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含椭圆孔或裂纹压电介质平面问题的基本解

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摘 要

应用复变函数的方法, 并基于精确的电边界条件, 导出了含一椭圆孔或裂纹的横观各向同性压电体在任意集中力和集中电荷作用下的复变函数解, 即 Creen 函数解• 叠加该解, 得到了裂纹表面作用任意集中载荷或分布载荷时的一般解• 这些解不但澄清了从前文献中一些不合理的结果, 同时也为应用边界元法求解更复杂的压电介质断裂力学问题提供了基本解•

关键词 压电介质 椭圆孔 裂纹 平面问题 基本解
中图分类号 O343, O482

§1. 引 言

众所周知, 在弹性分析中基本解或 Green 函数解占据重要的地位, 因为这些解不仅可以用来构造许多重要问题的解析解, 同时还可作为边界元法的基本解, 以求解更复杂的弹性平衡问题•

随着压电材料在工程中的广泛应用, 有关压电体内的 Green 函数问题越来越引起人们的关注• 文献[1~ 8]分别研究了压电介质三维 Green 函数问题• 文献[9, 10, 11]给出了二维压电 Green 函数问题的解, 此外, 文献[12~ 14]分别获得了在半平面或半空间压电介质表面作用集中载荷时的解• 但值得指出的是, 这些研究均是针对无孔洞介质的情形•

本文应用复变函数的方法, 针对于含一椭圆孔或裂纹的横观各向同性压电介质, 给出了其在任意集中载荷作用下二维问题的 Green 函数解• 因该解已预先考虑了孔或裂纹的存在, 故当其被用作边界元法的基本解时, 无需再考虑孔周或裂纹表面(高应力集中区) 的边界条件, 从而可获得高精度解•

同时, 利用上述基本解, 本文还给出了含一裂纹压电介质场强因子的一般解, 并由此验证了从前文献中某些结果的正确性•

§2. 基 本 公 式

考虑一横观各向同性压电体, 取其各向同性平面与 x_1_x_3 平行, 下面我们研究其 x_1_x_2 平面内的二维问题, 根据文献[15], $x_1- x_2$ 平面内的应力 σ_{ij}, 位移 u_i, 电场 E_i, 电位移 D_i 和电势

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$\varphi(z_k)$ 可表示为[15]:

$$\langle \sigma_{22}, \sigma_{12}, \sigma_{11} \rangle = 2\mathrm{Re}\sum_{k=1}^{3}\langle 1, -\mu_k, \mu_k^2\rangle \phi_k(z_k), \quad z_k = x_1 + \mu_k x_2 \tag{2.1}$$

$$\langle u_1, u_2\rangle = 2\mathrm{Re}\sum_{k=1}^{3}\langle p_k, q_k\rangle \varphi_k(z_k), \quad \mathrm{Im}\,\mu_k > 0 \tag{2.2}$$

$$\langle E_1, E_2\rangle = 2\mathrm{Re}\sum_{k=1}^{3}\kappa_k\langle 1, \mu_k\rangle \phi_k(z_k) \tag{2.3}$$

$$\langle D_1, D_2\rangle = 2\mathrm{Re}\sum_{k=1}^{3}\lambda_k\langle \mu_k, -1\rangle \phi_k(z_k) \tag{2.4}$$

$$\varphi(z_k) = -2\mathrm{Re}\sum_{k=1}^{3}\kappa_k\varphi_k(z_k) \tag{2.5}$$

$$p_k = a_{11}\mu_k^2 + a_{12} - b_{21}\lambda_k, \quad q_k = (a_{11}\mu_k^2 + a_{22} - b_{22}\lambda_k)/\mu_k$$

$$\lambda_k = -\frac{(b_{21}+b_{13})\mu_k^2 + b_{22}}{\delta_{11}\mu_k^2 + \delta_{22}}, \quad \kappa_k = (b_{13} + \delta_{11}\lambda_k)\mu_k, \quad \phi_k(z_k) = d\varphi_k(z_k)/dz_k$$

其中, a_{ij}, b_{ij}, δ_{ij} 分别为弹性常数、压电常数和介电常数; μ_k 为互不相等的复常数, 由特征方程确定[15]•

力, 电位移的边界条件可表示为:

$$2\mathrm{Re}\sum_{k=1}^{3}\varphi_k(z_k) = \mp\int_0^s t_2\,ds \tag{2.6}$$

$$2\mathrm{Re}\sum_{k=1}^{3}\mu_k\varphi_k(z_k) = \pm\int_0^s t_1\,ds \tag{2.7}$$

$$2\mathrm{Re}\sum_{k=1}^{3}\lambda_k\varphi_k(z_k) = \mp\int_0^s D_n\,ds \tag{2.8}$$

其中, t_1, t_2 为边界上外力的直角坐标分量; D_n 为电位移的法向分量; s 为弧长. 对于外(内)边界, 等式右端取上(下)部符号•

§3. 复势函数基本解

如图1所示, x_1-x_2 平面被一自由椭圆孔 L 削弱, 仅在点 z_0 处受任意集中力 $T_{10}+iT_{20}$ 和集中电荷 Q_0 作用, 并认为孔内被均匀的空气所充满•

3.1 孔内的场解

设孔内的电势 $\varphi^c(z)$ 为:

$$\varphi^c(z) = -2\mathrm{Re}\,\varphi_0(z) \tag{3.1}$$

其中, $\varphi_0(z)$ 为在 L 内部解析的待定函数, 那么孔内的电场 (E_1^0, E_2^0) 和电位移 (D_1^0, D_2^0) 可分别表示为

$$\left.\begin{aligned} E_1^0 &= -\frac{\partial\varphi^c}{\partial x} = 2\mathrm{Re}\,\phi_0(z), \\ E_2^0 &= -\frac{\partial\varphi^c}{\partial y} = -2\mathrm{Im}\,\phi_0(z) \end{aligned}\right\} \tag{3.2}$$

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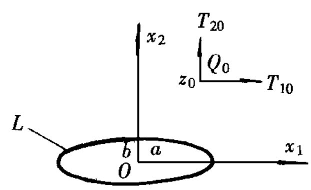

图1 椭圆孔外受集中载

$$D_1^0 = 2\varepsilon_0 \mathrm{Re}\,\phi_0(z), \quad D_2^0 = -2\varepsilon_0 \mathrm{Im}\,\phi_0(z) \tag{3.3}$$

其中, ε_0 为空气的介电常数, $\phi_0(z) = d\varphi_0(z)/dz$ •

又由高斯定理得:

$$\int_L D_n^0 ds = \int_L D_1^0 dx_2 - D_2^0 dx_1 \tag{3.4}$$

把式(3.3)代入式(3.4)右边, 整理得:

$$\int_L D_n^0 ds = 2\varepsilon_0 \mathrm{Im}\,\varphi_0(z) \tag{3.5}$$

引入将 L 外部映射到 ζ 平面单位圆 γ 外部的保角映射函数:

$$z = R_0(\zeta + m_0\zeta^{-1}), \quad R_0 = (a+b)/2, \quad m_0 = (a-b)/(a+b)$$

则全纯函数 $\varphi_0(z)$ 在 L 内部可展为 Faber 级数[16]

$$\Phi_0(\zeta) = \sum_{n=1}^{\infty} a_n^0(\zeta^n + m^n\zeta^n) \tag{3.6}$$

其中, $\varphi_0[z(\zeta)] = \Phi_0(\zeta)$, a_n^0 为待定系数•

3.2 孔外的场解

在孔外, 复势函数可表示为:

$$\varphi_k(z_k) = A_k \ln(z_k - z_{k_0}) + \varphi_{k_0}(z_k) \tag{3.7}$$

其中, $z_{k_0} = x_{10} + \mu_k x_{20}$; $\varphi_{k_0}(z_k)$ 为 L_k(L_k 由 L 经映射: $z_k = x_1 + \mu_k x_2$ 所得) 外部全纯的函数, 且 $\varphi_{k_0}(\infty) = 0$; A_k 为复常数•

为求 A_k, 把式(3.7)代入式(2.6)~(2.8), (2.2), (2.5), 并计算绕 Z_0 点旋转一周时相关量的增量, 则由静力平衡条件、高斯定理($\oint D_n ds = Q_0$)、电势和位移单值条件可得确定 A_k 的线性方程组:

$$\left.\begin{aligned} &\mathrm{Im}\sum_{k=1}^{3} A_k = -\frac{T_{20}}{4\pi}, \quad \mathrm{Im}\sum_{k=1}^{3}\mu_k A_k = \frac{T_{10}}{4\pi}, \quad \mathrm{Im}\sum_{k=1}^{3}\lambda_k A_k = \frac{Q_0}{4\pi} \\ &\mathrm{Im}\sum_{k=1}^{3}\kappa_k A_k = 0, \quad \mathrm{Im}\sum_{k=1}^{3} p_k A_k = 0, \quad \mathrm{Im}\sum_{k=1}^{3} q_k A_k = 0 \end{aligned}\right\} \tag{3.8a~f}$$

在孔周, 力、电位移及电势的连续条件可表示为:

$$\left.\begin{aligned} &2\mathrm{Re}\sum_{k=1}^{3}\varphi_k(z_k) = 0, \quad 2\mathrm{Re}\sum_{k=1}^{3}\mu_k\varphi_k(z_k) = 0 \\ &2\mathrm{Re}\sum_{k=1}^{3}\lambda_k\varphi_k(z_k) = 2\varepsilon_0 \mathrm{Im}\,\varphi_0(z) \end{aligned}\right\} \tag{3.9}$$

$$2\mathrm{Re}\sum_{k=1}^{3}\kappa_k\varphi_k(z_k) = 2\mathrm{Re}\,\varphi_0(z) \tag{3.10}$$

又因保角映射函数:

$$z_k = R_k(\zeta_k + m_k\zeta_k^{-1}), \quad R_k = (a - i\mu_k b)/2, \quad m_k = (a + i\mu_k b)/(a - i\mu_k b) \tag{3.11}$$

分别把 z_k 平面上椭圆孔 L_k 的外部, 保角映射为 ζ_k 平面上同一单位圆 γ 的外部, 则在 ζ_k 平面, 复势函数变为

$$\Phi_k(\zeta_k) = A_k \ln[z_k(\zeta_k) - z_{k_0}(\zeta_{k_0})] + \Phi_{k_0}(\zeta_k) \tag{3.12}$$

其中, ζ_{k_0} 为 z_{k_0} 的象, $|\zeta_{k_0}| > 1$; $\Phi_k(\zeta_k) = \varphi[z_k(\zeta_k)]$; $\Phi_{k_0}(\zeta_k)$ 为 γ 外部全纯的函数•

在单位圆 γ 上, $\zeta_k = \zeta = \sigma = e^{i\theta}$, 则由式(3.9),(3.10)得

$$\left.\begin{aligned} &2\mathrm{Re}\sum_{k=1}^{3}\Phi_k(\sigma) = 0, \quad 2\mathrm{Re}\sum_{k=1}^{3}\mu_k\Phi_k(\sigma) = 0 \\ &2\mathrm{Re}\sum_{k=1}^{3}\lambda_k\Phi_k(\sigma) = 2\varepsilon_0\mathrm{Im}\sum_{n=1}^{\infty}a_n^0(\sigma^n + m_0^n\sigma^{-n}) \end{aligned}\right\} \tag{3.13}$$

$$2\mathrm{Re}\sum_{k=1}^{3}\kappa_k\Phi_k(\sigma) = 2\varepsilon_0\mathrm{Re}\sum_{n=1}^{\infty}a_n^0(\sigma^n + m_0^n\sigma^{-n}) \tag{3.14}$$

在式(3.13),(3.14)的两边同乘 $d\sigma/\sigma - \zeta_k$, 然后在单位圆上取积分, 并应用文献[17]所给的积分公式得

$$\left.\begin{aligned} &\sum_{k=1}^{3}\Phi_{k_0}(\zeta) = f_1(\zeta), \quad \sum_{k=1}^{3}\mu_k\Phi_{k_0}(\zeta) = f_2(\zeta) \\ &\sum_{k=1}^{3}\lambda_k\Phi_{k_0}(\zeta) = f_D(\zeta) \end{aligned}\right\} \tag{3.15}$$

$$\sum_{k=1}^{3}\kappa_k\Phi_{k_0}(\zeta) = f_4(\zeta) + \sum_{n=1}^{\infty}(a_n^0 m_0^n + \overline{a_n^0})\zeta^{-n} \tag{3.16}$$

其中

$$f_D(\zeta) = f_3(\zeta) + i\varepsilon_0\sum_{n=1}^{\infty}(a_n^0 m_0^n - \overline{a_n^0})\zeta^{-n}$$

$$f_1(\zeta) = -\sum_{k=1}^{3}\left[A_k\ln\left(1 - \frac{m_k}{\zeta_{k_0}\zeta}\right) + \overline{A_k}\ln\left(1 - \frac{1}{\zeta_{k_0}\zeta}\right)\right]$$

$$f_2(\zeta) = -\sum_{k=1}^{3}\left[\mu_k A_k\ln\left(1 - \frac{m_k}{\zeta_{k_0}\zeta}\right) + \overline{\mu_k A_k}\ln\left(1 - \frac{1}{\zeta_{k_0}\zeta}\right)\right]$$

$$f_3(\zeta) = -\sum_{k=1}^{3}\left[\lambda_k A_k\ln\left(1 - \frac{m_k}{\zeta_{k_0}\zeta}\right) + \overline{\lambda_k A_k}\ln\left(1 - \frac{1}{\zeta_{k_0}\zeta}\right)\right]$$

$$f_4(\zeta) = -\sum_{k=1}^{3}\left[\kappa_k A_k\ln\left(1 - \frac{m_k}{\zeta_{k_0}\zeta}\right) + \overline{\kappa_k A_k}\ln\left(1 - \frac{1}{\zeta_{k_0}\zeta}\right)\right]$$

引入矩阵

$$C = \begin{bmatrix} 1 & 1 & 1 \\ \mu_1 & \mu_2 & \mu_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix}, \quad \Lambda = C_{\varphi}^{-1} = [\Lambda_{kj}]$$

则由式(3.15)解得

$$\Phi_{k_0}(\zeta) = \sum_{j=1}^{3}\Lambda_{kj}f_j + i\varepsilon_0\Lambda_{k3}\sum_{n=1}^{\infty}(a_n^0 m_0^n - \overline{a_n^0})\zeta^{-n} \tag{3.17}$$

为确定 a_n^0, 把式(3.17)代入式(3.16), 并利用下列关系:

$$\ln\left(1 - \frac{m_k}{\zeta_{k_0}\zeta}\right) = -\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{m_k}{\zeta_{k_0}\zeta}\right)^n, \quad \ln\left(1 - \frac{1}{\zeta_{k_0}\zeta}\right) = -\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{\zeta_{k_0}\zeta}\right)^n$$

然后比较 ζ_k^n 的系数得

$$a_n^0 m_0^n(1 - i\varepsilon_0\rho_0) + \overline{a_n^0}(1 + i\varepsilon_0\rho_0) = c_n^0 \tag{3.18}$$

其中

$$\rho_0 = \sum_{k=1}^{3}\kappa_k\Lambda_k, \qquad c_n^0 = \sum_{k=1}^{3}\sum_{j=1}^{3}\kappa_k\Lambda_{kj}f_{jn} - f_{4n} \tag{3.19}$$

$$f_{1n}=\frac{1}{n}\sum_{k=1}^{3}\left[A_k\left(\frac{m_k}{\zeta_{k_0}}\right)^n+\overline{A_k}\left(\frac{1}{\overline{\zeta_{k_0}}}\right)^n\right],\quad f_{2n}=\frac{1}{n}\sum_{k=1}^{3}\left[A_k\mu_k\left(\frac{m_k}{\zeta_{k_0}}\right)^n+\overline{A_k\mu_k}\left(\frac{1}{\overline{\zeta_{k_0}}}\right)^n\right] \tag{3.20a, b}$$

$$f_{3n}=\frac{1}{n}\sum_{k=1}^{3}\left[A_k\lambda_k\left(\frac{m_k}{\zeta_{k_0}}\right)^n+\overline{A_k\lambda_k}\left(\frac{1}{\overline{\zeta_{k_0}}}\right)^n\right],\quad f_{4n}=\frac{1}{n}\sum_{k=1}^{3}\left[A_k\kappa_k\left(\frac{m_k}{\zeta_{k_0}}\right)^n+\overline{A_k\kappa_k}\left(\frac{1}{\overline{\zeta_{k_0}}}\right)^n\right] \tag{3.20c, d}$$

式(3.18)及其共轭式可给出 a_n^0 的解为

$$a_n^0=\frac{\overline{c_n^0}-m_0^nc_n^0+i\varepsilon_0(\overline{c_n^0}\rho_0-m_0^nc_n^0\overline{\rho_0})}{\Delta} \tag{3.21}$$

其中

$$\Delta=(1-m_0^{2n})(1+\varepsilon_0^2\rho_0\overline{\rho_0})-2\varepsilon_0(1+m_0^{2n})\mathrm{Im}\rho_0 \tag{3.22}$$

若令 $\beta_n=a_n^0m_0^n-\overline{a_n^0}$, 则有

$$\beta_n=\frac{2m_0^n\overline{c_n^0}-c_n^0[1+m_0^{2n}-i\varepsilon_0\overline{\rho_0}(1-m_0^{2n})]}{\Delta} \tag{3.23}$$

把式(3.17)代入式(3.12),我们最终获得 Green 函数:

$$\Phi_k(\zeta_k)=A_k\ln[z_k(\zeta_k)-z_{k_0}(\zeta_{k_0})]+\sum_{j=1}^{3}\Lambda_{kj}f_j+i\varepsilon_0\Lambda_{k3}\sum_{n=1}^{\infty}\beta_n\zeta_k^{-n} \tag{3.24}$$

§4. 有关裂纹问题的解

4.1 裂纹外作用力任一集中载时的解

在式(3.22),(3.23)中令 $m_0=m_k=1$, 则有

$$\Delta=-4\varepsilon_0\mathrm{Im}\rho_0,\quad \beta_n=\frac{i\mathrm{Im}c_n^0}{\varepsilon_0\mathrm{Im}\rho_0} \tag{4.1}$$

把式(4.1)代入式(3.24)得

$$\Phi_k(\zeta_k)=A_k\ln[z_k(\zeta_k)-z_{k_0}(\zeta_{k_0})]+\sum_{j=1}^{3}\Lambda_{kj}f_j-\frac{\Lambda_{k3}}{\mathrm{Im}\rho_0}\mathrm{Im}\sum_{n=1}^{\infty}c_n^0\zeta_k^{-n} \tag{4.2}$$

式(4.2)表明对于裂纹问题, $\Phi_k(\zeta_k)$ 与 ε_0 无关, 故介质内的场解与 ε_0 无关·

在 $x_1=a$ 处,场强因子可表示为:

$$\langle k_1,k_2,k_D\rangle=\sqrt{2\pi}\lim_{z_k\to a}(z_k-a)^{1/2}\langle\sigma_{22},\sigma_{12},D_2\rangle \tag{4.3}$$

把式(2.1),(2.4)代入式(4.3)得

$$\langle k_1,k_2,k_D\rangle=\sqrt{2\pi}\lim_{z_k\to a}(z_k-a)^{1/2}\langle 1,-\mu_k,-\lambda_k\rangle\frac{d\varphi_{k_0}(z_k)}{dz_k} \tag{4.4}$$

在 ζ_k 平面, 式(4.4)可化为

$$\langle k_1,k_2,k_D\rangle=2\sqrt{\frac{\pi}{a}}\lim_{\zeta_k\to 1}\mathrm{Re}\sum_{k=1}^{3}\langle 1,-\mu_k,-\lambda_k\rangle\frac{d\Phi_{k_0}(\zeta_k)}{d\zeta_k} \tag{4.5}$$

利用式(3.15),则由式(4.5)得

$$\langle k_1,k_2,k_D\rangle=2\sqrt{\frac{\pi}{a}}\langle f_{10}(1),-f_{20}(1),-f_{D0}(1)\rangle \tag{4.6}$$

其中

$$\left.\begin{aligned}
f_{10}(1) &= \lim_{\zeta_k \to 1} f_1'(\zeta_k) = 2\mathrm{Re}\sum_{k=1}^{3}\frac{A_k}{1-\zeta_{k_0}} \\
f_{20}(1) &= \lim_{\zeta_k \to 1} f_2'(\zeta_k) = 2\mathrm{Re}\sum_{k=1}^{3}\frac{\mu_k A_k}{1-\zeta_{k_0}} \\
f_{D0}(1) &= \lim_{\zeta_k \to 1} f_{D_0}'(\zeta_k) = f_{30}(1) + 2\varepsilon_0 \mathrm{Im}\sum_{n=1}^{\infty} n a_n^0 \\
f_{30}(1) &= \lim_{\zeta_k \to 1} f_3'(\zeta_k) = 2\mathrm{Re}\sum_{k=1}^{3}\frac{\lambda_k A_k}{1-\zeta_{k_0}}
\end{aligned}\right\} \tag{4.7a~b}$$

利用式(3.21),(4.1)得

$$\mathrm{Im}\sum_{n=1}^{\infty} n a_n^0 = \frac{1}{2\varepsilon_0 \mathrm{Im}\rho_0}\mathrm{Im}\sum_{n=1}^{\infty} n c_n^0 \tag{4.8}$$

把式(3.19)代入式(4.8)可得

$$\mathrm{Im}\sum_{n=1}^{\infty} n a_n^0 = \frac{1}{2\varepsilon_0 \mathrm{Im}\rho_0}\mathrm{Im}\left[-\sum_{k=1}^{3}\sum_{j=1}^{3}\kappa_k \Lambda_{kj} f_{j_0}(1) + f_{40}(1)\right] \tag{4.9}$$

其中

$$f_{40}(1) = \lim_{\zeta_k \to 1} f_4'(\zeta_k) = 2\mathrm{Re}\sum_{k=1}^{3}\frac{\kappa_k A_k}{1-\zeta_{k_0}} \tag{4.10}$$

注意到$f_{j0}(1)$, $f_{40}(1)$均为实数,同时又可证明对于横观各向同性体, $\sum_{k=1}^{3}\kappa_k \Lambda_{k2}$也为实数(见附录A),则由式(4.9)得

$$\mathrm{Im}\sum_{n=1}^{\infty} n a_n^0 = -\frac{1}{2\varepsilon_0}\left[\frac{\mathrm{Im}\sum_{k=1}^{3}\kappa_k \Lambda_{k1}}{\mathrm{Im}\sum_{k=1}^{3}\kappa_k \Lambda_{k1}} f_{10}(1) + f_{30}(1)\right] \tag{4.11}$$

把式(4.11)代入式(4.7c)

$$f_{D0}(1) = -\frac{\mathrm{Im}\sum_{k=1}^{3}\kappa_k \Lambda_{k1}}{\mathrm{Im}\sum_{k=1}^{3}\kappa_k \Lambda_{k3}} f_{10}(1) \tag{4.12}$$

把式(4.12)代入式(4.6),并利用:

$$\frac{1}{1-\zeta_{k_0}} = \frac{1}{2}\left[1-\sqrt{\frac{z_{k_0}+a}{z_{k_0}-a}}\right]$$

可得

$$\left.\begin{aligned}
k_1 &= 2\sqrt{\frac{\pi}{a}}\mathrm{Re}\sum_{k=1}^{3} A_k\left(k1-\sqrt{\frac{z_{k_0}+a}{z_{k_0}-a}}\right) \\
k_2 &= -2\sqrt{\frac{\pi}{a}}\mathrm{Re}\sum_{k=1}^{3}\mu_k A_k\left(1-\sqrt{\frac{z_{k_0}+a}{z_{k_0}-a}}\right)
\end{aligned}\right\} = 0 \tag{4.13}$$

$$k_D=\frac{\operatorname{Im}\sum_{k=1}^{3}\kappa_k\Lambda_{k1}}{\operatorname{Im}\sum_{k=1}^{3}\kappa_k\Lambda_{k3}}k_1 \tag{4.14}$$

4.2 裂纹上表面作用任一集中载时的解

当集中载作用在裂纹上表面 $z_{k_0}=x_0$ 处时, 式(4.13) 可化为

$$k_1=2\sqrt{\frac{\pi}{a}}\operatorname{Re}\sum_{k=1}^{3}A_k\left(1+i\sqrt{\frac{a+x}{a-x}}\right),\quad k_2=-2\sqrt{\frac{\pi}{a}}\operatorname{Re}\sum_{k=1}^{3}\mu_kA_k\left(1+i\sqrt{\frac{a+x}{a-x}}\right) \tag{4.15}$$

4.3 裂纹上下表面作用一对自平衡集中载时的解

根据式(4.13), 由叠加原理得:

$$k_1=-2\sqrt{\frac{\pi}{a}}\sqrt{\frac{a+x}{a-x}}\operatorname{Im}\sum_{k=1}^{3}A_k,\quad k_2=2\sqrt{\frac{\pi}{a}}\sqrt{\frac{a+x}{a-x}}\operatorname{Im}\sum_{k=1}^{3}\mu_kA_k \tag{4.16}$$

把式(3.8a、b) 代入式(4.16) 得

$$k_1=\frac{T_{20}}{2\sqrt{\pi a}}\sqrt{\frac{a+x}{a-x}},\quad k_2=\frac{T_{10}}{2\sqrt{\pi a}}\sqrt{\frac{a+x}{a-x}} \tag{4.17}$$

式(4.17) 和(4.14) 表明此时 k_D 与电载荷无关•

4.4 裂纹上下表面作用均布外载(σ_{21}^0, σ_{22}^0, D_2^0) 时的解

$$k_1=\sqrt{\pi a}\sigma_{22}^0,\quad k_2=\sqrt{\pi a}\sigma_{12}^0,\quad k_D=\frac{\operatorname{Im}\sum_{k=1}^{3}\kappa_k\Lambda_{k_1}}{\operatorname{Im}\sum_{k=1}^{3}\kappa_k\Lambda_{k_3}}\sqrt{\pi a}\sigma_{22}^0 \tag{4.18}$$

当裂纹上下表面仅作用等值反号的均布电荷 q_n 时

$$k_1=k_2=k_D=0 \tag{4.19}$$

实际上, 此时 $A_k=0$, 由此得 $\Phi_k(\zeta_k)=0$, 即此时介质内的场强均为零, 而在裂纹内部电场为 $E_2^0=q_n/\varepsilon_0$•

4.5 无限远处作用均布外载(σ_{11}^∞, σ_{21}^∞, σ_{22}^∞, D_1^∞, D_2^∞) 时的解

$$k_1=\sqrt{\pi a}\sigma_{22}^\infty,\quad k_2=\sqrt{\pi a}\sigma_{12}^\infty,\quad k_D=\frac{\operatorname{Im}\sum_{k=1}^{3}\kappa_k\Lambda_{k_1}}{\operatorname{Im}\sum_{k=1}^{3}\kappa_k\Lambda_{k_3}}\sqrt{\pi a}\sigma_{22}^\infty \tag{4.20}$$

而文献[18] 的结果为 $k_D=\sqrt{\pi a}D_2^\infty$, 由式(3.2), (3.3), (3.6), (3.21) 可得此时裂纹内部的电场为

$$D_2^0=D_2^\infty-\frac{\operatorname{Im}\sum_{k=1}^{3}\kappa_k\Lambda_{k_1}}{\operatorname{Im}\sum_{k=1}^{3}\kappa_k\Lambda_{k_3}}\sigma_{22}^\infty,\quad E_2^0=\frac{D_2^0}{\varepsilon_0} \tag{4.21}$$

式(4.21) 表明此时在裂纹内部的电场为常数, 显然 $D_2^0\neq 0$

§5. 结　　论

应用复变函数的方法, 研究了含一椭圆孔或裂纹横观各向同性压电体的二维问题, 给出了

Green 函数解和场强因子基本解• 结果表明:

(1) 电位移的奇异性取决于应力的奇异性•

(2) 均匀的电载荷既不会引起电场的奇异性,也不会引起应力的奇异性•

(3) 在均匀外载作用下,裂纹表面的电位移法向分量为一常数,而不等于零•

(4) 空气的介电常数 ε_0 仅对裂纹内部的电场有影响,而介质内的场强与 ε_0 无关•

(5) 对于裂纹问题,在裂纹表面假设 $D_n = 0$ 可能会导致错误的结果•

附录 A

$$\sum_{k=1}^{3}\kappa_k\Lambda_{k2} = \sum_{k=1}^{3}(b_{13}+\delta_{11}\lambda_k)\mu_k\Lambda_{k2} = b_{13}\sum_{k=1}^{3}\mu_k\Lambda_{k2}+\delta_{11}\sum_{k=1}^{3}\lambda_k\mu_k\Lambda_{k2} \tag{A1}$$

其中

$$\Lambda = [\Lambda_{kj}] = \frac{1}{\Delta}\begin{bmatrix}\mu_2\lambda_3-\mu_3\lambda_2 & \lambda_2-\lambda_3 & \mu_3-\mu_2\\ \mu_3\lambda_1-\mu_1\lambda_3 & \lambda_3-\lambda_1 & \mu_1-\mu_3\\ \mu_1\lambda_2-\mu_2\lambda_1 & \lambda_1-\lambda_2 & \mu_2-\mu_1\end{bmatrix} \tag{A2}$$

$$\Delta = (\lambda_2-\lambda_3)\mu_1+(\lambda_3-\lambda_1)\mu_2+(\lambda_1-\lambda_2)\mu_3 \tag{A3}$$

由式(A2)易验证

$$\sum_{k=1}^{3}\mu_k\Lambda_{k2} = 1 \tag{A5}$$

$$\sum_{k=1}^{3}\lambda_k\mu_k\Lambda_{k2} = \frac{1}{\Delta}[(\mu_1\lambda_1\lambda_2-\mu_1\lambda_1\lambda_3)+(\mu_2\lambda_2\lambda_3-\mu_3\lambda_2\lambda_3)-(\mu_2\lambda_2\lambda_1-\mu_3\lambda_3\lambda_1)] \tag{A6}$$

对于横观各向同性压电材料,文献[15]的研究结果表明:

$$\mathrm{Re}\,\mu_1 = \mathrm{Im}\,\lambda_1 = 0,\quad \mu_3 = -\overline{\mu_2},\quad \lambda_3 = \overline{\lambda_2} \tag{A7}$$

利用式(A7),式(A6),(A3)可化为

$$\sum_{k=1}^{3}\lambda_k\mu_k\Lambda_{k2} = \frac{1}{\Delta}[(\mu_1\lambda_1\lambda_2+\overline{\mu_1\lambda_1\lambda_2})+(\mu_2\lambda_2\overline{\lambda_2}+\overline{\mu_2\lambda_2}\lambda_2)-(\mu_2\lambda_2\lambda_1+\overline{\mu_2\lambda_2\lambda_1})] \tag{A8}$$

$$\Delta = (\lambda_2\mu_1+\overline{\lambda_2\mu_1})+(\overline{\lambda_2}\mu_2+\lambda_2\overline{\mu_2})-(\lambda_1\mu_2+\overline{\lambda_1\mu_2}) \tag{A9}$$

综合式(A1, A5, A8, A9)知 $\sum_{k=1}^{3}\kappa_k\Lambda_{k2}$ 为实数•

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The Fundamental Solutions for the Plane Problem in Piezoelectric Media with an Elliptic Hole or a Crack

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Abstract

Based on the complex potential method, the Green's functions of the plane problem in transversely isotropic piezoelectric media with an elliptic hole are obtained in terms of exact electric boundary conditions at the rim of the hole. When the elliptic hole degenerates into a crack, the fundamental solutions for the field intensity factors are given. The general solutions for concentrated and distributed loads applied on the surface of the hole or crack are produced through the superposition of the fundamental solustions. With the aid of these soltuions, some erroneous results provided previously in other works are pointed out. More important is that these solutions can be used as the fundamental solutions of boundary method to solve more practical problems in piezoelectric media.

Key words piezoelectric media, elliptic hole, crack, plane strain, fundamental solution