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修正多重尺度法在圆薄板大挠度弯曲问题中的应用及其渐近性研究(Ⅰ)

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摘 要

本文应用修正多重尺度法研究圆板在铰链和简单支承条件下的大挠度弯曲• 作出其级数解, 分析其边界层效应和证明其渐近性•

关键词 圆板 大挠度弯曲 边界层效应 渐近性 修正多重尺度法
中图分类号 O302

§1. 引 言

关于圆薄板大挠度弯曲问题的研究已有较长的历史[1- 4]• 特别是提出以中心挠度与板厚度之比作为摄动参数的钱伟长方法之后, 由于这种方法只须应用正值摄动方法就能求得足够精确的渐近解, 因而促进了摄动方法在研究板、壳大挠度弯曲问题中的应用• 但是当中心挠度与板厚度之比较大时, 所得级数解的高阶项就可能大于其低阶项, 失去其渐近性, 而往往就会给出不合理的结果[5]•

本文试图应用修正多重尺度法研究此问题, 作出形式解并研究其渐近性• 在 1993 年, 乔宗椿曾应用此方法作出薄板边缘固支下的形式解[6], 只取用级数解的首项就得到文[1] 的结果• 本文拟先研究铰链支承的情形, 再研究薄板边缘切向力为零的简单支承的情形• 在后一情形, 文[6] 所采用的级数解的形式失效, 而必须按摄动参数的分数次幂展开• 本文对文[6] 作适当的补充和修正•

§2. 控 制 方 程

设圆薄板的半径为 a, 厚度为 h; 取板的中心作为极坐标系(r, θ) 的原点, 以 N_r 和 N_t 分别表示板中面的径向和横向薄膜应力; 又设薄板作用有密度为 q 的均匀的横向载荷, 我们知道其挠度 w 和 N_r, N_t 决定于下面的冯•卡门方程组:

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$$\left.\begin{aligned} &D\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\frac{dw}{dr} = N_r\frac{dw}{dr} + \frac{qr}{2} \\ &r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}(r^2N_r) + \frac{Eh}{2}\left(\frac{dw}{dr}\right)^2 = 0 \\ &N_t = d(rN_r)/dr \end{aligned}\right\} \tag{2.1}$$

近解

其中 E 是薄板的杨氏模数, $D = Eh^3/12(1-\sigma^2)$ 是薄板的抗弯刚度, σ 是泊松比• 如果引进应力函数 $F(r)$,

$$N_r = (h/r)dF/dr,\quad N_t = hd^2F/dr^2 \tag{2.2}$$

则(2.1)化为

$$\left.\begin{aligned} &D\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}r\frac{dw}{dr}\right) = \frac{h}{r}\frac{dF}{dr}\frac{dw}{dr} + \frac{qr}{2} \\ &\frac{1}{E}\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}r\frac{dF}{dr} + \frac{1}{2r}\left(\frac{dw}{dr}\right)^2\right) = 0 \end{aligned}\right\} \tag{2.3}$$

控 制

再引进无量纲量

$$w = w/a,\quad r = r/a,\quad F = F/Ea^2,\quad q = qa/Eh \tag{2.4}$$

又得到无量纲的方程组:

$$\left.\begin{aligned} &\varepsilon^2 L_1 w = \frac{1}{r}\frac{dw}{dr}\Phi + \frac{qr}{2} \\ &L_2\Phi = \frac{-1}{2r}\left(\frac{dw}{dr}\right)^2 \end{aligned}\right\} \tag{2.5}$$

rk

nW

其中记 $\varepsilon^2 = h^2/12a^2(1-\sigma^2)$

$$L_1 = \frac{d^3}{dr^3} + \frac{1}{r}\frac{d^2}{dr^2} - \frac{1}{r^2}\frac{d}{dr},\quad L_2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} \tag{2.6}$$

$\Phi = dF/dr$, 并略去了无量纲记号“~”•

§3. 多重尺度下的算子展开式

为显示解在边界 $r = 1$ 的急变性态, 在该邻域引进两种尺度变量

$$\xi = u(r)/\varepsilon^p,\quad \eta = r \tag{3.1}$$

其中 $u(r)$ 是待定函数, 要求在边界上取零值, 在邻域内部取正值; p 是待定常数, 用以选择变量 ξ 的尺度• 在多重尺度变量 ξ 和 η 下, 关于 r 的各阶导数可按 ε^p 的量级分解成

$$\frac{d^i}{dr^i} = \varepsilon^{-ip}\sum_{j=0}^{i}\varepsilon^{jp}\delta_{r^i}^{(j)}\qquad (i = 1, 2, 3, \cdots) \tag{3.2}$$

其中记

$$\left.\begin{aligned} &\delta_r^{(0)} = u_r\partial/\partial\xi,\quad \delta_r^{(1)} = \partial/\partial\eta \\ &\delta_{r^2}^{(0)} = u_r^2\frac{\partial^2}{\partial\xi^2},\quad \delta_{r^2}^{(1)} = 2u_r\frac{\partial^2}{\partial\xi\partial\eta} + u_{rr}\frac{\partial}{\partial\xi},\quad \delta_{r^2}^{(2)} = \frac{\partial^2}{\partial\eta^2} \\ &\delta_{r^3}^{(0)} = u_r^3\frac{\partial^3}{\partial\xi^3},\quad \delta_{r^3}^{(1)} = 3u_r^2\frac{\partial^3}{\partial\xi^2\partial\eta} + 3u_ru_{rr}\frac{\partial^2}{\partial\xi^2} \\ &\delta_{r^3}^{(2)} = 3u_r\frac{\partial^3}{\partial\xi\partial\eta^2} + 3u_{rr}\frac{\partial^2}{\partial\xi\partial\eta} + u_{rrr}\frac{\partial}{\partial\xi} \\ &\delta_{r^3}^{(3)} = \partial^3/\partial\eta^3 \end{aligned}\right\} \tag{3.3}$$

将分解式(3.3)式代入(2.6),得到关于算子 L_1 和 L_2 的分解式:

$$L_1 \equiv \varepsilon^{-3p}\sum_{j=0}^{3}\varepsilon^{jp}M_j,\quad L_2 \equiv \varepsilon^{-2p}\sum_{j=0}^{2}\varepsilon^{jp}N_j \tag{3.4}$$

其中记

$$\left.\begin{array}{l} M_0=\delta_r^{(0)3},\ M_1=\delta_r^{(1)3}+\delta_r^{(0)2}/\eta,\ M_2=\delta_r^{(2)3}+\delta_r^{(1)2}/\eta-\delta_r^{(0)}/\eta^2 \\ M_3=\delta_r^{(3)3}+\delta_r^{(2)2}/\eta-\delta_r^{(1)}/\eta^2 \\ N_0=\delta_r^{(0)2},\ N_1=\delta_r^{(1)2}+\delta_r^{(0)}/\eta,\ N_2=\delta_r^{(2)2}+\delta_r^{(1)}/\eta-1/\eta^2 \end{array}\right\} \tag{3.5}$$

§4. 边缘铰链支承的情形

边界条件是

$$\left.\begin{array}{ll} w=0,\ \dfrac{d^2w}{dr^2}+\dfrac{\sigma}{r}\dfrac{dw}{dr}=0,\ r\dfrac{dN_r}{dr}+(1-\sigma)N_r=0, & \text{当 } r=a \\ dw/dr,\ N_r\ \text{取有限值}, & \text{当 } r\to 0 \end{array}\right\} \tag{4.1}$$

在无量纲变量和应力函数的表示下是

$$\left.\begin{array}{ll} w=0,\ \dfrac{d^2w}{dr^2}+\dfrac{\sigma}{r}\dfrac{dw}{dr}=0,\ \dfrac{d\phi}{dr}-\dfrac{\sigma}{r}\phi=0, & \text{当 } r=1 \\ dw/dr,\ \phi/r\ \text{取有限值}, & \text{当 } r\to 0 \end{array}\right\} \tag{4.2}$$

在上式中略去了无量纲记号"~"•

假设边值问题(2.5)、(4.2)的解具有展开式

$$W=\sum_{n=0}^{\infty}\left[\varepsilon^{np}w_n(r)+\varepsilon^{(n+\alpha)p}\psi(r)v_n(\xi,\eta)\right] \tag{4.3}$$

$$\Phi=\sum_{n=0}^{\infty}\left[\varepsilon^{np}\phi_n(r)+\varepsilon^{(n+\beta)p}\psi(r)h_n(\xi,\eta)\right] \tag{4.4}$$

其中 α,β 是待定常数,$\psi(r)$ 是无限次可微的截断函数(cut_off function),用以消除边界层校正项 v_n, h_n 对邻域外部的影响,并保持 W 和 Φ 的可微性,

$$\psi(r)=\begin{cases}0, & \text{当 } 0\leqslant r\leqslant 1/3\\ 1, & \text{当 } 2/3\leqslant r\leqslant 1\end{cases}$$

将(4.3)和(4.4)式代入边值问题(2.5)、(4.2),并比较 ε^p 的同次幂系数• 当 $0\leqslant r\leqslant 1/3$ 时,w_n 和 ϕ_n,$(n=0,1,2,\dots)$ 应满足递推方程组:

$$\frac{1}{r}\frac{dw_0}{dr}\phi_0=\frac{-qr}{2},\quad L_0\phi_0=\frac{-1}{2r}\left(\frac{dw_0}{dr}\right)^2 \tag{4.5}$$

$$\frac{dw_0}{dr}\phi_1-\frac{dw_1}{dr}\phi_0=0,\quad L_2\phi_1=\frac{-1}{r}\frac{dw_0}{dr}\frac{dw_1}{dr} \tag{4.6}$$

$$\left.\begin{array}{l} \dfrac{dw_0}{dr}\phi_n+\dfrac{dw_n}{dr}\phi_0=rL_1w_{n-2}-\displaystyle\sum_{j=1}^{n-1}\dfrac{dw_j}{dr}\phi_{n-j} \\ L_2\phi_n=\dfrac{-1}{r}\dfrac{dw_0}{dr}\dfrac{dw_n}{dr}-\dfrac{1}{2r}\displaystyle\sum_{j=1}^{n-1}\dfrac{dw_j}{dr}\dfrac{dw_{n-j}}{dr}\qquad (n=2,3,\dots) \end{array}\right\} \tag{4.7}$$

为了确定 v_n 和 h_n 再考察边界层邻域 $2/3 \leqslant r \leqslant 1$。考虑到递推方程组(4.5)～(4.7)，从控制方程(2.5)有

$$
\left\{
\begin{aligned}
& \varepsilon^{2-3p+j\alpha p}\left(\sum_{j=0}^{3}\varepsilon^{jp}M_j\right)\left(\sum_{n=0}^{\infty}\varepsilon^{np}v_n\right) \\
& \quad = \frac{1}{\eta}\varepsilon^{-p+\alpha p}\left[\left(\delta_r^{(0)}+\varepsilon^{p}\delta_r^{(1)}\right)\sum_{n=0}^{\infty}\varepsilon^{np}v_n\right]\sum_{n=0}^{\infty}\varepsilon^{np}\phi_n \\
& \quad + \frac{1}{\eta}\varepsilon^{-p+(\alpha+\beta)p}\left[\left(\delta_r^{(0)}+\varepsilon^{p}\delta_r^{(1)}\right)\sum_{n=0}^{\infty}\varepsilon^{np}v_n\right]\sum_{n=0}^{\infty}\varepsilon^{np}h_n \\
& \quad + \frac{1}{\eta}\varepsilon^{\beta p}\left(\sum_{n=0}^{\infty}\varepsilon^{np}\frac{dw_n}{dr}\right)\sum_{n=0}^{\infty}\varepsilon^{np}h_n
\end{aligned}
\right. \tag{4.8a}
$$

$$
\begin{aligned}
& \varepsilon^{-2p+\beta p}\left(\sum_{j=0}^{2}\varepsilon^{jp}N_j\right)\sum_{n=0}^{\infty}\varepsilon^{np}h_n \\
& \quad = \frac{-1}{2\eta}\left\{\left[2\varepsilon^{-p+\alpha p}\left(\delta_r^{(0)}+\varepsilon^{p}\delta_r^{(1)}\right)\sum_{n=0}^{\infty}\varepsilon^{np}v_n\right]\sum_{n=0}^{\infty}\varepsilon^{np}\frac{dw_n}{dr}\right. \\
& \quad \left. + \varepsilon^{-2p+2\alpha p}\left[\left(\delta_r^{(0)}+\varepsilon^{p}\delta_r^{(1)}\right)\left(\sum_{n=0}^{\infty}\varepsilon^{np}v_n\right)^2\right]\right\}
\end{aligned} \tag{4.8b}
$$

从边界条件(4.1)有

$$
\sum_{n=0}^{\infty}\varepsilon^{np}w_n\Big|_{r=1}+\varepsilon^{\alpha p}\sum_{n=0}^{\infty}\varepsilon^{np}v_n\Big|_{\xi=0,\,\eta=1}=0 \tag{4.9a}
$$

$$
\begin{aligned}
& \sum_{n=0}^{\infty}\varepsilon^{np}\frac{d^2w_n}{dr^2}\bigg|_{r=1}+\varepsilon^{(-2+\alpha)p}\left(\sum_{j=0}^{2}\varepsilon^{jp}\delta_{r^2}^{(j)}\right)\sum_{n=0}^{\infty}\varepsilon^{np}v_n\Big|_{\xi=0,\,\eta=1} \\
& \quad + \sigma\left[\sum_{n=0}^{\infty}\varepsilon^{np}\frac{dw_n}{dr}\bigg|_{r=1}+\varepsilon^{(-1+\alpha)p}\left(\delta_r^{(0)}+\varepsilon^{p}\delta_r^{(1)}\right)\sum_{n=0}^{\infty}\varepsilon^{np}v_n\Big|_{\xi=0,\,\eta=1}\right]=0
\end{aligned} \tag{4.9b}
$$

$$
\begin{aligned}
& \sum_{n=0}^{\infty}\varepsilon^{np}\frac{d\phi_n}{dr}\bigg|_{r=1}+\varepsilon^{(-2+\beta)p}\left(\sum_{j=0}^{2}\varepsilon^{jp}\delta_{r^2}^{(j)}\right)\sum_{n=0}^{\infty}\varepsilon^{np}h_n\Big|_{\xi=0,\,\eta=1} \\
& \quad - \sigma\left[\sum_{n=0}^{\infty}\varepsilon^{np}\phi_n\Big|_{r=1}+\varepsilon^{\beta p}\sum_{n=0}^{\infty}\varepsilon^{np}h_n\Big|_{\xi=0,\,\eta=1}\right]=0
\end{aligned} \tag{4.9c}
$$

为了求得较正项 v_0 的边界条件，从(4.9b)知可取 $\alpha=2$；为了求得 v_0, h_0 的控制方程，从(4.8a)知可取 $p=1$；从(4.8b)知可取 $\beta=3$。再比较 ε 的同次幂的系数，从递推关系式(4.5)～(4.9)知 w_n, ϕ_n, v_n, h_n $(n=0, 1, 2, \cdots)$ 确定于下面的递推边值问题：

$$
\left\{
\begin{aligned}
& \frac{1}{r}\frac{dw_0}{dr}\phi_0=\frac{-qr}{2},\quad L_2\phi_0=\frac{-1}{2r}\left(\frac{dw_0}{dr}\right)^2 && (4.10\text{a}) \\
& M_0v_0-\frac{\phi_0}{\eta}\delta_r^{(0)}v_0=0,\quad N_0h_0=\frac{-1}{2\eta}\left(2\frac{dw_0}{dr}\delta_r^{(0)}v_0\right) && (4.10\text{b}) \\
& w_0\big|_{r=1}=0,\ \left(\frac{d\phi_0}{dr}-\sigma\phi_0\right)\bigg|_{r=1}=0,\ \delta_{r^2}^{(0)}v_0\big|_{\xi=0,\,\eta=1}=-\left(\frac{d^2w_0}{dr^2}+\sigma\frac{dw_0}{dr}\right)\bigg|_{r=1} && (4.10\text{c}) \\
& \lim_{r\to 0}\frac{dw_0}{dr}<\infty,\ \lim_{r\to 0}\frac{\phi_0}{r}<\infty,\ \lim_{\xi\to\infty}v_0=0,\ \lim_{\xi\to\infty}h_0=0 && (4.10\text{d})
\end{aligned}
\right.
$$

$$\begin{cases} \dfrac{dw_0}{dr}\phi_1+\dfrac{dw_1}{dr}\phi_0=0,\quad L_2\phi_1+\dfrac{1}{r}\dfrac{dw_0}{dr}\dfrac{dw_1}{dr}=0 & (4.11a)\\ M_0v_1-\dfrac{\phi_0}{\eta}\delta_r^{(0)}v_1=-M_1v_0+\dfrac{\phi_0}{\eta}\delta_r^{(1)}v_0+\dfrac{\phi_1}{\eta}\delta_r^{(0)}v_0 & (4.11b)\\ N_0h_1=-N_1h_0-\dfrac{1}{2\eta}\left[2\dfrac{dw_0}{dr}(\delta_r^{(0)}v_1+\delta_r^{(1)}v_0)+2\dfrac{dw_1}{dr}\delta_r^{(0)}v_0+(\delta_r^{(0)}v_0)^2\right] & (4.11c)\\ w_1|_{r=1}=0,\quad \left(\dfrac{d\phi_1}{dr}-\sigma\phi_1\right)\Big|_{r=1}=0 & \\ \delta_{r^2}^{(0)}v_1|_{\xi=0,\eta=1}=-\left(\dfrac{d^2w_1}{dr^2}+\sigma\dfrac{dw_1}{dr}\right)\Big|_{r=1}-\delta_{r^2}^{(1)}v_0|_{\xi=0,\eta=1} & \\ \qquad -\sigma\delta_r^{(0)}v_0|_{\xi=0,\eta=1} & (4.11d)\\ \lim\limits_{r\to0}\dfrac{dw_1}{dr}<\infty,\quad \lim\limits_{r\to0}\dfrac{\phi_1}{r}<\infty,\quad \lim\limits_{\xi\to\infty}v_1=0,\quad \lim\limits_{\xi\to\infty}h_1=0 & (4.11e) \end{cases}$$

$$\begin{cases} \dfrac{dw_0}{dr}\phi_i+\dfrac{dw_i}{dr}\phi_0=rL_1w_{i-2}-\sum\limits_{j=1}^{i-1}\dfrac{dw_j}{dr}\phi_{i-j} & (4.12a)\\ L_2\phi_i-\dfrac{1}{r}\dfrac{dw_0}{dr}\dfrac{dw_i}{dr}=\dfrac{-1}{2r}\sum\limits_{j=1}^{i-1}\dfrac{dw_j}{dr}\dfrac{dw_{i-j}}{dr} & (4.12b)\\ M_0v_i-\dfrac{\phi_0}{\eta}\delta_r^{(0)}v_i=-\sum\limits_{j=1}^{3}M_jv_{i-j}+\dfrac{1}{\eta}\Big[\phi_i\delta_r^{(0)}v_0+\sum\limits_{j=0}^{i}\phi_{i-j}\delta_r^{(1)}v_{j-1} & \\ \qquad +\sum\limits_{j=0}^{i-2}\Big(\phi_{i-j-1}\delta_r^{(0)}v_{j+1}+\dfrac{dw_j}{dr}h_{i-2-j}+\sum\limits_{k=0}^{1}\delta_r^{(k)}v_{j-k}h_{i-3-j}\Big)\Big] & (4.12c)\\ N_0h_i=-\sum\limits_{j=1}^{2}N_jh_{i-j}-\dfrac{1}{2\eta}\sum\limits_{j=0}^{i}\sum\limits_{k=0}^{1}\Big(2\dfrac{dw_j}{dr}\delta_r^{(k)}v_{i-j-k}+\sum\limits_{m=0}^{1}\delta_r^{(k)}v_{j-k}\,\delta_r^{(m)}v_{i-j-m-1}\Big) & (4.12d)\\ w_i|_{r=1}=-v_{i-2}|_{\xi=0,\eta=1},\quad \left(\dfrac{d\phi_i}{dr}-\sigma\phi_i\right)\Big|_{r=1}=\left(-\sum\limits_{k=0}^{1}\delta_r^{(k)}h_{i-k-2}+\sigma h_{i-3}\right)\Big|_{\xi=0,\eta=1} & \\ \left(\sum\limits_{k=0}^{2}\delta_{r^2}^{(k)}v_{i-k}+\dfrac{1}{\eta}\sum\limits_{m=0}^{1}\delta_r^{(m)}v_{i-m-1}\right)\Big|_{\xi=0,\eta=1}=-\left(\dfrac{d^2w_i}{dr^2}+\dfrac{\sigma}{r}\dfrac{dw_i}{dr}\right)\Big|_{r=1} & (4.12e)\\ \lim\limits_{r\to0}\dfrac{dw_i}{dr}<\infty,\quad \lim\limits_{r\to0}\dfrac{\phi_i}{r}<\infty,\quad \lim\limits_{\xi\to\infty}v_i=0,\quad \lim\limits_{\xi\to\infty}h_i=0 & (4.12f) \end{cases}$$

$$(i=3,4,\dots)$$

其中负下标的量取作零, 并记 $v_n=\phi(r)v_n,\ h_n=\phi(r)h_n$•

为简单起见, 下面只求出精确到 $O(\varepsilon^2)$ 的级数解•

从(4.10a)的第一方程解出 dw_0/dr, 代入第二方程, 得到关于 ϕ_0 的常微分方程

$$\frac{d^2\phi_0}{dr^2}+\frac{1}{r}\frac{d\phi_0}{dr}-\frac{\phi_0}{r^2}=\frac{-q^2r^3}{8\phi_0^2} \qquad (4.13)$$

考虑到关于 ϕ_0/r 当 $r\to0$ 是有界的条件, 假设 ϕ_0 具有级数解

$$\phi_0=q^{2/3}\sum_{i=1}^{\infty}a_ir^i \qquad (4.14)$$

将上式代入方程(4.13), 求得各系数为

$$\left.\begin{aligned} &a_{2i}=0,\quad (i=1,2,\cdots);\ a_3=\frac{-1}{64a_1^2},\ a_5=\frac{-2}{3\times 64^2a_1^5}\\ &a_7=\frac{-13}{18\times 64^3a_1^8},\ a_9=\frac{-17}{18\times 64^4a_1^{11}},\ \cdots \end{aligned}\right\}\tag{4.15}$$

其中 a_1 是待定常数, 由边界条件 $d\phi_0(1)/dr-\sigma\phi_0(1)=0$ 确定· 所以

$$\phi_0=q^{2/3}a_1rf(cr^2)\tag{4.16}$$

其中记 $c=1/32a_1^3$, 和

$$f(x)=1-\frac{x}{2}-\frac{x^2}{6}-\frac{13}{144}x^3-\frac{17}{288}x^4+\cdots\tag{4.17}$$

已证明级数(4.17)是收敛的[1]· 将(4.16)式代入(4.13)经积分得

$$w_0=\frac{-q^{1/3}r^2}{4a_1}g_0(cr^2)+k$$

其中 k 是积分常数, 和

$$g_0(x)=1+\frac{x}{4}+\frac{5}{36}x^2+\frac{55}{576}x^3+\frac{7}{96}x^4+\cdots\tag{4.18}$$

再由(4.10c)中关于 w_0 的边界条件和 $k=q^{1/3}g_0(c)/4a_1$, 所以

$$w_0=\frac{1}{4a_1}q^{1/3}[g_0(c)-r^2g_0(cr^2)]\tag{4.19}$$

下面再求边界层校正项· 从(4.10b)的第一方程有

$$u_r^3\frac{\partial^3v_0}{\partial\xi^3}-\frac{\phi_0}{\eta}u_r\frac{\partial v_0}{\partial\xi}=0$$

为了使上面方程取最简单的形式, 可取待定函数 $u(r)$ 使 $u_r^2=\phi_0/\eta$, 即取

$$u(r)=\int_r^1\sqrt{\frac{\phi_0}{r}}dr\tag{4.20}$$

而得到 v_0 的最简单的控制方程

$$\frac{\partial^3v_0}{\partial\xi^3}-\frac{\partial v_0}{\partial\xi}=0\tag{4.21}$$

它的满足条件 $\lim\limits_{r\to0}v_0=0$ 的解为

$$v_0=A_0(\eta)e^{-\xi}\tag{4.22}$$

其中 $A_0(\eta)$ 是待定函数· 从(4.10c)中关于 v_0 的边界条件得到 $A_0(\eta)$ 的边界条件

$$A_0(1)=\frac{q}{2}\,\frac{2\phi_0(1)-\phi_0'(1)+\sigma\phi_0(1)}{\phi_0^3(1)}=\frac{q}{\phi_0^2(1)}=\frac{1}{a_1^2q^{1/3}}f^{-2}(c)\tag{4.23}$$

为了得到 $A_0(\eta)$ 的控制方程, 根据多重尺度法的步骤, 必须令方程(4.11b)的右端为零, 为此还须再求出 ϕ_1·

从(4.11a)的第一方程解出 dw_1/dr 再代入第二方程, 得到 ϕ_1 的控制方程

$$\frac{d^2\phi_1}{dr^2}+\frac{1}{r}\frac{d\phi_1}{dr}-\frac{1}{r^2}\phi_1=\frac{1}{4a_1^3}f^{-3}(cr^2)\phi_1\tag{4.24}$$

根据条件 $\lim\limits_{r\to0}(\phi_1/r)<\infty$, 假设方程(4.24)具有级数解

$$\phi_1=\sum_{i=1}^{\infty}b_ir^i$$

经代入方程(4.24)和比较 r 的同次幂的系数知

$$\phi_1 = b_1 r f_1(cr^2)$$

其中 b_1 是待定常数, 和

$$f_1(x) = 1 + x + \frac{5}{6}x^2 + \frac{13}{18}x^3 + \frac{187}{288}x^4 + \frac{259}{432}x^5 + \frac{20485}{36288}x^6 + \cdots$$

再根据 ϕ_1 的边界条件知 $b_1 = 0$, 所以 $\phi_1 \equiv 0$• 从方程(4.11b) 的右端为零, 得到 $A_0(\eta)$ 的控制方程

$$\frac{dA_0}{d\eta} + \left(\frac{3}{4}\frac{\phi_0'}{\phi_0} - \frac{1}{4\eta}\right)A_0 = 0$$

它的满足条件(4.23) 的解为

$$A_0(\eta) = a_1^{-2}q^{-1/3}f^{-5/4}(c)\eta^{-1/2}f^{-3/4}(c\eta^2) \tag{4.25}$$

将(4.25) 式代回(4.22) 式就求得 $v_0(\xi, \eta)$,

$$v_0(\xi, \eta) = a_1^{-2}q^{-1/3}f^{-5/4}(c)\eta^{1/2}f^{-3/4}(c\eta^2)e^{-\xi} \tag{4.26}$$

其中 $a_1 = (1/32c)^{1/3}$, 和 c 是代数方程 $\phi_0'(1) - \sigma\phi_0(1) = 0$ 即

$$2cf'(c) + (1 - \sigma)f(c) = 0 \tag{4.27}$$

的根• 再从(4.10b) 的第二方程得到 h_0 的控制方程:

$$\frac{\partial^2 h_0}{\partial \xi^2} = \frac{1}{\eta}\frac{dw_0}{dr}\frac{A_0(\eta)}{u_r}e^{-\xi}$$

对 ξ 积分两次, 并考虑到 h_0 的边界条件得

$$h_0 = \frac{1}{\eta}\frac{dw_0}{dr}\frac{A_0(\eta)}{u_r}e^{-\xi} \tag{4.28}$$

为了求得准确到 $O(\varepsilon^2)$ 的解, 还需再求出 w_1, ϕ_1 和 w_2, ϕ_2• 它们分别确定于边值问题

$$\begin{cases} \dfrac{dw_0}{dr}\phi_1 + \dfrac{dw_1}{dr}\phi_0 = 0,\quad L_2\phi_1 + \dfrac{1}{r}\dfrac{dw_0}{dr}\dfrac{dw_1}{dr} = 0 \\ w_1|_{r=1} = 0,\quad \left(\dfrac{d\phi_1}{dr} - \sigma\phi_1\right)\Big|_{r=1} = 0,\quad \lim\limits_{r\to 0}\dfrac{dw_1}{dr} < \infty,\quad \lim\limits_{r\to 0}\dfrac{\phi_1}{r} < \infty \end{cases}$$

$$\begin{cases} \dfrac{dw_0}{dr}\phi_2 + \dfrac{dw_2}{dr}\phi_0 = rL_1w_0 - \dfrac{dw_1}{dr}\phi_1,\quad L_2\phi_1 + \dfrac{1}{r}\dfrac{dw_0}{dr}\dfrac{dw_2}{dr} = \dfrac{-1}{2r}\left(\dfrac{dw_1}{dr}\right)^2 \\ w_2|_{r=1} = -v_0|_{\xi=0,\eta=4},\quad \left(\dfrac{d\phi_2}{dr} - \sigma\phi_2\right)\Big|_{r=1} = (-\delta_r^{(0)}h_0)|_{\xi=0,\eta=1} \\ \lim\limits_{r\to 0}\dfrac{dw_2}{dr} < \infty,\quad \lim\limits_{r\to 0}\dfrac{\phi_2}{r} < \infty \end{cases}$$

应用前面求 w_0, ϕ_0 的方法可以求得 $w_1 \equiv \phi_1 \equiv 0$, 和

$$\phi_2 = 4cr[H(c)f_2(cr^2) - \psi_2(cr^2)]$$

$$w_2 = a_1^{-2}q^{-1/3}\left\{\frac{1}{32}a^{-3}[r^2g_2(cr^2) - g_2(c)] - f^{-2}(c)\right\}$$

其中 c 是代数方程(4.27) 的根, 和

$$f_2(x) = 1 + x + \frac{5}{6}x^2 + \frac{13}{18}x^3 + \frac{187}{288}x^4 + \frac{259}{432}x^5 + \frac{20485}{36288}x^6 + \cdots$$

$$\psi_2(x) = x + \frac{3}{2}x^2 + \frac{67}{36}x^3 + \frac{35}{16}x^4 + \frac{10849}{4320}x^5 + \frac{1032721}{362880}x^6 + \cdots$$

$$H(c) = \frac{(1-\sigma)\psi_2(c) + 2c\psi_2'(c) - 4f^{-3}(c)}{(1-\sigma)f_2(c) + 2cf_2'(c)}$$

$$g_2(x) = (-1 + H(c)) + (-2 + H(c))x + \left(-\frac{35}{12} + \frac{35}{36}H(c)\right)x^2 + \left(-\frac{2225}{576} + \frac{275}{288}H(c)\right)x^3 + \left(-\frac{781}{160} + \frac{455}{480}H(c)\right)x^4 + \left(-\frac{207167}{34560} + \frac{205}{216}H(c)\right)x^5 + \left(-\frac{36671129}{5080320} + \frac{323969}{338688}H(c)\right)x^6 + \cdots$$

在下一节再考察圆板的边缘是简单支承的情形• 将发现如果仍采用 ε 的整数次幂的级数解, 在边界将出现奇性而失效, 而必须按 ε 的分数次幂展开•

§ 5. 边缘简单支承的情形

在无量纲变量和应力函数的表示下, 边界条件是

$$\begin{cases} w = 0, \quad \dfrac{d^2 w}{dr^2} + \dfrac{\sigma}{r}\dfrac{dw}{dr} = 0, \quad \phi(r) = 0, \quad 当\ r = 1 & (5.1) \\ \lim\limits_{r\to 0}\dfrac{dw}{dr} < \infty, \quad \lim\limits_{r\to 0}\dfrac{\phi}{r} < \infty & (5.2) \end{cases}$$

(已略去无量纲记号“~”)• 仍假设边值问题(2.5)、(5.1)、(5.2)的解具有展开式(4.3)、(4.4)• 当 $0 \leqslant r \leqslant 1/3$ 时 w_n 和 ϕ_n $(n = 0, 1, 2, \cdots)$ 仍满足递推方程组(4.5)、(4.7); 当 $2/3 \leqslant r \leqslant 1$ 时满足递推方程组(4.8), 只是边界条件代替以

$$\begin{cases} \sum\limits_{n=0}^{\infty}\varepsilon^{np} w_n \big|_{r=1} + \varepsilon^{\alpha p}\sum\limits_{n=0}^{\infty}\varepsilon^{np} v_n \big|_{\xi=0,\eta=1} = 0 & (5.3a) \\ \sum\limits_{n=0}^{\infty}\varepsilon^{np}\dfrac{d^2 w_n}{dr^2}\Big|_{r=1} + \varepsilon^{(-2+\alpha)p}\left(\sum\limits_{j=0}^{2}\varepsilon^{jp}\delta_{r^2}^{(j)}\right)\sum\limits_{n=0}^{\infty}\varepsilon^{np} v_n \big|_{\xi=0,\eta=1} \\ \quad + \sigma\left[\sum\limits_{n=0}^{\infty}\varepsilon^{np}\dfrac{dw_n}{dr}\Big|_{r=1} + \varepsilon^{(-1+\alpha)p}\left(\delta_r^{(0)} + \varepsilon^p\delta_r^{(1)}\right)\sum\limits_{n=0}^{\infty}\varepsilon^{np} v_n \big|_{\xi=0,\eta=1}\right] = 0 & (5.3b) \\ \sum\limits_{n=0}^{\infty}\varepsilon^{np}\phi_n \big|_{r=1} + \varepsilon^{\beta p}\sum\limits_{n=0}^{\infty}\varepsilon^{np} h_n \big|_{\xi=0,\eta=1} = 0 & (5.3c) \end{cases}$$

为了求得 v_0 的边界条件, 从(5.3b) 知可取 $\alpha = 2$; 为了求得 v_0, h_0 的控制方程, 从(4.8a) 知可取 $p = 2/3$; 又从(4.8b) 知可取 $\beta = 3$, 然后比较 $\varepsilon^{2/3}$ 的同次幂的系数, 可得关于 w_n, ϕ_n, v_n, h_n $(n = 0, 1, 2, \cdots)$ 的递推边值问题(在边界层内, 视 $\varepsilon^{-2/3}\phi_0$ 与 v_0 是同阶量):

$$\begin{cases} \dfrac{1}{r}\dfrac{dw_0}{dr}\phi_0 = -\dfrac{qr}{2}, \quad L_2\phi_0 = -\dfrac{1}{2r}\left(\dfrac{dw_0}{dr}\right)^2 & (5.4a) \\ M_0 v_0 - \dfrac{1}{\eta}\left(\varepsilon^{-2/3}\phi_0\right)\delta_r^{(0)} v_0 = 0, \quad N_0 h_0 = -\dfrac{1}{2\eta}\left(2\dfrac{dw_0}{dr}\delta_r^{(0)} v_0\right) & (5.4b) \\ w_0 \big|_{r=1} = 0, \quad \phi_0 \big|_{r=1} = 0, \quad \delta_{r^2}^{(0)} v_0 \big|_{\xi=0,\eta=1} = -\left(\dfrac{d^2 w_0}{dr^2} + \sigma\dfrac{dw_0}{dr}\right)\Big|_{r=1} & (5.4c) \\ \lim\limits_{r\to 0}\dfrac{dw_0}{dr} < \infty, \quad \lim\limits_{r\to 0}\dfrac{\phi_0}{r} < \infty, \quad \lim\limits_{\xi\to\infty} v_0 = 0, \quad \lim\limits_{\xi\to\infty} h_0 = 0 & (5.4d) \end{cases}$$

$$\begin{cases} \dfrac{dw_0}{dr}\phi_1 + \dfrac{dw_1}{dr}\phi_0 = 0, \quad L_2\phi_1 + \dfrac{1}{r}\dfrac{dw_0}{dr}\dfrac{dw_1}{dr} = 0 & (5.5a) \\ M_0 v_1 - (\varepsilon^{-2/3}\phi_0)\dfrac{1}{\eta}\delta_r^{(0)}v_1 = -M_1 v_0 + \dfrac{1}{\eta}\dfrac{dw_0}{dr}h_0 & \\ \quad + \dfrac{1}{\eta}[(\varepsilon^{-2/3}\phi_1)\delta_r^{(0)}v_0 + (\varepsilon^{-2/3}\phi_0)\delta_r^{(1)}v_0] & (5.5b) \\ N_0 h_1 = -N_1 h_0 - \dfrac{1}{2\eta}\left[2\dfrac{dw_0}{dr}(\delta_r^{(0)}v_1 + \delta_r^{(1)}v_0) + 2\dfrac{dw_1}{dr}\delta_r^{(0)}v_0 + (\delta_r^{(0)}v_0)^2\right. & (5.5c) \\ w_1|_{r=1} = 0, \quad \phi_1|_{r=1} = 0, & \\ \quad \delta_{r^2}^{(0)}v_1|_{\xi=0,\eta=1} = -(\delta_{r^2}^{(1)}v_0 + \sigma\delta_r^{(0)}v_0)|_{\xi=0,\eta=1} - \left(\dfrac{d^2w_1}{dr^2} + \sigma\dfrac{dw_2}{dr}\right)\Bigg|_{r=1} & (5.5d) \\ \lim\limits_{r\to 0}\dfrac{dw_1}{dr} < \infty, \quad \lim\limits_{r\to 0}\dfrac{\phi_1}{r} < \infty, \quad \lim\limits_{\xi\to\infty}v_1 = 0, \quad \lim\limits_{\xi\to\infty}h_1 = 0 & (5.5e) \end{cases}$$

$$\begin{cases} \dfrac{dw_0}{dr}\phi_n + \dfrac{dw_n}{dr}\phi_0 = rL_1 w_{n-3} - \sum\limits_{j=1}^{n-1}\dfrac{dw_j}{dr}\phi_{n-j} & (5.6a) \\ L_2\phi_n + \dfrac{1}{r}\dfrac{dw_0}{dr}\dfrac{dw_n}{dr} = \dfrac{-1}{2r}\sum\limits_{j=1}^{n-1}\dfrac{dw_j}{dr}\dfrac{dw_{n-j}}{dr} & (5.6b) \\ M_0 v_n - (\varepsilon^{-2/3}\phi_0)\dfrac{1}{\eta}\delta_r^{(0)}v_n = -\sum\limits_{j=1}^{3}M_j v_{n-j} + \dfrac{1}{\eta}\left\{\sum\limits_{j+k=n-1}\dfrac{dw_j}{dr}h_k\right. & \\ \quad \left. + \sum\limits_{j+k=n,(j\neq 0)}(\varepsilon^{-2/3}\phi_j)\delta_r^{(0)}v_k + \sum\limits_{j+k=n-1}(\varepsilon^{-2/3}\phi_j)\delta_r^{(1)}v_k + \sum\limits_{i=0}^{1}\sum\limits_{j+k=n-2-i}h_j\delta_r^{(i)}v_k\right\} & (5.6c) \\ N_0 h_n = -\sum\limits_{j=1}^{2}N_j h_{n-j} - \dfrac{1}{2\eta}\sum\limits_{j=0}^{n}\sum\limits_{k=0}^{1}\left[2\dfrac{dw_j}{dr}\delta_r^{(k)}v_{n-j-k}\right. & \\ \quad \left. + \sum\limits_{m=0}^{1}(\delta_r^{(k)}v_{j-k})(\delta_r^{(m)}v_{n-j-m-1})\right] \quad) & (5.6d) \\ w_n|_{r=1} = -v_{n-2}|_{\xi=0,\eta=1}, \quad \phi_n|_{r=1} = -h_{n-3}|_{\xi=0,\eta=1} & \\ \quad \delta_{r^2}^{(0)}v_n|_{\xi=0,\eta=1} = -\sum\limits_{k=1}^{2}\delta_{r^2}^{(k)}v_{n-k}|_{\xi=0,\eta=1} - \sigma\sum\limits_{m=0}^{1}\delta_r^{(m)}v_{n-m-1}|_{\xi=0,\eta=1} & \\ \qquad - \left(\dfrac{d^2w_n}{dr^2} + \sigma\dfrac{dw_n}{dr}\right)\Bigg|_{r=1} & (5.6e) \\ \lim\limits_{r\to 0}\dfrac{dw_n}{dr} < \infty, \quad \lim\limits_{r\to 0}\dfrac{\phi_n}{r} < \infty, \quad \lim\limits_{\xi\to\infty}v_n = 0, \quad \lim\limits_{\xi\to\infty}h_n = 0 & (5.6f) \end{cases}$$

$$(n = 2, 3, \cdots)$$

下面我们只求出准确到 $O(\varepsilon^{4/3})$ 的级数解·

从(5.4a) 的第一方程解出 ϕ_0 代入第二方程得

$$\left\{ d - r^3\dfrac{d^3w_0}{dr^3} - \left(5r^2\dfrac{d^2w_0}{dr^2} + 3r\dfrac{dw_0}{dr}\right)\dfrac{dw_0}{dr} + 2r^3\left(\dfrac{dw_0}{dr}\right)^2 = \dfrac{1}{q}\left(\dfrac{dw_0}{dr}\right)^5 \right. \tag{5.7}$$

假设具有级数解

$$(\quad w_0 = q^{1/3}\left(a_0 + \sum_{i=1}^{\infty} a_i r^i\right.$$

代入方程(5.7),经比较 r 的同次幂的系数知

$$w_0 = q^{1/3}[a_0 + a_2 r^2 g(cr^2)] \tag{5.8}$$

其中记 $c = -2a_2^3$, 和

$$g(x) = 1 + \frac{1}{4}x + \frac{5}{36}x^2 + \frac{55}{576}x^3 + \frac{7}{96}x^4 + \frac{205}{3456}x^5 + \frac{17051}{338688}x^6 + \cdots$$

从 w_0 的边界条件(5.4c) 知, $a_0 = -a_2 g(c)$, (5.8) 式又可写成

$$w_0 = \alpha q^{1/3}[g(c) - r^2 g(cr^2)] \tag{5.9}$$

式中记 $\alpha = -a_2$ 是待定常数, 和 $c = 2\alpha^3$• 将(5.9) 式代回(5.4a) 的第一方程, 又求得

$$\phi_0 = \frac{-qr^2}{2}\left(\frac{dw_0}{dr}\right)^{-1} = \alpha_1 q^{2/3} rf(cr^2) \tag{5.10}$$

其中记 $\alpha_1 = 1/4\alpha$, 和

$$f(x) = 1 - \frac{1}{2}x - \frac{1}{6}x^2 - \frac{13}{144}x^3 - \frac{17}{288}x^4 - \frac{37}{864}x^5 - \frac{1205}{36288}x^6 + \cdots$$

从 ϕ_0 的边界条件知

$$f(c) = 0 \tag{5.11}$$

从代数方程(5.11) 解出 c, 随之求出 α, 再代回(5.9) 和(5.10) 式就完全确定了 w_0 和 ϕ_0• 下面再求校正项 v_0 和 h_0•

从(5.4b) 的第一方程有

$$u_r^2 \frac{\partial^3 v_0}{\partial \xi^3} - (\varepsilon^{-2/3}\phi_0)\frac{1}{\eta}\frac{\partial v_0}{\partial \xi} = 0 \tag{5.12}$$

为使上面方程取最简单的形式, 可取待定函数 $u(r)$ 使 $u_r^2 = \phi_0/\eta u$, 即取

$$u(r) = \left(\frac{3}{2}\int_r^1 \sqrt{\phi_0/r}\, dr\right)^{2/3} \quad - \tag{5.13}$$

得到

$$\frac{\partial^3 v_0}{\partial \xi^3} - \xi\frac{\partial v_0}{\partial \xi} = 0 \tag{5.14}$$

这是关于 $\partial v_0/\partial \xi$ 的 Airy 方程, 具有解

$$v_0 = b_1(\eta)\int_0^{\xi} A_i(t)\,dt + b_2(\eta)\int_0^{\xi} B_i(t)\,dt + b_3(\eta)$$

其中 $A_i(t)$ 和 $B_i(t)$ 分别是第一类和第二类 Airy 函数; $b_1(\eta)$, $b_2(\eta)$, $b_3(\eta)$ 是待定函数• 因 $\lim_{\xi\to\infty} v_0 = 0$, 所以 $b_2(\eta) = 0$, 和

$$b_3(\eta) = -b_1(\eta)\int_0^{\infty} A_i(t)\,dt$$

即

$$v_0 = -b_1(\eta)\int_{\xi}^{\infty} A_i(t)\,dt \approx \frac{-b_1(\eta)}{2\sqrt{\pi}}\int_{\xi}^{\infty} t^{-1/4}\exp\left(\frac{-2}{3}t^{3/2}\right)dt$$

$$= \frac{-2b_1(\eta)}{\sqrt{6\pi}}\int_{\sqrt{2/3}\,\xi^{3/4}}^{\infty} e^{-x^2}\,dx \tag{5.15}$$

再从(5.4b) 的第二方程有

$$\frac{\partial^2 h_0}{\partial \xi^2} = -\frac{1}{\eta u_r(\eta)}\frac{dw_0}{dr}\frac{\partial v_0}{\partial \xi}$$

经对 ξ 积分两次, 并考虑到边界条件 $\lim\limits_{\xi\to\infty} h_0 = 0$ 得

$$h_0 = \frac{-2b_1(\eta)w_{0,r}}{\sqrt{6\pi}u_r(\eta)}\int_\xi^\infty\int_{\sqrt{2/3}t^{3/4}}^\infty e^{-x^2}dx\,dt \tag{5.16}$$

式中 $b_1(\eta)$ 是待定函数. 为了确定 $b_1(\eta)$ 再考虑下一边值问题 (5.5a) ~ (5.5e).

从(5.5a) 的第一方程解出 Φ_1 再代入第二方程, 得

$$\begin{aligned} & r^2\left\{\frac{d^3w_1}{dr^3}f^2(cr^2)+4\frac{d^2w_1}{dr^2}f(cr^2)f'(cr^2)+2\frac{dw_1}{dr}[(f'(cr^2))^2\right. \\ & \left.+f(cr^2)f''(cr^2)]\right\}+r\left[2\frac{d^2w_1}{dr^2}f^2(cr^2)+\frac{dw_1}{dr}2f(cr^2)f'(cr^2)\right] \\ & -\frac{dw_1}{dr}\left[(f^2(cr^2)-\frac{r^2}{4\alpha^3}f^{-1}(cr^2)\right]=0 \end{aligned} \tag{5.17}$$

假设 w_1 具有展开式(考虑到边界条件(5.5e)):

$$w_1 = B_0+\sum_{i=2}^\infty B_i r^i \tag{5.18}$$

代入方程(5.17), 并令 r 的各次幂的系数为零, 求出 $B_0, B_2, \cdots$ 再代回(5.18) 式得

$$w_1 = B_0 + B_2r^2g_1(cr^2) \tag{5.18a}$$

其中 B_0, B_2 是待定常数, 和

$$g_1(x) = 1+x+\frac{35}{36}x^2+\frac{275}{288}x^3+\frac{91}{96}x^4+\frac{205}{216}x^5+\frac{323969}{338688}x^6+\cdots$$

又从边界条件(5.5d) 知 $B_0 = -B_2g_1(c)$, 所以

$$w_1 = B_2[r^2g_1(cr^2)-g_1(c)] \tag{5.19}$$

将 w_1 再代入(5.5a) 的第一方程, 又求得 Φ_1:

$$\Phi_1 = 4\alpha^2B_2q^{1/3}rf_1(cr^2) \tag{5.20}$$

其中

$$f_1(x) = 1+x+\frac{5}{6}x^2+\frac{13}{18}x^3+\frac{187}{288}x^4+\frac{259}{432}x^5+\frac{20485}{36288}x^6+\cdots$$

又根据 Φ_1 的边界条件知 $B_2 = 0$, 所以

$$w_1\equiv 0,\quad \Phi_1\equiv 0 \tag{5.21}$$

为了确定(5.15) 和(5.16) 式中的 $b_1(\eta)$, 可令方程(5.5b) 的右端为零, 得到关于 $b_1(\eta)$ 的微分方程

$$\frac{-2}{\sqrt{6\pi}}\left[-\left(\delta_{r^3}^{(1)}+\frac{1}{\eta}\delta_{r^2}^{(0)}+\frac{1}{\eta}(\varepsilon^{-2/3}_\pi\Phi_0)\,\delta_r^{(1)}\right)\left(b_1(\eta)\int_{\sqrt{2/3}\xi^{3/4}}^\infty e^{-x^2}dx\right)+\frac{1}{\eta}\frac{dw_0}{dr}h_0=0\right.$$

因为只要求级数解精确到 $O(\varepsilon^{4/3})$, 可以取 $b_1(\eta)$ 使 $db_1(\eta)/d\eta = 0$, 即

$$b_1(\eta) = b_1(1) = \frac{-1}{u_r^2(1)A_i'(0)}\left(\frac{d^2w_0(1)}{dr^2}+\sigma\frac{dw_0(1)}{dr}\right) \tag{5.22}$$

下面再求出 w_2 和 Φ_2, 它们确定于边值问题:

$$\begin{cases} \dfrac{dw_0}{dr}\Phi_2+\dfrac{dw_2}{dr}\Phi_0=0,\quad L_2\Phi_2+\dfrac{1}{r}\dfrac{dw_0}{dr}\dfrac{dw_2}{dr}=0 & (5.23a) \\ w_2|_{r=1}=-v_0|_{\xi=0,\eta=1},\quad \Phi_2|_{r=1}=0 & (5.23b) \\ \lim\limits_{r\to 0}\dfrac{dw_2}{dr}<\infty,\ \lim\limits_{r\to 0}\dfrac{\Phi_2}{r}<\infty & (5.23c) \end{cases}$$

重复前面求 w_1 和 Φ_1 的步骤知道(参见(5.18a) 和(5.20) 式)

$$w_2 = D_0 + D_2 r^2 g_1(cr^2),\quad \Phi_2 = 4\alpha^2 D_2 q^{1/3} r f_1(cr^2)$$

式中 D_0, D_2 是待定常数• 又根据 Φ_2 的边界条件知 $D_2 = 0$, 即

$$w_2 = D_0 = -v_0|_{\xi=0,\eta=1} = \frac{b_1(1)}{\sqrt{6}},\quad \Phi_2 = 0 \tag{5.24}$$

边值问题(2.5)、(5.1)、(5.2) 的准确到 $O(\varepsilon^{4/3})$ 的解是

$$w = w_0 + \varepsilon^{4/3} w_2 + \varepsilon^{4/3}\psi(r) v_0(\xi, \eta) + O(\varepsilon^{6/3})$$

$$\Phi = \Phi_0 + \varepsilon^{6/3}\psi(r) h_0(\xi, \eta) + O(\varepsilon^{6/3})$$

关于级数解的渐近性质将在部分(Ⅱ) 讨论•

§6. 数 值 分 析

作为一个例子, 考察 $E = 10^4\text{kg/mm}^2$, $\sigma = 0.3$, $a = 76\text{mm}$ 的情形• 它们的挠度曲线如图 1, 2, 3, 4 所示:

(1) 铰链支承情形

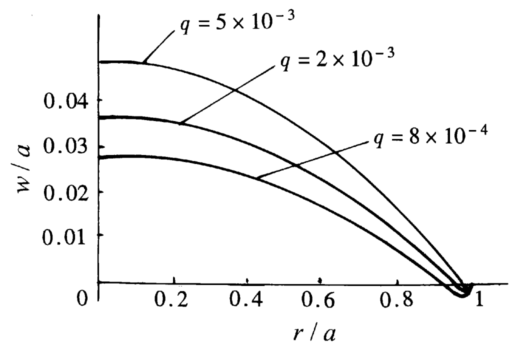

图 1　$h = 0.1\text{mm}$

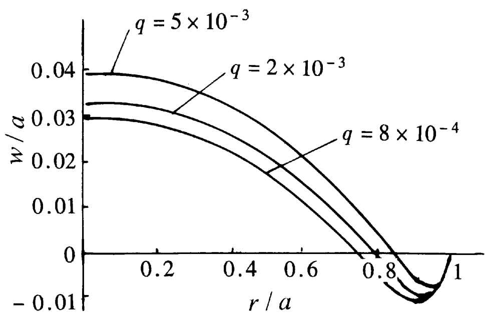

图 2　$h = 0.3\text{mm}$

(2) 简单支承情形

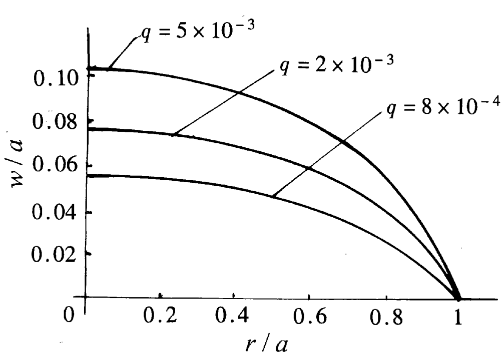

图 3　$h = 0.1\text{mm}$

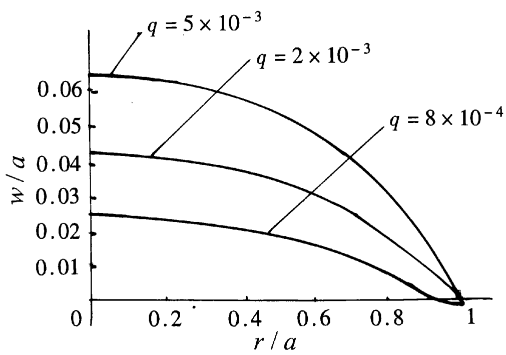

图 4　$h = 0.3\text{mm}$

从以上挠度曲线可以看出:

1. 随着板的厚度增加, 边界层的效应增加, 和板的中心挠度减小;

2. 当边界条件改变时, 边界层的效应也改变, 边缘铰链支承板的边界层效应强于边缘简单支承板的边界层效应, 因前者板的边缘在板平面内不能移动故•

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Application of the Modified Method of Multiple Scales to the Bending Problems for Circular Thin Plate at Very Large Deflection and the Asymptotics of Solutions(Ⅰ)

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Abstract

In this paper, the modified method of multiple scales is applied to study the bending problems for circular thin plate with large deflection under the hinged and simply supported edge conditions. The series solutions are constructed, the boundary layer effects are analysed and their asymptotics are proved.

Key words circular plate, large deflection, boundary layer effect, asymptotics, modified method of multiple scales