol{\Omega}_0(x_1) - \boldsymbol{\Psi}_0(x_1)]^- = 2\Delta\boldsymbol{\phi}_1 \tag{2.8b}$$

根据[21], 式(2.8)的一般解为:

$$\Omega_0(z) = \boldsymbol{B}\boldsymbol{F}_0(z) = \frac{1}{2}\Delta\phi_{,1} - \frac{1}{2}\left\langle\frac{z}{X(z)}\right\rangle\Delta\phi_{,1} + \left\langle\frac{1}{X(z)}\right\rangle\boldsymbol{c}^0, \tag{2.9}$$

式中

$$X(z) = \sqrt{z^2 - a^2}$$

$$\left\langle\frac{z}{X(z)}\right\rangle = \mathrm{diag}\left[\frac{z_1}{X(z_1)}, \frac{z_2}{X(z_2)}, \frac{z_3}{X(z_3)}, \frac{z_4}{X(z_4)}\right],$$

$$\left\langle\frac{1}{X(z)}\right\rangle = \mathrm{diag}\left[\frac{1}{X(z_1)}, \frac{1}{X(z_2)}, \frac{1}{X(z_3)}, \frac{1}{X(z_4)}\right],$$

这里, $\boldsymbol{c}^0$ 是待定常矢量; $X(z)$ 是在沿 L 割开的平面内的一个单值分支, 它在无限远处满足关系 $z^{-1}X(z) \approx 1$, 且在裂纹上(+)、下(−) 表面有不同的值(分支)的[1]:

$$X^+(x_1) = -X^-(x_1) = \mathrm{i}\sqrt{a^2 - x_1^2}, \quad (\text{在 } L \text{ 上})\cdot \tag{2.10}$$

为求 c^0, 令 z 沿 Γ 旋转一周, 这里 Γ 代表贴近裂纹表面的顺时针闭围道(此时 $z \to x_1$), 那么根据[22] 可得:

$$\oint_\Gamma D_2 \mathrm{d}z = 0, \qquad \oint_\Gamma \boldsymbol{E}_1 \mathrm{d}z = 0, \tag{2.11}$$

综合式(2.2), (2.11), 并考虑到力平衡条件和位移单值条件, 可得:

$$\oint_\Gamma u_{,1} \mathrm{d}z = 0, \qquad \oint_\Gamma \phi_{,1} \mathrm{d}z = 0, \tag{2.12}$$

把式(2.2) 代入式(2.12) 得:

$$\begin{bmatrix} \boldsymbol{A} & \overline{\boldsymbol{A}} \\ \boldsymbol{B} & \overline{\boldsymbol{B}} \end{bmatrix}\begin{bmatrix} \oint_\Gamma \boldsymbol{F}(z)\mathrm{d}z \\ \oint_\Gamma \overline{\boldsymbol{F}(z)}\mathrm{d}z \end{bmatrix} = 0, \tag{2.13}$$

注意到式(1.5), 式(2.13) 意指:

$$\oint_\Gamma \boldsymbol{F}(z)\mathrm{d}z = 0, \tag{2.14}$$

把式(2.3), (2.9) 代入式(2.14), 然后再应用留数定理得:

$$\boldsymbol{c}^0 = 0, \tag{2.15}$$

再把式(2.15), (2.9) 代入式(2.3), 即得所研究问题的复势函数解:

$$\boldsymbol{B}\boldsymbol{F}(z) = \boldsymbol{B}\boldsymbol{c}^\infty + \frac{1}{2}\Delta\phi_{,1} - \frac{1}{2}\left\langle\frac{z}{X(z)}\right\rangle\Delta\phi_{,1}\cdot \tag{2.16}$$

为求 $\boldsymbol{D}_2^0$, 让我们应用在裂纹表面上的电场切向分量连续条件:

$$\boldsymbol{E}_1^\pm = \boldsymbol{E}_1^0 \quad (-a \leqslant x_1 \leqslant a), \tag{2.17}$$

由式(2.2a), 并注意到式(1.6) 得:

$$\boldsymbol{u}_{,1} = \boldsymbol{u}_{,1}^\infty + 2\mathrm{Re}[-\mathrm{i}\boldsymbol{Y}\boldsymbol{B}\boldsymbol{F}_0(x_1)], \tag{2.18}$$

其中, $\boldsymbol{u}_{,1}^\infty$ 为无限远处的广义应变矢量·

在裂纹上表面, 将式(2.9) 代入式(2.18), 并注意到式(1.9), (2.10) 得:

$$\boldsymbol{u}_{,1} = \boldsymbol{u}_{,1}^\infty + \frac{1}{2}\boldsymbol{H}^{\mathrm{I}}\Delta\phi_{,1} + \frac{1}{2}\frac{x_1}{\sqrt{a^2 - x_1^2}}\boldsymbol{H}^{\mathrm{R}}\Delta\phi_{,1} \quad (-a \leqslant x_1 \leqslant a), \tag{2.19}$$

综合式(2.19), (2.7), (1.10), (1.3a), (1.2) 即得裂纹上表面电场的切向分量为:

$$E_1^+ = E_1^\infty + \frac{1}{2}\boldsymbol{H}_{41}^{\mathrm{I}}\boldsymbol{t}_2^\infty - \frac{x_1}{\sqrt{a^2 - x_1^2}}[H_{41}^{\mathrm{R}}t_2^\infty + H_{44}^{\mathrm{R}}(D_2^\infty - D_2^0)], \tag{2.20}$$

其中, $\boldsymbol{t}_2^\infty = (\sigma_{21}^\infty, \sigma_{22}^\infty, \sigma_{23}^\infty)^{\mathrm{T}}$ 代表无限远处的机械载荷矢量•

注意到式(2.10),同理可得在裂纹下表面电场的切向分量的表达式:

$$\boldsymbol{E}_1^- = \boldsymbol{E}_1^\infty + \frac{1}{2}\boldsymbol{H}_{41}^{\mathrm{I}}\boldsymbol{t}_2^\infty + \frac{x_1}{2\sqrt{a^2 - x_1^2}}[\boldsymbol{H}_{41}^{\mathrm{R}}\boldsymbol{t}_2^\infty + \boldsymbol{H}_{44}^{\mathrm{R}}(\boldsymbol{D}_2^\infty - \boldsymbol{D}_2^0)] \tag{2.21}$$

由式(2.20),(2.21),(2.17)得:

$$\boldsymbol{H}_{41}^{\mathrm{R}}\boldsymbol{t}_2^\infty + \boldsymbol{H}_{44}^{\mathrm{R}}(\boldsymbol{D}_2^\infty - \boldsymbol{D}_2^0) = \boldsymbol{0}, \tag{2.22}$$

$$\boldsymbol{E}_1^0 = \boldsymbol{E}_1^\infty + \frac{1}{2}\boldsymbol{H}_{41}^{\mathrm{I}}\boldsymbol{t}_2^\infty, \tag{2.23}$$

再由式(2.22),即得确定 $\boldsymbol{D}_2^0$ 的方程:

$$\boldsymbol{D}_2^\infty - \boldsymbol{D}_2^0 = -\frac{1}{\boldsymbol{H}_{44}^{\mathrm{R}}}\boldsymbol{H}_{41}^{\mathrm{R}}\boldsymbol{t}_2^\infty, \tag{2.24}$$

式(2.24)表明,裂纹表面的电位移法向分量并不为零,它取决于无限远处的机械载荷和电载荷,而与空气的介电常数无关•

根据式(2.23),(2.24),(2.1),我们可获得裂纹内的场强为:

$$\boldsymbol{E}_1^0 = \boldsymbol{E}_1^\infty + \frac{1}{2}\boldsymbol{H}_{41}^{\mathrm{I}}\boldsymbol{t}_2^\infty, \qquad \boldsymbol{D}_2^0 = \boldsymbol{D}_2^\infty + \frac{1}{\boldsymbol{H}_{44}^{\mathrm{R}}}\boldsymbol{H}_{41}^{\mathrm{R}}\boldsymbol{t}_2^\infty, \tag{2.25a}$$

$$\boldsymbol{E}_2^0 = \frac{\boldsymbol{D}_2^\infty}{\varepsilon_0} + \frac{1}{\varepsilon_0\boldsymbol{H}_{44}^{\mathrm{R}}}\boldsymbol{H}_{41}^{\mathrm{R}}\boldsymbol{t}_2^\infty, \qquad \boldsymbol{D}_1^0 = \varepsilon_0\boldsymbol{E}_1^0• \tag{2.25b}$$

在裂纹尖端 $(x_1 = a)$,场强度因子可定义为:

$$\boldsymbol{K} = [K_{\mathrm{II}}, K_{\mathrm{I}}, K_{\mathrm{III}}, K_{\mathrm{IV}}]^{\mathrm{T}} = \lim_{r\to 0}\sqrt{2\pi r}[\sigma_{21}, \sigma_{22}, \sigma_{23}, D_2]^{\mathrm{T}} = \lim_{r\to 0}\sqrt{2\pi r}\,\phi_{,1}, \tag{2.26}$$

其中, r 为距裂纹尖端$(x_1 = a)$的距离•

把式(2.2b)和式(2.16),(2.7)代入式(2.26)得:

$$\boldsymbol{K} = \sqrt{\pi a}[\sigma_{21}^\infty, \sigma_{22}^\infty, \sigma_{23}^\infty, D_2^\infty - D_2^0]^{\mathrm{T}}, \tag{2.27}$$

如果我们假设 $\boldsymbol{D}_2^0 = \boldsymbol{0}$,则由式(2.27)得:

$$\boldsymbol{K} = \sqrt{\pi a}[\sigma_{21}^\infty, \sigma_{22}^\infty, \sigma_{23}^\infty, D_2^\infty]^{\mathrm{T}},$$

此即为曾前根据非穿透裂纹假设所得到的结果[2~7,9]•

又由式(2.27),(2.24)得:

$$K_{\mathrm{IV}} = -\sqrt{\pi a}\,\frac{1}{\boldsymbol{H}_{44}^{\mathrm{R}}}\boldsymbol{H}_{41}^{\mathrm{R}}\boldsymbol{t}_2^\infty, \tag{2.28}$$

式(2.28)表明电位移强度因子取决于材料常数和机械载荷,而与电载荷无关•

同时,式(2.28)又可写成:

$$\boldsymbol{K}_{\mathrm{IV}}\boldsymbol{H}_{44}^{\mathrm{R}} + \boldsymbol{H}_{41}^{\mathrm{R}}\boldsymbol{K}_e = 0, \tag{2.29}$$

其中, $\boldsymbol{K}_e = (K_{\mathrm{II}}, K_{\mathrm{I}}, K_{\mathrm{III}})^{\mathrm{T}}$ 代表机械应力强度因子矢量•

根据文献[3],能量释放率 G 可表示为:

$$4\boldsymbol{G} = \boldsymbol{K}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{R}}\boldsymbol{K} = [\boldsymbol{K}_e^T, \boldsymbol{K}_{\mathrm{IV}}]\begin{bmatrix}\boldsymbol{H}_{11}^{\mathrm{R}} & \boldsymbol{H}_{14}^{\mathrm{R}} \\ \boldsymbol{H}_{41}^{\mathrm{R}} & \boldsymbol{H}_{44}^{\mathrm{R}}\end{bmatrix}\begin{bmatrix}\boldsymbol{K}_e \\ \boldsymbol{K}_{\mathrm{IV}}\end{bmatrix} = \\ \boldsymbol{K}_e^{\mathrm{T}}\boldsymbol{H}_{11}^{\mathrm{R}}\boldsymbol{K}_e + \boldsymbol{K}_{\mathrm{IV}}\boldsymbol{H}_{41}^{\mathrm{R}}\boldsymbol{K}_e + \boldsymbol{K}_e^{\mathrm{T}}\boldsymbol{H}_{14}^{\mathrm{R}}\boldsymbol{K}_{\mathrm{IV}} + \boldsymbol{H}_{44}^{\mathrm{R}}\boldsymbol{K}_{\mathrm{IV}}^2, \tag{2.30}$$

应用式(2.29),式(2.30)可简化为:

$$4\boldsymbol{G} = \boldsymbol{K}_e^{\mathrm{T}}\boldsymbol{H}_{11}^{\mathrm{R}}\boldsymbol{K}_e + \left(-\frac{1}{\boldsymbol{H}_{44}^{\mathrm{R}}}\right)\boldsymbol{K}_e^{\mathrm{T}}\boldsymbol{K}_e(\boldsymbol{H}_{14}^{\mathrm{R}})^{\mathrm{T}}\boldsymbol{H}_{14}^{\mathrm{R}} \tag{2.31}$$

故式(2.31)又可进一步写为:

$$\boldsymbol{G} = \boldsymbol{G}_e + \Delta\boldsymbol{G}, \tag{2.32}$$

其中

$$\boldsymbol{G}_e = \frac{1}{4}\boldsymbol{K}_e^{\mathrm{T}}\boldsymbol{H}_{11}^{\mathrm{R}}\boldsymbol{K}_e, \qquad \Delta\boldsymbol{G} = \frac{1}{4}\left(-\frac{1}{\boldsymbol{H}_{44}^{\mathrm{R}}}\right)K_e^{\mathrm{T}}\boldsymbol{K}_e(\boldsymbol{H}_{14}^{\mathrm{R}})^{\mathrm{T}}\boldsymbol{H}_{14}^{\mathrm{R}} \tag{2.33}$$

注意式(1.11),显然有 $\boldsymbol{G}_e \geqslant 0$, $\Delta\boldsymbol{G} \geqslant 0$, 因此 $\boldsymbol{G} \geqslant 0$, 即能量释放率总是正的, 而且从式(2.33),(1.10b),(1.7)可以发现 $\boldsymbol{G}_e$ 仅仅取决于机械弹性常数和机械载荷, 即 $\boldsymbol{G}_e$ 代表纯弹性各向异性材料的能量释放率, 因此, 式(2.32)还表明在压电材料内, 裂纹的扩展力通常要大于纯弹性材料内的值•

当无限远处仅存在电载荷, 既 $t_2^{\infty} = 0$ 时, 综合式(2.7),(2.24),(2.16)可得:

$$\boldsymbol{F}(z) = \boldsymbol{c}^{\infty}, \tag{2.34}$$

式(2.34)表明此时在介质内, 应力场为零(但变形是均匀的); 能量释放率也为零(非负); 而电场是均匀的(非奇异), 且等于无限远处的值; 同时在裂纹内, 电场分量可表示为:

$$\boldsymbol{E}_1^0 = \boldsymbol{E}_1^{\infty}, \qquad \boldsymbol{E}_2^0 = \frac{\boldsymbol{D}_2^{\infty}}{\varepsilon_0} = \frac{\varepsilon_{22}}{\varepsilon_0}\boldsymbol{E}_2^{\infty}, \tag{2.35}$$

式(2.35)与最近 Sosa 和 Khutoryansky 所获得的结果[17]是一致的•

最后, 应指出的是, 根据上述结果, 我们发现场强因子和能量释放率仅与外载和实矩阵 H^{R}, H^{I}, 亦即仅与(由式(1.12) $\boldsymbol{S}$, $\boldsymbol{H}$, $\boldsymbol{L}$ 有关, 故本文解对于退化材料的情形(此时 $\boldsymbol{A}$, $\boldsymbol{B}$ 可能奇异)也成立•

3 结 论

应用 Stroh 公式, 研究了含裂纹压电材料的二维问题, 得到了场强度因子和能量释放率的精确解• 结果表明, 非穿透裂纹假设并不适用于裂纹问题的分析• 凡是基于此假设所获得的有关裂纹问题的解, 都含有某些病态的结果, 故都需做适当的修正•

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A Exact Solution of Crack Problems in Piezoelectric Materials

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Abstract: An assumption that the normal component of the electric displacement on crack faces is thought of as being zero is widely used in analyzing the fracture mechanics of piezoelectric materials. However, it is shown from the available experiments that the above assumption will lead to erroneous results. In this paper, the two_dimensional problem of a piezoelectric material with a crack is studied based on the exact electric boundary condition on the crack faces. Stroh formalism is used to obtain the closed_form solutions when the material is subjected to uniform loads at infinity. It is shown from these solutions that: (i) the stress intensity factor is the same as that of isotropic material, while the intensity factor of the electric displacement depends on both material properties and the mechanical loads, but not on the electric load. (ii) the energy release rate in a piezoelectric material is larger than that in a pure elastic_anisotropic material, i e, it is always positive, and independent of the electric loads. (iii) the field solutions in a piezoelectric material are not related to the dielectric constant of air or vacuum inside the crack

Key words: piezoelectric material; plane problem; crack; energy release rate; exact solution