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扁薄锥壳非对称大变形问题

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摘 要

本文用双参数摄动法研究了扁锥壳非对称大变形问题, 求得了在线性载荷作用下的扁锥壳变形的三次近似解析解并绘出了摄动点的挠度与载荷的特征曲线• 应用本文方法还可对其它板壳的非轴对称大变形问题进行讨论• 本文通过算例对平板及不同初挠度的扁锥壳大挠度变形进行了讨论•

关键词 扁薄锥壳 非对称 大变形 双参数摄动法
中图分类号 O405

§ 1. 引 言

扁薄锥壳非对称问题的研究在理论方面及应用方面都有着重要的意义• 扁锥壳作为压力容器的一类部件由于其加工制造的难度较小, 在工程中使用非常普遍• 关于扁锥壳非轴对称非线性问题的研究, 就我们目前掌握的资料中, 尚无发现• 王新志教授等人最近几年在研究圆薄板和扁球壳的非对称非线性问题方面做了大量的工作[1－4]• 随着在压力容器设计中常规设计法逐渐要被应力分类设计法所代替的发展趋势, 要求对扁锥壳非轴对称非线性大变形问题进行较精确的应力及变形计算• 我们把扁锥壳看作具有初挠度的圆薄板, 首先采用双参数摄动法把具有初挠度的板的基本方程变为一系列可逐个求解的线性偏微分方程组, 对其中的每一个偏微分方程可采用 Fourier 级数化为常微分方程进行求解•

本文采用的双参数摄动法对于非轴对称问题具有较广的适用性• 本文的结果可为有关工程问题的优化设计提供理论依据•

§ 2. 基本方程及边界条件

任意扁薄壳看作具有初挠度的薄板, 设初挠度为 $w_0 = w_0(x, y) = w_0(r, \theta)$ 可得:
中面变形几何方程[5, 6]

$$la \quad \varepsilon_r = \frac{\partial u_r}{\partial r} + \frac{\partial w_0}{\partial r}\frac{\partial w}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 \tag{2.1}$$

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$$\varepsilon_\theta = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} + \frac{1}{r^2}\frac{\partial w_0}{\partial \theta}\frac{\partial w}{\partial \theta} + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 \tag{2.2}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial w_0}{\partial r}\frac{\partial w}{\partial \theta} + \frac{1}{r}\frac{\partial w_0}{\partial \theta}\frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta} \tag{2.3}$$

$$\kappa_r = -\frac{\partial^2 w}{\partial r^2} \tag{2.4}$$

$$\kappa_\theta = -\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} - \frac{1}{r}\frac{\partial w}{\partial r} \tag{2.5}$$

$$\chi_{r\theta} = -\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial w}{\partial \theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right) \tag{2.6}$$

本构方程

$$N_r = \frac{Et}{1-\mu^2}(\varepsilon_r + \mu\varepsilon_\theta) \tag{2.7}$$

$$M_r = D(\kappa_r + \mu\kappa_\theta) \tag{2.8}$$

$$N_\theta = \frac{Et}{1-\mu^2}(\varepsilon_\theta + \mu\varepsilon_r) \tag{2.9}$$

$$M_\theta = D(\kappa_\theta + \mu\kappa_r) \tag{2.10}$$

$$N_{r\theta} = \frac{Et}{2(1+\mu)}\gamma_{r\theta} \tag{2.11}$$

$$M_{r\theta} = D(1-\mu)\chi_{r\theta} \tag{2.12}$$

其中 $D = Et^3/[12(1-\mu^2)]$, E 为板的弹性模量, t 为板的厚度, μ 为材料的泊松比•

受力平衡方程

$$\frac{N_r - N_\theta}{r} + \frac{\partial N_r}{\partial r} + \frac{1}{r}\frac{\partial N_{r\theta}}{\partial \theta} = 0 \tag{2.13}$$

$$\frac{1}{r}\frac{\partial N_\theta}{\partial \theta} + \frac{2N_{r\theta}}{r} + \frac{\partial N_{r\theta}}{\partial r} = 0 \tag{2.14}$$

$$-\frac{Q_r}{r} - \frac{\partial Q_r}{\partial r} - \frac{1}{r}\frac{\partial Q_\theta}{\partial \theta} = N_r\frac{\partial^2 w}{\partial r^2} + 2N_{r\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right) + N_\theta\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) + q \tag{2.15}$$

$$-\frac{1}{r}\frac{\partial M_\theta}{\partial \theta} - \frac{2M_{r\theta}}{r} - \frac{\partial M_{r\theta}}{\partial r} + Q_\theta = 0 \tag{2.16}$$

$$\frac{M_r}{r} + \frac{\partial M_r}{\partial r} + \frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta} - \frac{M_\theta}{r} - Q_r = 0 \tag{2.17}$$

其中 $w = w_0 + w$, q 为外载集度• 把(2.1) ~ (2.12)、(2.16)、(2.17) 代入方程(2.13) ~ (2.15) 后化简得

基本方程

$$\begin{aligned} & r\frac{\partial^2 u_r}{\partial r^2} + \frac{\partial u_r}{\partial r} - \frac{u_r}{r} + \frac{1+\mu}{2}\frac{\partial^2 u_\theta}{\partial r \partial \theta} - \frac{3-\mu}{2r}\frac{\partial u_\theta}{\partial \theta} + \frac{1-\mu}{2r}\frac{\partial^2 u_r}{\partial \theta^2} \\ & = -2\left\{ r\frac{\partial}{\partial r}\left[\frac{\partial w_0}{\partial r}\frac{\partial w}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 + \frac{\mu}{r^2}\frac{\partial w_0}{\partial \theta}\frac{\partial w}{\partial \theta} + \frac{\mu}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2\right]\right. \\ & \quad + (1-\mu)\left[\frac{\partial w_0}{\partial r}\frac{\partial w}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 - \frac{1}{r^2}\frac{\partial w_0}{\partial \theta}\frac{\partial w}{\partial \theta} - \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2\right] \\ & \quad \left. + \frac{1}{2r}\frac{\partial}{\partial \theta}\left[\frac{\partial w_0}{\partial r}\frac{\partial w}{\partial \theta} + \frac{\partial w_0}{\partial \theta}\frac{\partial w}{\partial r} + \frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta}\right]\right\} \end{aligned} \tag{2.18}$$

$$\frac{2}{r}\frac{\partial^2 u_\theta}{\partial\theta^2}+\frac{3-\mu}{r}\frac{\partial u_r}{\partial\theta}+(1+\mu)\frac{\partial^2 u_r}{\partial r\partial\theta}+(1-\mu)\left(\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}+r\frac{\partial^2 u_\theta}{\partial r^2}\right)$$

$$=-\left\{(1-\mu)r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w_0}{\partial r}\frac{\partial w}{\partial\theta}+\frac{1}{r}\frac{\partial w_0}{\partial\theta}\frac{\partial w}{\partial r}+\frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial\theta}\right)\right.$$

$$+\frac{2(1-\mu)}{r}\left(\frac{\partial w_0}{\partial r}\frac{\partial w}{\partial\theta}+\frac{\partial w}{\partial r}\frac{\partial w_0}{\partial\theta}+\frac{\partial w}{\partial r}\frac{\partial w}{\partial\theta}\right)+2\frac{\partial}{\partial\theta}\left[\frac{1}{r^2}\frac{\partial w_0}{\partial\theta}\frac{\partial w}{\partial\theta}\right.$$

$$\left.\left.+\frac{1}{2r^2}\left(\frac{\partial w}{\partial\theta}\right)^2+\mu\frac{\partial w_0}{\partial r}\frac{\partial w}{\partial r}+\frac{\mu}{2}\left(\frac{\partial w}{\partial r}\right)^2\right]\right\} \tag{2.19}$$

$$\nabla^4 w=\frac{12}{t^2}\left\{\left(\frac{\partial^2 w_0}{\partial r^2}+\frac{\partial^2 w}{\partial r^2}\right)\left[\frac{\partial u_r}{\partial r}+\frac{\mu}{r}u_r+\frac{\mu}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial w_0}{\partial r}\frac{\partial w}{\partial r}+\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2\right.\right.$$

$$\left.+\frac{\mu}{r^2}\frac{\partial w_0}{\partial\theta}\frac{\partial w}{\partial\theta}+\frac{\mu}{2r^2}\left(\frac{\partial w}{\partial\theta}\right)^2\right]+\frac{1-\mu}{r^2}\left(\frac{\partial u_r}{\partial\theta}+r\frac{\partial u_\theta}{\partial r}-u_\theta+\frac{\partial w_0}{\partial r}\frac{\partial w}{\partial\theta}\right.$$

$$\left.+\frac{\partial w}{\partial r}\frac{\partial w_0}{\partial\theta}+\frac{\partial w}{\partial r}\frac{\partial w}{\partial\theta}\right)\left(\frac{\partial^2 w_0}{\partial r\partial\theta}+\frac{\partial^2 w}{\partial r\partial\theta}-\frac{1}{r}\frac{\partial w_0}{\partial\theta}-\frac{1}{r}\frac{\partial w}{\partial\theta}\right)$$

$$+\frac{1}{r}\left[\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}+\mu\frac{\partial u_r}{\partial r}+\frac{1}{r^2}\frac{\partial w_0}{\partial\theta}\frac{\partial w}{\partial\theta}+\frac{1}{2r^2}\left(\frac{\partial w}{\partial\theta}\right)^2+\mu\frac{\partial w_0}{\partial r}\frac{\partial w}{\partial r}\right.$$

$$\left.\left.+\frac{\mu}{2}\left(\frac{\partial w}{\partial r}\right)^2\right]\left(\frac{\partial w_0}{\partial r}+\frac{\partial w}{\partial r}+\frac{1}{r}\frac{\partial^2 w_0}{\partial\theta^2}+\frac{1}{r}\frac{\partial^2 w}{\partial\theta^2}\right)+q\frac{1-\mu^2}{Et}\right\} \tag{2.20}$$

其中 $\nabla^4=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2$

引入下列无量纲量

$$x=\frac{r}{R},\quad u=\frac{Ru_r}{t^2},\quad v=\frac{Ru_\theta}{t^2},\quad W=\frac{w}{t},\quad W_0=\frac{w_0}{t},\quad q=\frac{R^4}{Dt}q,\quad y=\theta$$

把基本方程无量纲化,并代入本问题(如图 1 所示)的初挠度(扁锥面方程)

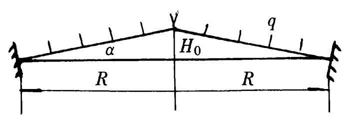


图 1 扁薄锥壳结构示意图

$$w_0=w_0(r)=(R-r)\tan\alpha\approx(R-r)\alpha$$

即 $$W_0=\frac{w_0}{t}=\frac{R\alpha}{t}(1-x)$$

则

$$\frac{\partial W_0}{\partial x}=-\frac{R\alpha}{t}=C,\quad \frac{\partial W_0}{\partial y}=0,\quad \frac{\partial^2 W_0}{\partial x^2}=0,\quad \frac{\partial^2 W_0}{\partial x\partial y}=0,\quad \frac{\partial^2 W_0}{\partial y^2}=0$$

可得本问题的基本方程

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}-\frac{u}{x}+\frac{1+\mu}{2}\frac{\partial^2 v}{\partial x\partial y}-\frac{3-\mu}{2x}\frac{\partial v}{\partial y}+\frac{1-\mu}{2x}\frac{\partial^2 u}{\partial y^2}$$

$$=-\left\{x\frac{\partial}{\partial x}\left[C\frac{\partial W}{\partial x}+\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2+\frac{\mu}{2x^2}\left(\frac{\partial W}{\partial y}\right)^2\right]+(1-\mu)\left[C\frac{\partial W}{\partial x}\right.\right.$$

$$\left.\left.+\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2-\frac{1}{2x^2}\left(\frac{\partial W}{\partial y}\right)^2+\frac{1}{2x}\left(C\frac{\partial^2 W}{\partial y^2}+\frac{\partial^2 W}{\partial x\partial y}\frac{\partial W}{\partial y}+\frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial y^2}\right)\right]\right\} \tag{2.21}$$

$$\frac{2}{x}\frac{\partial^2 v}{\partial y^2}+\frac{3-\mu}{x}\frac{\partial u}{\partial y}+(1+\mu)\frac{\partial^2 u}{\partial x\partial y}+(1-\mu)\left(\frac{\partial v}{\partial x}-\frac{v}{x}+x\frac{\partial^2 v}{\partial x^2}\right)$$

$$=-\left\{(1-\mu)x\frac{\partial}{\partial x}\left(\frac{1}{x}C\frac{\partial W}{\partial y}+\frac{1}{x}\frac{\partial W}{\partial x}\frac{\partial W}{\partial y}\right)+\frac{2(1-\mu)}{x}\left(C\frac{\partial W}{\partial y}+\frac{\partial W}{\partial x}\frac{\partial W}{\partial y}\right)\right.$$

$$\left.+2\frac{\partial}{\partial y}\left[\frac{1}{2x^2}\left(\frac{\partial W}{\partial y}\right)^2+\mu C\frac{\partial W}{\partial x}+\frac{\mu}{2}\left(\frac{\partial W}{\partial x}\right)^2\right]\right\} \tag{2.22}$$

$$\therefore \nabla_0^4 W = 12\left\{\frac{\partial^2 W}{\partial x^2}\left[\frac{\partial u}{\partial x}+\frac{\mu}{x}u+\frac{\mu}{x}\frac{\partial v}{\partial y}+C\frac{\partial W}{\partial x}+\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2+\frac{\mu}{2x^2}\left(\frac{\partial W}{\partial y}\right)^2\right]\right.$$

$$+\frac{1-\mu}{x^2}\left(\frac{\partial u}{\partial y}+x\frac{\partial v}{\partial x}-v+C\frac{\partial W}{\partial y}+\frac{\partial W}{\partial x}\frac{\partial W}{\partial y}\right)\left(\frac{\partial^2 W}{\partial x\partial y}-\frac{1}{x}\frac{\partial W}{\partial y}\right)$$

$$+\frac{1}{x}\left[\frac{1}{x}\frac{\partial v}{\partial y}+\frac{u}{x}+\mu\frac{\partial u}{\partial x}+\frac{1}{2x^2}\left(\frac{\partial W}{\partial y}\right)^2+\mu C\frac{\partial W}{\partial x}+\frac{\mu}{2}\left(\frac{\partial W}{\partial x}\right)^2\right]$$

$$\left.\cdot\left(C+\frac{\partial W}{\partial x}+\frac{1}{x}\frac{\partial^2 W}{\partial y^2}\right)\right\}+q \tag{2.23}$$

其中 $\nabla_0^4=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2$

为了便于说明求解方法, 在这里只讨论周边固支的情况•

边界条件

$$x\to 0,\quad W,\ u,\ v,\ \frac{\partial W}{\partial x},\ \frac{\partial u}{\partial x},\ \frac{\partial v}{\partial x}\ \text{有限} \tag{2.24}$$

$$x=1,\quad W=u=v=\frac{\partial W}{\partial x}=0 \tag{2.25}$$

§3. 问题的求解

由于本问题是具有初挠度的非线性问题, 我们采用双参数摄动法求解本问题• 首先我们可取某点挠度 w_m 的无量纲化作为摄动参数 ε, 再取与初挠度相关的 C 作为第二个摄动参数 η, W, u, v, q 可写为 ε 和 η 的幂级数形式

$$W=\sum_{i+j=1}^{\infty}\sum_{i=1}^{i+j}W_{ij}\varepsilon^i\eta^j \tag{3.1}$$

$$u=\sum_{i+j=1}^{\infty}\sum_{i=1}^{i+j}u_{ij}\varepsilon^i\eta^j \tag{3.2}$$

$$v=\sum_{i+j=1}^{\infty}\sum_{i=1}^{i+j}v_{ij}\varepsilon^i\eta^j \tag{3.3}$$

$$q=\sum_{i+j=1}^{\infty}\sum_{i=1}^{i+j}q_{ij}\varepsilon^i\eta^j=q\sum_{i+j=1}^{\infty}\sum_{i=1}^{i+j}\alpha_{ij}\varepsilon^i\eta^j \tag{3.4}$$

其中 α_{ij} 为与 x, y 无关的常数• 在摄动点 (x_0, y_0) 无量纲挠度为

$$\varepsilon=\sum_{i+j=1}^{\infty}\sum_{i=1}^{i+j}W_{ij}(x_0,y_0)\varepsilon^i\eta^j$$

本问题取锥壳中心点为摄动点, 则有摄动条件

$$W_{ij}(0,y)=\begin{cases}1, & i=1\ 且\ j=0\\ 0, & i\neq 1\ 或\ j\neq 0\end{cases} \tag{3.5}$$

(3.1)~(3.4) 代入本问题的基本方程和边界条件, 并取 $\varepsilon^a\eta^b(a\geqslant 1,\ b\geqslant 0)$ 的同次幂系数, 得到如下问题

$$x\frac{\partial^2 u_{a,b}}{\partial x^2}+\frac{\partial u_{a,b}}{\partial x}-\frac{u_{a,b}}{x}+\frac{1-\mu}{2x}\frac{\partial^2 u_{a,b}}{\partial y^2}+\frac{1}{2}\left[(1+\mu)\frac{\partial^2 v_{a,b}}{\partial x\partial y}-\frac{3-\mu}{x}\frac{\partial v_{a,b}}{\partial y}\right]$$

$$=-\left[x\frac{\partial}{\partial x}\frac{\partial W_{a,b-1}}{\partial x}+(1-\mu)\frac{\partial W_{a,b-1}}{\partial x}+\frac{1-\mu}{2x}\frac{\partial^2 W_{a,b-1}}{\partial y^2}\right]$$

$$+\sum_{\substack{j=0\\l=b-j}}^{b}\sum_{\substack{i=1\\k=a-i}}^{a-1}\left[\frac{x}{2}\frac{\partial}{\partial x}\left(\frac{\partial W_{i,j}}{\partial x}\frac{\partial W_{k,l}}{\partial x}+\frac{\mu}{x^2}\frac{\partial W_{i,j}}{\partial y}\frac{\partial W_{k,l}}{\partial y}+\frac{1-\mu}{2}\left(\frac{\partial W_{i,j}}{\partial x}\frac{\partial W_{k,l}}{\partial x}\right.\right.\right.$$

$$\left.\left.-\frac{1}{x^2}\frac{\partial W_{i,j}}{\partial y}\frac{\partial W_{k,l}}{\partial y}\right)+\frac{1-\mu}{2x}\left(\frac{\partial^2 W_{i,j}}{\partial x\partial y}\frac{\partial W_{k,l}}{\partial y}+\frac{\partial W_{i,j}}{\partial x}\frac{\partial^2 W_{k,l}}{\partial y^2}\right)\right] \tag{3.6}$$

$$x\frac{\partial^2 v_{a,b}}{\partial x^2}+\frac{\partial v_{a,b}}{\partial x}-\frac{v_{a,b}}{x}+\frac{2}{(1-\mu)x}\frac{\partial^2 v_{a,b}}{\partial y^2}+\frac{1}{1-\mu}\left[(1+\mu)\frac{\partial^2 u_{a,b}}{\partial x\partial y}+\frac{3-\mu}{x}\frac{\partial u_{a,b}}{\partial y}\right]$$

$$=-\left[x\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial W_{a,b-1}}{\partial y}\right)+\frac{2}{x}\frac{\partial W_{a,b-1}}{\partial y}+\frac{2\mu}{1-\mu}\frac{\partial^2 W_{a,b-1}}{\partial x\partial y}\right.$$

$$+\sum_{\substack{j=0\\l=b-j}}^{b}\sum_{\substack{i=1\\k=a-i}}^{a-1}\left[x\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial W_{i,j}}{\partial x}\frac{\partial W_{k,l}}{\partial y}+\frac{2}{x}\frac{\partial W_{i,j}}{\partial x}\frac{\partial W_{k,l}}{\partial y}\right.\right.$$

$$\left.\left.\left.+\frac{1}{1-\mu}\frac{\partial}{\partial y}\left(\frac{1}{x^2}\frac{\partial W_{i,j}}{\partial y}\frac{\partial W_{k,l}}{\partial y}+\mu\frac{\partial W_{i,j}}{\partial x}\frac{\partial W_{k,l}}{\partial x}\right)\right]\right] \tag{3.7}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{1}{x}\frac{\partial}{\partial x}+\frac{1}{x^2}\frac{\partial^2}{\partial y^2}\right)^2 W_{a,b}=\frac{12}{x}\left(\mu\frac{\partial u_{a,b-1}}{\partial x}+\frac{u_{a,b-1}}{x}+\frac{1}{x}\frac{\partial v_{a,b-1}}{\partial y}\right)+\frac{12\mu}{x}\frac{\partial W_{a,b-2}}{\partial x}$$

$$+12\sum_{\substack{j=0\\l=b-j}}^{b}\sum_{\substack{i=1\\k=a-i}}^{a-1}\left[\frac{\partial^2 W_{i,j}}{\partial x^2}\left(\frac{\partial u_{k,l}}{\partial x}+\frac{\mu}{x}u_{k,l}+\frac{\mu}{x}\frac{\partial v_{k,l}}{\partial y}\right)+\frac{1-\mu}{x^2}\left(\frac{\partial^2 W_{i,j}}{\partial x\partial y}-\frac{1}{x}\frac{\partial W_{i,j}}{\partial y}\right)\right.$$

$$\left.\cdot\left(\frac{\partial u_{k,l}}{\partial y}+x\frac{\partial v_{k,l}}{\partial x}-v_{k,l}\right)+\frac{1}{x}\left(\frac{\partial W_{i,j}}{\partial x}+\frac{1}{x}\frac{\partial^2 W_{i,j}}{\partial y^2}\right)\left(\frac{1}{x}\frac{\partial v_{k,l}}{\partial y}+\frac{u_{k,l}}{x}+\mu\frac{\partial u_{k,l}}{\partial x}\right)\right]$$

$$+\sum_{\substack{j=0\\l=b-j-1}}^{b-1}\sum_{\substack{i=1\\k=a-i}}^{a-1}\left[\frac{\partial^2 W_{i,j}}{\partial x^2}\frac{\partial W_{k,l}}{\partial x}+\frac{1-\mu}{x^2}\left(\frac{\partial^2 W_{i,j}}{\partial x\partial y}-\frac{1}{x}\frac{\partial W_{i,j}}{\partial y}\right)\frac{\partial W_{k,l}}{\partial y}\right.$$

$$\left.+\frac{1}{2x}\left(\frac{1}{x^2}\frac{\partial W_{i,j}}{\partial y}\frac{\partial W_{k,l}}{\partial y}+\mu\frac{\partial W_{i,j}}{\partial x}\frac{\partial W_{k,l}}{\partial x}\right)+\frac{\mu}{x}\left(\frac{\partial W_{i,j}}{\partial x}+\frac{1}{x}\frac{\partial^2 W_{i,j}}{\partial y^2}\right)\frac{\partial W_{k,l}}{\partial x}\right]$$

$$+\sum_{\substack{l=0\\n=b-l-j}}^{b}\sum_{j=0}^{b-l}\sum_{\substack{k=1\\m=a-k-i}}^{a-2}\sum_{i=1}^{a-k-1}12\left[\frac{\partial^2 W_{i,j}}{\partial x^2}\left(\frac{1}{2}\frac{\partial W_{k,l}}{\partial x}\frac{\partial W_{m,n}}{\partial x}+\frac{\mu}{2x^2}\frac{\partial W_{k,l}}{\partial y}\frac{\partial W_{m,n}}{\partial y}\right)\right.$$

$$+\frac{1-\mu}{x^2}\left(\frac{\partial^2 W_{i,j}}{\partial x\partial y}-\frac{1}{x}\frac{\partial W_{i,j}}{\partial y}\right)\frac{\partial W_{k,l}}{\partial x}\frac{\partial W_{m,n}}{\partial y}+\frac{1}{x}\left(\frac{\partial W_{i,j}}{\partial x}+\frac{1}{x}\frac{\partial^2 W_{i,j}}{\partial y^2}\right)$$

$$\left.\cdot\left(\frac{1}{2x^2}\frac{\partial W_{k,l}}{\partial y}\frac{\partial W_{m,n}}{\partial y}+\frac{\mu}{2}\frac{\partial W_{k,l}}{\partial x}\frac{\partial W_{m,n}}{\partial x}\right)\right]+q_{a,b} \tag{3.8}$$

边界条件

$$x\to 0,\quad W_{a,b},\ u_{a,b},\ v_{a,b},\ \frac{\partial W_{a,b}}{\partial x},\ \frac{\partial u_{a,b}}{\partial x},\ \frac{\partial v_{a,b}}{\partial x}\ \text{有限} \tag{3.9}$$

$$x=1,\quad W_{a,b}=u_{a,b}=v_{a,b}=\frac{\partial W_{a,b}}{\partial x}=0 \tag{3.10}$$

设方程(3.6)~(3.8)的右边分别为 $F_{1ab}(x,y)$, $F_{2ab}(x,y)$, $F_{3ab}(x,y)$, 取各函数的 Fourier 展式

$$u_{ab}=\sum_{-\infty}^{+\infty}[u_{rk}+iu_{ik}]e^{iky},\quad F_{1ab}(x,y)=\sum_{-\infty}^{+\infty}[F_{1rk}+iF_{1ik}]e^{iky}$$

$$v_{ab}=\sum_{-\infty}^{+\infty}[v_{rk}+iv_{ik}]e^{iky},\quad F_{2ab}(x,y)=\sum_{-\infty}^{+\infty}[F_{2rk}+iF_{2ik}]e^{iky}$$

$$W_{ab}=\sum_{-\infty}^{+\infty}[W_{rk}+iW_{ik}]e^{iky},\quad F_{3ab}(x,y)=\sum_{-\infty}^{+\infty}[F_{3rk}+iF_{3ik}]e^{iky}$$

$$q_{ab} = \sum_{-\infty}^{+\infty} [q_{rk} + iq_{ik}] e^{iky}$$

其中 $u_{rk}, u_{ik}, v_{rk}, v_{ik}, W_{rk}, W_{ik}$ 均为x 的实函数, 且已知

$$[F_{1rk} + iF_{1ik}] = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_{1ab}(x, y) e^{-iky} dy$$

$$[F_{2rk} + iF_{2ik}] = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_{2ab}(x, y) e^{-iky} dy$$

$$[F_{3rk} + iF_{3ik}] = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_{3ab}(x, y) e^{-iky} dy$$

$$[q_{rk} + iq_{ik}] = \frac{1}{2\pi}\int_{-\pi}^{\pi} q_{ab} e^{-iky} dy = \frac{\alpha_{ab}}{2\pi}\int_{-\pi}^{\pi} \overset{+}{q} e^{-iky} dy$$

代入方程(3.6) ~ (3.8) 和边界条件(3.9)、(3.10) 并利用 e^{iky} 的正交性及复函数的性质可得如下问题

$$\left[x\frac{d^2}{dx^2} + \frac{d}{dx} - \frac{1}{x} - \frac{(1-\mu)k^2}{2x}\right] u_{rk} - \frac{k}{2}\left[(1+\mu)\frac{d}{dx} - \frac{3-\mu}{x}\right] v_{ik} = F_{1rk} \tag{3.11a}$$

$$\left[x\frac{d^2}{dx^2} + \frac{d}{dx} - \frac{1}{x} - \frac{(1-\mu)k^2}{2x}\right] u_{ik} + \frac{k}{2}\left[(1+\mu)\frac{d}{dx} - \frac{3-\mu}{x}\right] v_{rk} = F_{1ik} \tag{3.11b}$$

$$\left[x\frac{d^2}{dx^2} + \frac{d}{dx} - \frac{1}{x} - \frac{2k^2}{(1-\mu)x}\right] v_{rk} - \frac{k}{1-\mu}\left[(1+\mu)\frac{d}{dx} + \frac{3-\mu}{x}\right] u_{ik} = F_{2rk} \tag{3.11c}$$

$$\left[x\frac{d^2}{dx^2} + \frac{d}{dx} - \frac{1}{x} - \frac{2k^2}{(1-\mu)x}\right] v_{ik} + \frac{k}{1-\mu}\left[(1+\mu)\frac{d}{dx} + \frac{3-\mu}{x}\right] u_{rk} = F_{2ik} \tag{3.11d}$$

$$\left[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{k^2}{x^2}\right]^2 W_{rk} = q_{rk} + F_{3rk} \tag{3.11e}$$

$$\left[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{k^2}{x^2}\right]^2 W_{ik} = q_{ik} + F_{3ik} \tag{3.11f}$$

边界条件

$$x \to 0, \quad W_{rk},\ W_{ik},\ u_{rk},\ u_{ik},\ v_{rk},\ v_{ik},\ \frac{dW_{rk}}{dx},\ \frac{dW_{ik}}{dx}\ \text{有限} \tag{3.12a}$$

$$x = 1, \quad W_{rk} = W_{ik} = u_{rk} = u_{ik} = v_{rk} = v_{ik} = \frac{dW_{rk}}{dx} = \frac{dW_{ik}}{dx} = 0$$

$$(k = 0, \pm 1, \pm 2, \pm 3, \ldots) \tag{3.12b}$$

对方程组(3.11) 中(a)、(b)、(c)、(d) 进行消元可得

$$\mathscr{L}(u_{rk}) = \left[x\frac{d^2}{dx^2} + 3\frac{d}{dx} + \frac{2k^2}{(\mu-1)x}\right] F_{1rk} - \left[\frac{(\mu+1)k}{2(\mu-1)}\frac{d}{dx} + \frac{k}{x}\right] F_{2ik} \tag{3.13a}$$

$$\mathscr{L}(u_{ik}) = \left[x\frac{d^2}{dx^2} + 3\frac{d}{dx} + \frac{2k^2}{(\mu-1)x}\right] F_{1ik} + \left[\frac{(1+\mu)k}{2(\mu-1)}\frac{d}{dx} + \frac{k}{x}\right] F_{2rk} \tag{3.13b}$$

$$(1-\mu)\mathscr{L}(v_{rk}) = \left[x\frac{d^2}{dx^2} + 3\frac{d}{dx} + \frac{(\mu-1)k^2}{2x}\right] F_{2rk} + \left[k(1+\mu)\frac{d}{dx} + \frac{4k}{x}\right] F_{1ik} \tag{3.13c}$$

$$(1-\mu)\mathscr{L}(v_{ik}) = \left[x\frac{d^2}{dx^2} + 3\frac{d}{dx} + \frac{(\mu-1)k^2}{2x}\right] F_{2ik} - \left[k(1+\mu)\frac{d}{dx} + \frac{4k}{x}\right] F_{1rk} \tag{3.13d}$$

$$\nabla_0^4 W_{rk} = q_{rk} + F_{3rk} \tag{3.13e}$$

$$\nabla_0^4 W_{ik} = q_{ik} + F_{3ik} \tag{3.13f}$$

其中算子

$$\mathscr{L}(H)=\left[x^2\frac{d^4}{dx^4}+6x\frac{d^3}{dx^3}+(5-2k^2)\frac{d^2}{dx^2}-\frac{1+2k^2}{x}\frac{d}{dx}+\frac{1-2k^2+k^4}{x^2}\right]H$$
$$=x^{\sqrt{k^2}}\frac{d}{dx}x^{\left(\sqrt{(k-1)^2}-3\sqrt{k^2}-1+S\right)/2}\frac{d}{dx}x^{1-S}\frac{d}{dx}x^{\left(3-3\sqrt{(k-1)^2}+\sqrt{k^2}+S\right)/2}\frac{d}{dx}\left(x^{\sqrt{(k-1)^2}}H\right)$$
$$\nabla_0^4(H)=\left[\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}-\frac{k^2}{x^2}\right]^2H=x^{k-1}\frac{d}{dx}x^{1-2k}\frac{d}{dx}x^{2k-1}\frac{d}{dx}x^{1-2k}\frac{d}{dx}(x^kH)$$
$$S=\sqrt{2}\sqrt{1+3k+k^2+\sqrt{(k-1)^2}-3\sqrt{k^2}+\sqrt{(k-1)^2}\sqrt{k^2}}$$

可对(3.13)中各方程进行积分求解 u_{rk}, u_{ik}, v_{rk}, v_{ik}, W_{rk} 和 W_{ik}, 由边界条件(3.12), 回代方程组(3.11), 同时注意到在 $x=0$ 时 W 的值不受 y 的影响, 即 $\partial W/\partial y|_{x=0}=0$, 亦即 W_{rk} 和 W_{ik} 在 $k\neq 0$ 时 x^0 的系数为 0, 可确定积分常数·

§4. 算　　例

设扁薄锥壳的周边加紧, 其凹面承受的法向载荷为

$$q=q(r,\theta)=\gamma R\left(1-\frac{r}{R}\sin\theta\right)\qquad k$$

其中 γ 为某种液体的比重· 无量纲化后的载荷为

$$q=\frac{R^4}{Dt}q=\frac{\gamma R^5}{Dt}(1-x\sin y)=A(1-x\sin y)$$

展为 Fourier 级数

$$q_{rk}=\begin{cases}A\alpha_{a,b}, & (k=0)\\ 0, & (k=\pm1,\pm2,\pm3,\dots)\end{cases},\qquad q_{ik}=\begin{cases}\pm\dfrac{A\alpha_{a,b}x}{2}, & (k=\pm1)\\ 0, & (k=0,\pm2,\pm3,\dots)\end{cases}$$

4.1 一次近似的边值问题结果

$$u_{1,0}=0,\qquad v_{1,0}=0 \tag{4.1a, b}$$
$$W_{1,0}=(1-x^2)^2\left(1-\frac{x\sin(y)}{3}\right),\qquad \alpha_{1,0}=\frac{64}{A} \tag{4.1c, d}$$

4.2 二次近似的边值问题结果

$$u_{2,0}=\frac{x^2-1}{540(3-\mu)}\Big\{(-1326+1261\mu-273\mu^2)x+(3354-1919\mu+267\mu^2)x^3+(-1626+791\mu-83\mu^2)x^5+(-126+51\mu-3\mu^2)x^7+[(-24+25\mu-3\mu^2)x+(156-97\mu+15\mu^2)x^3+(-264+133\mu-15\mu^2)x^5+(126-57\mu+5\mu^2)x^7]\cos(2y)-6[(2+19\mu-3\mu^2)+(213-143\mu+24\mu^2)x^2+(-447+227\mu-26\mu^2)x^4+(228-103\mu+9\mu^2)x^6]\sin(y)\Big\} \tag{4.2a}$$

$$v_{2,0}=\frac{x^2-1}{540(3-\mu)}\Big\{6[(2+19\mu-3\mu^2)+(27-27\mu+6\mu^2)x^2+(-33+23\mu-4\mu^2)x^4+(12-7\mu+\mu^2)x^6]\cos(y)-[(-24+25\mu-3\mu^2)x+(63-39\mu+6\mu^2)x^3+(-57+31\mu-4\mu^2)x^5+(18-9\mu+\mu^2)x^7]\sin(2y)\Big\} \tag{4.2b}$$

$$W_{2,0}=0,\qquad \alpha_{2,0}=0 \tag{4.2c, d}$$

$$u_{1,1}=\frac{4}{15}(x^2-x)(6-4\mu-4x+\mu x-4x^2+\mu x^2)+\frac{x-1}{315(3-\mu)}[4(9-17\mu+2\mu^2)+(-279+37\mu+8\mu^2)x+(-387+393\mu-88\mu^2)x^2+(327-181\mu+24\mu^2)x^3+(327-181\mu+24\mu^2)x^4]\sin(y) \tag{4.3a}$$

$$v_{1,1}=\frac{x-1}{315(3-\mu)}[4(9-17\mu+2\mu^2)+(-279+37\mu+8\mu^2)x+(261-15\mu$$
$$-24\mu^2)x^2+(-33-\mu+4\mu^2)x^3+(-33-\mu+4\mu^2)x^4]\cos(y) \tag{4.3b}$$

$$W_{1,1}=0,\qquad \alpha_{1,1}=0 \tag{4.3c, d}$$

4.3 三次近似的边值问题结果

$$u_{3,0}=0,\qquad v_{3,0}=0 \tag{4.4a, b}$$

$$W_{3,0}=\sum_{i=0}^{5}\frac{B_{2i}^{(1)}x^{2i+4}}{16(i+1)^2(i+2)^2}-\sum_{i=0}^{5}\frac{B_{2i}^{(1)}x^2}{16(i+1)^2(i+2)}+\sum_{i=0}^{5}\frac{B_{2i}^{(1)}}{16(i+1)(i+2)^2}$$
$$+\frac{A\alpha_{3,0}}{64}(1-x^2)^2-\left[\sum_{i=0}^{5}\frac{B_{2i+1}^{(3)}x^{2i+5}}{16(i+1)(i+2)^2(i+3)}-\sum_{i=0}^{5}\frac{B_{2i+1}^{(3)}x^3}{16(i+1)(i+2)(i+3)}\right.$$
$$\left.+\sum_{i=0}^{5}\frac{B_{2i+1}^{(3)}x}{16(i+2)^2(i+3)}+\frac{A\alpha_{3,0}}{192}x(1-x^2)^2\right]\sin(y)$$
$$+\left[\sum_{i=1}^{5}\frac{B_{2i}^{(2)}x^{2i+4}}{16i(i+1)(i+2)(i+3)}-\sum_{i=1}^{5}\frac{B_{2i}^{(2)}x^4}{16i(i+2)(i+3)}\right.$$
$$\left.+\sum_{i=1}^{5}\frac{B_{2i}^{(2)}x^2}{16(i+1)(i+2)(i+3)}\right]\cos(2y)-\left[\sum_{i=1}^{5}\frac{B_{2i+1}^{(4)}x^{2i+5}}{16i(i+1)(i+3)(i+4)}\right.$$
$$\left.-\sum_{i=1}^{5}\frac{B_{2i+1}^{(4)}x^5}{16i(i+3)(i+4)}+\sum_{i=1}^{5}\frac{B_{2i+1}^{(4)}x^3}{16(i+1)(i+3)(i+4)}\right]\sin(3y) \tag{4.4c}$$

$$\alpha_{3,0}=\frac{-4}{A}\sum_{i=0}^{5}\frac{B_{2i}^{(1)}}{(i+1)(i+2)^2} \tag{4.4d}$$

$$u_{2,1}=0,\qquad v_{2,1}=0 \tag{4.5a, b}$$

$$W_{2,1}=\sum_{i=-1}^{7}\frac{B_i^{(5)}x^{i+4}}{(i+2)^2(i+4)^2}-\sum_{i=-1}^{7}\frac{B_i^{(5)}x^2}{2(i+2)^2(i+4)}+\sum_{i=-1}^{7}\frac{B_i^{(5)}}{2(i+2)(i+4)^2}$$
$$+\frac{A\alpha_{2,1}}{64}(1-x^2)x^2-\left[\sum_{i=0}^{6}\frac{B_i^{(7)}x^{i+4}}{(i+1)(i+3)^2(i+5)}-\sum_{i=0}^{6}\frac{B_i^{(7)}x^3}{2(i+1)(i+3)(i+5)}\right.$$
$$\left.+\sum_{i=0}^{6}\frac{B_i^{(7)}x}{2(i+3)^2(i+5)}+\frac{A\alpha_{2,1}}{192}x(1-x^2)^2\right]\sin(y)+\left[\sum_{i=-1}^{7}\frac{B_i^{(6)}x^{i+4}}{i(i+2)(i+4)(i+6)}\right.$$
$$\left.-\sum_{i=-1}^{7}\frac{B_i^{(6)}x^4}{2i(i+4)(i+6)}+\sum_{i=-1}^{7}\frac{B_i^{(6)}x^2}{2(i+2)(i+4)(i+6)}\right]\cos(2y) \tag{4.5c}$$

$$\alpha_{2,1}=\frac{-32}{A}\sum_{i=-1}^{7}\frac{B_i^{(5)}}{(i+2)(i+4)^2} \tag{4.5d}$$

$$u_{1,2}=0,\qquad v_{1,2}=0 \tag{4.6a, b}$$

$$W_{1,2}=\sum_{i=-1}^{2}\frac{B_i^{(8)}x^{i+4}}{(i+2)^2(i+4)^2}-\sum_{i=-1}^{2}\frac{B_i^{(8)}x^2}{2(i+2)^2(i+4)}+\sum_{i=-1}^{2}\frac{B_i^{(8)}}{2(i+2)(i+4)^2}$$
$$+\frac{A\alpha_{1,2}}{64}(1-x^2)^2-\left[\sum_{i=0}^{3}\frac{B_i^{(9)}x^{i+4}}{(i+1)(i+3)^2(i+5)}-\sum_{i=0}^{3}\frac{B_i^{(9)}x^3}{2(i+1)(i+3)(i+5)}\right.$$
$$\left.+\sum_{i=0}^{3}\frac{B_i^{(9)}x}{2(i+3)^2(i+5)}+\frac{A\alpha_{1,2}}{192}x(1-x^2)^2\right]\sin(y) \tag{4.6c}$$

$$\alpha_{1,2}=\frac{-32}{A}\sum_{i=-1}^{2}\frac{B_i^{(8)}}{(i+2)(i+4)^2} \tag{4.6d}$$

其中

$$B_0^{(1)} = \frac{-8(1+\mu)(457-273\mu)}{45},$$

$$B_2^{(1)} = \frac{16(1+\mu)(2929-3356\mu+795\mu^2)}{45(3-\mu)},$$

$$B_4^{(1)} = \frac{-64(1-\mu^2)(311-104\mu)}{15(3-\mu)},$$

$$B_6^{(1)} = 256(1-\mu^2),$$

$$B_8^{(1)} = \frac{-104(1-\mu^2)}{3},$$

$$B_{10}^{(1)} = \frac{-176(1-\mu^2)}{15},$$

$$B_2^{(2)} = \frac{64(1-\mu)(40+29\mu-12\mu^2)}{45(3-\mu)},$$

$$B_4^{(2)} = \frac{-8(1-\mu^2)(1243-397\mu)}{45(3-\mu)},$$

$$B_6^{(2)} = 104(1-\mu^2),$$

$$B_8^{(2)} = \frac{-200(1-\mu^2)}{3},$$

$$B_{10}^{(2)} = \frac{728(1-\mu^2)}{45},$$

$$B_1^{(3)} = \frac{8(4995-469\mu-4251\mu^2+1197\mu^3)}{135(3-\mu)},$$

$$B_3^{(3)} = \frac{4(12555-2989\mu-11811\mu^2+3717\mu^3)}{45(\mu-3)},$$

$$B_5^{(3)} = \frac{4576(1-\mu^2)}{9},$$

$$B_7^{(3)} = \frac{-2944(1-\mu^2)}{9},$$

$$B_9^{(3)} = \frac{1144(1-\mu^2)}{15},$$

$$B_{11}^{(3)} = \frac{28(1-\mu^2)}{15},$$

$$B_3^{(4)} = \frac{-4(1-\mu)(165+122\mu-51\mu^2)}{135(3-\mu)},$$

$$B_5^{(4)} = \frac{8(1-\mu^2)(303-97\mu)}{135(3-\mu)},$$

$$B_7^{(4)} = -8(1-\mu^2),$$

$$B_9^{(4)} = \frac{664(1-\mu^2)}{135},$$

$$B_{11}^{(4)} = \frac{-52(1-\mu^2)}{45},$$

$$B_{-1}^{(5)} = \frac{(1+\mu)(457-273\mu)}{45},$$

$$B_5^{(5)} = \frac{-752(1-\mu^2)}{9}$$

$$B_6^{(5)} = 0$$

$$B_7^{(5)} = \frac{-181(1-\mu^2)}{35}$$

$$B_{-1}^{(6)} = \frac{(1-\mu)(21+10\mu-3\mu^2)}{45(3-\mu)}$$

$$B_0^{(6)} = 0$$

$$B_1^{(6)} = \frac{-2(1-\mu^2)(31-9\mu)}{15(3-\mu)}$$

$$B_2^{(6)} = \frac{-512(1-\mu^2)(27-8\mu)}{315(3-\mu)}$$

$$B_3^{(6)} = \frac{338(1-\mu^2)}{15}$$

$$B_4^{(6)} = \frac{128(1-\mu^2)(27-8\mu)}{63(3-\mu)}$$

$$B_5^{(6)} = \frac{-10132(1-\mu^2)}{315}$$

$$B_6^{(6)} = 0$$

$$B_7^{(6)} = \frac{479(1-\mu^2)}{63}$$

$$B_0^{(7)} = \frac{-4(1-\mu^2)(31-9\mu)}{5(3-\mu)}$$

$$B_1^{(7)} = \frac{-1024(1+\mu)(45-49\mu+11\mu^2)}{105(3-\mu)}$$

$$B_2^{(7)} = \frac{592(1-\mu^2)}{3}$$

$$B_3^{(7)} = \frac{512(1+\mu)(45-49\mu+11\mu^2)}{35(3-\mu)}$$

$$B_4^{(7)} = \frac{-4772(1-\mu^2)}{15}$$

$$B_5^{(7)} = 0$$

$$B_6^{(7)} = \frac{528(1-\mu^2)}{7}$$

$$B_{-1}^{(8)} = \frac{-32(1+\mu)(3-2\mu)}{5}$$

$$B_0^{(8)} = 32(1-\mu^2)$$

$$B_1^{(8)} = 0$$

$$B_2^{(8)} = \frac{-64(1-\mu^2)}{5}$$

$B_0^{5)} = \dfrac{256(1+\mu)(3-2\mu)}{5}$,　　$B_0^{9)} = \dfrac{32(1-\mu^2)(27-8\mu)}{35(3-\mu)}$

$B_1^{5)} = -228(1-\mu^2)$,　　$B_1^{9)} = \dfrac{-64(1-\mu^2)}{5}$

$B_2^{5)} = \dfrac{-256(1+\mu)(1107+1099\mu+244\mu^2)}{315(3-\mu)}$,　$B_2^{9)} = 0$

$B_3^{5)} = \dfrac{3710(1-\mu^2)}{9}$,　　$B_3^{9)} = \dfrac{32(1-\mu^2)}{7}$

$B_4^{5)} = \dfrac{-128(1-\mu^2)(27-8\mu)}{105(3-\mu)}$

把计算结果代入载荷级数中得到

$$q = (\alpha_{1,0}\varepsilon + \alpha_{1,1}\varepsilon\eta + \alpha_{2,0}\varepsilon^2 + \alpha_{3,0}\varepsilon^3 + \alpha_{2,1}\varepsilon^2\eta + \alpha_{1,2}\varepsilon\eta^2 + \cdots)q$$

即　$$\alpha_{3,0}\varepsilon^3 + (\alpha_{2,1}\eta)\varepsilon^2 + (\alpha_{1,0} + \alpha_{1,2}\eta^2)\varepsilon - 1 \bullet 0 \qquad (4.7)$$

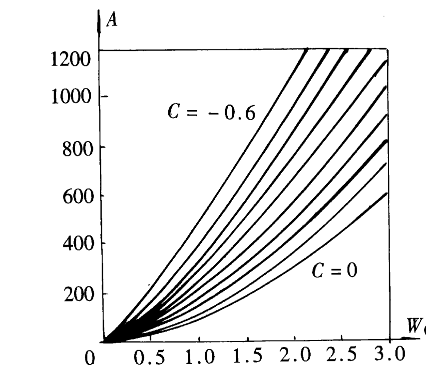
从中解出 ε 的实数解(其中 η 为已知初挠度参数)，代入下式

$$W \bullet W_{1,0}\varepsilon + W_{2,0}\varepsilon^2 + W_{1,1}\varepsilon\eta + W_{3,0}\varepsilon^3 + W_{2,1}\varepsilon^2\eta + W_{1,2}\varepsilon\eta^2 \qquad (4.8)$$

可得扁锥壳非对称的大变形挠度•

4.4　特征曲线

我们取材料的泊松比 $\mu = 0.3$，并给定点(这里给 $A = 64$ 时挠度最大点 $x_0 = 0.14$, $y_0 = -\pi/2$)，可得到该点挠度与载荷 q 和初挠度参数 C 关系的特征曲线(如图 2 所示)•



曲线自下而上依次为 $C = -0.2\sqrt{i}$ ($i = 0, 1, 2, \cdots, 9$)

图 2　载荷挠度特征曲线图

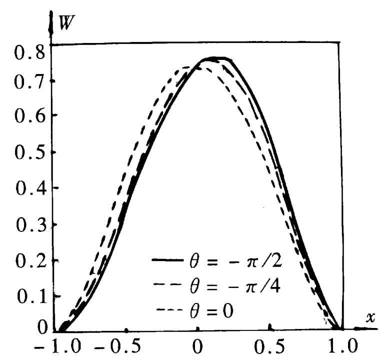


图 3　剖面挠曲线图

仅为了说明扁薄锥壳在给定载荷和给定初挠度下的挠度规律，我们不妨取 $\mu = 0.3$、$A = 64$、$C = -0.1$，从式(4.8)中可计算出壳体各点的三次近似的挠度值，为了清楚地看明各点位移的规律，我们给出了在 $\theta = 0$、$-\pi/4$、$-\pi/2$ 的三个剖面图(如图 3)•

§5. 结　　论

1. 本文给出了扁锥壳的三次近似摄动解•　通过式(4.1)及图 2 分别可以看出，一次近似摄动解与载荷呈线性关系，而三次近似解中显示了其非线性的特征•　$C = 0$ 为不考虑初挠度影响的圆板非对称大挠度问题的近似解，随着初挠度的增加，初挠度对变形的影响不断增大•

2. 在运算中我们发现，非对称问题的挠度最大点随载荷的增加而从中心向外逼近某一值•

载荷越大,非对称现象越明显• 而且随摄动次数的增加挠度最大点不断变化而趋于稳定•

3. 最大挠度不仅随着载荷的增加而增加, 而且随着初挠度的增加而下降, 这说明初挠度对壳体的承载能力有影响• 在扁壳范围内, 初挠度愈大承载能力愈强•

4. 本文所采用的双参数摄动法具有较普遍的适用性, 对于扁壳体的大变形问题效果很好; 本文所采用的 Fourier 级数的方法对于非对称问题是一种很好的求解手段•

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Non_Symmetrical Large Deformation of a Shallow Thin Conical Shell

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Abstract

In this paper, non_symmetrical large deformation problem of a shallow conical shell is studied by two_parameter perturbation method. The third_order approximate analytical soluion of the deformation of a shallow conical shell subjected to linear loads is obtained and the characteristic curves of loaddeflection on a perturbing point are portrayed. The similar questions of other kind of shell and plate can be discussed by using this paper's method. As the examples, the large deflection of plate and shallow conical shells with different initial deflections is discussed.

Key words shallow thin conical shell, non_symmetrical, large deformation, two_parameter perturbation method