应用数学和力学, 第 19 卷 第 8 期(1998 年 8 月)
Applied Mathematics and Mechanics

应用数学和力学编委会编
重庆出版社出版

层合板一个新的高阶理论解析解*

范业立[①]　林芳勇[①]

(黄小清推荐, 1997 年 2 月 24 日收到)

摘　要

本文以复合材料的 Reddy 高阶理论为基础, 引进一个位移函数 Φ, 将原来求解的微分方程组转化为一个高阶微分方程, 得到了四边简支情况下的 Navier 型解, 和一对边简支另一对边任意情况下的 Levy 型解. 文中列举了算例进行比较, 其数值结果和文献上已有结果相吻合, 表明本文采用的解法是可靠的. Reddy 高阶理论未知数不多, 但精度比一阶剪切变形理论要好, 计算时无需用剪切修正系数, 计算较为简单.

关键词　层合板　高阶理论　解析解
中图分类号　TB301

§1. 引　言

复合材料由于它具有优越的材料物理性能, 日益受到工程技术人员的关注, 从而复合材料叠层结构理论也得到不断发展, 由经典理论发展到一阶剪切理论, 由高阶理论发展到弹性理论. Whitney 和 Leissa[1] 找到了经典理论层合板弯曲问题的 Navier 型解. Whitney 和 Pagano[2] 找到了一阶剪切变形理论的 Navier 解. Pagano[3] 用三维弹性理论求解层合板问题求得了解析解. Reddy[4] 等人使用状态空间法给对称层合板的一阶剪切变形理论求得了 Levy 型解. 而 A. A. Khdeir[5] 等人对 Reddy 高阶理论进行研究, 使用状态空间法解常微分方程, 求得了该理论 Levy 型解. 关于层合板问题的求解, 国内也有学者从事这方面的研究工作; 叶开沅等人[6] 求得了四边固定层合板自由振动时的解析解. 闻立洲[7] 求得了一阶剪切变形理论任意边界条件下自由振动的解析解.

本文以 Reddy 高阶理论为基础, 引进一位移函数 Φ, 将正交对称层合板情况下三个基本方程转化为一个高阶微分方程, 求得了四边简支边条件下的 Navier 型解, 及一对边简支另一对边任意的 Levy 解. 为了检验所提方法的正确性, 文中对几个算例进行了数值计算与比较, 所得结果和文献上已有结果吻合, 说明了本文解法是可靠的. 本文方法简单、通用、使用也方便.

§2. 基本方程的简化

在小变形、无湿热效应, 只有横向载荷作用下 Reddy[8] 高阶理论的基本方程为:

* 广东省科学基金资助项目

① 华南理工大学工程力学系, 广州 510641

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} \\ L_{12} & L_{22} & L_{23} & L_{24} & L_{25} \\ L_{13} & L_{23} & L_{33} & L_{34} & L_{35} \\ L_{14} & L_{24} & L_{34} & L_{44} & L_{45} \\ L_{15} & L_{25} & L_{35} & L_{45} & L_{55} \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \\ \phi_x \\ \phi_y \\ w_0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{2.1}$$

其中: L_{ij} 为微分算子, 且有 $L_{ij}=L_{ji}$. 引入高阶合力

$$P_i=\int_{-h/2}^{h/2}\sigma_i z^3 dz \quad (i=1,2,6); \qquad (R_1,R_2)=\int_{-h/2}^{h/2} z^2(\sigma_5,\sigma_4)\,dz \tag{2.2}$$

和叠层刚度

$$(A_{ij},B_{ij},D_{ij},E_{ij},F_{ij},H_{ij})=\int_{-h/2}^{h/2}Q_{ij}(1,z,z^2,z^3,z^4,z^6)\,dz \qquad (i,j=1,2,6) \tag{2.3}$$

$$(A_{ij},D_{ij},F_{ij})=\int_{-h/2}^{h/2}Q_{ij}(1,z^2,z^4)\,dz \qquad (i,j=4,5) \tag{2.4}$$

该理论的边界条件表示如下:

w 或 Q_n; $\partial w/\partial n$ 或 P_n; ϕ_n 或 M_n; ϕ_{ns} 或 M_{ns}; 在边界 r 上

这里 r 是层合板中面 Ω 的边界, 且有

$$\left.\begin{aligned} & M_n=M_1n_x^2+M_2n_y^2+2M_6n_xn_y;\quad M_{ns}=(M_2-M_1)n_xn_y+M_6(n_x^2-n_y^2) \\ & P_n=P_1n_x^2+P_2n_y^2+2P_6n_xn_y;\quad P_{ns}=(P_2-P_1)n_xn_y+P_6(n_x^2-n_y^2) \\ & Q_n=Q_1n_x+Q_2n_y+\frac{4}{3h^2}\left(\frac{\partial P_{ns}}{\partial s}+\frac{\partial P_n}{\partial n}\right);\quad \bar{M}_i=M_i-\frac{4}{3h^2}P_i \qquad (i=1,2,6) \\ & \bar{Q}_i=Q_i-\frac{4}{h^2}R_i \qquad (i=1,2);\quad \frac{\partial}{\partial n}=n_x\frac{\partial}{\partial x}+n_y\frac{\partial}{\partial y};\quad \frac{\partial}{\partial s}=n_x\frac{\partial}{\partial y}-n_y\frac{\partial}{\partial x} \end{aligned}\right\} \tag{2.5}$$

其中

$$\left.\begin{aligned} M_1 &= D_{11}\frac{\partial\phi_x}{\partial x}+D_{12}\frac{\partial\phi_y}{\partial y}+F_{11}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+F_{12}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right) \\ M_2 &= D_{12}\frac{\partial\phi_x}{\partial x}+D_{22}\frac{\partial\phi_y}{\partial y}+F_{12}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+F_{22}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right) \\ M_6 &= D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)+F_{66}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right) \\ Q_2 &= A_{44}\left(\phi_y+\frac{\partial w}{\partial y}\right)+D_{44}\left(-\frac{4}{h^2}\right)\left(\phi_y+\frac{\partial w}{\partial y}\right) \\ Q_1 &= A_{55}\left(\phi_x+\frac{\partial w}{\partial x}\right)+D_{55}\left(-\frac{4}{h^2}\right)\left(\phi_x+\frac{\partial w}{\partial x}\right) \\ P_1 &= F_{11}\left(\frac{\partial\phi_x}{\partial x}\right)+F_{12}\left(\frac{\partial\phi_y}{\partial y}\right)+H_{11}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+H_{12}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right) \\ P_2 &= F_{12}\left(\frac{\partial\phi_x}{\partial x}\right)+F_{22}\left(\frac{\partial\phi_y}{\partial y}\right)+H_{12}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+H_{22}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right) \\ P_6 &= F_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)+H_{66}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right) \\ R_2 &= D_{44}\left(\phi_y+\frac{\partial w}{\partial y}\right)+F_{44}\left(-\frac{4}{3h^2}\right)\left(\phi_y+\frac{\partial w}{\partial y}\right) \\ R_1 &= D_{55}\left(\phi_x+\frac{\partial w}{\partial x}\right)+F_{55}\left(-\frac{4}{3h^2}\right)\left(\phi_x+\frac{\partial w}{\partial x}\right) \end{aligned}\right\} \tag{2.6}$$

对于对称层合板，下列宏观刚度为零•

$$B_{ij}=E_{ij}=0\quad(\text{对于 } i,j=1,2,4,5,6);\quad A_{45}=D_{45}=H_{45}=0$$

$$A_{16}=A_{26}=D_{16}=D_{26}=H_{16}=H_{26}=F_{16}=F_{26}=0$$

则原方程可简化为:

$$\begin{bmatrix} L_{11} & L_{12} & 0 & 0 & 0 \\ L_{12} & L_{22} & 0 & 0 & 0 \\ 0 & 0 & L_{33} & L_{34} & L_{35} \\ 0 & 0 & L_{34} & L_{44} & L_{45} \\ 0 & 0 & L_{35} & L_{45} & L_{55} \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \\ \psi_x \\ \psi_y \\ w_0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ q \end{Bmatrix} \tag{2.1$'$}$$

在求解弯曲问题时只用到后三个方程，展开后可写为:

$$\left.\begin{aligned} L_{33}\psi_x+L_{34}\psi_y+L_{35}w_0&=0\\ L_{34}\psi_x+L_{44}\psi_y+L_{45}w_0&=0\\ L_{35}\psi_x+L_{45}\psi_y+L_{55}w_0&=q \end{aligned}\right\} \tag{2.7}$$

其中:

$$\left.\begin{aligned} L_{33}={}&\left(D_{11}-\frac{8}{3h^2}F_{11}+\frac{16}{9h^4}H_{11}\right)\frac{\partial^2}{\partial x^2}+\left(D_{66}-\frac{8}{3h^2}F_{66}+\frac{16}{9h^4}H_{66}\right)\frac{\partial^3}{\partial y^3}\\ &-\left(A_{55}-\frac{8}{h^2}D_{55}+\frac{16}{h^4}F_{55}\right)\\ L_{34}={}&\left[D_{12}+D_{66}-\frac{8}{3h^2}(F_{12}+F_{66})+\frac{16}{9h^4}(H_{12}+H_{66})\right]\frac{\partial^2}{\partial x\partial y}\\ L_{35}={}&-\left(A_{55}-\frac{8}{h^2}D_{55}+\frac{16}{h^4}F_{55}\right)\frac{\partial}{\partial x}-\frac{4}{3h^2}\left\{\left(F_{11}-\frac{4}{3h^2}H_{11}\right)\frac{\partial^3}{\partial x^3}\right.\\ &\left.+\left[F_{12}+2F_{66}-\frac{4}{3h^2}(H_{12}+2H_{66})\right]\frac{\partial^3}{\partial x\partial y^2}\right\}\\ L_{44}={}&\left(D_{66}-\frac{8}{3h^2}F_{66}+\frac{16}{9h^4}H_{66}\right)\frac{\partial^2}{\partial x^2}+\left(D_{22}-\frac{8}{3h^2}F_{22}+\frac{16}{9h^4}H_{22}\right)\frac{\partial^3}{\partial y^3}\\ &-\left(A_{44}-\frac{8}{h^2}D_{44}+\frac{16}{h^4}F_{44}\right)\\ L_{45}={}&-\left(A_{44}-\frac{8}{h^2}D_{44}+\frac{16}{h^4}F_{44}\right)\frac{\partial}{\partial x}\\ &-\frac{4}{3h^2}\left\{\left[F_{12}+2F_{66}-\frac{4}{3h^2}(H_{12}+2H_{66})\right]\frac{\partial^3}{\partial x^2\partial y}+\left(F_{22}+\frac{4}{3h^2}H_{22}\right)\frac{\partial^3}{\partial y^3}\right\}\\ L_{55}={}&-\left(A_{55}-\frac{8}{h^2}D_{55}+\frac{16}{h^4}F_{55}\right)\frac{\partial^2}{\partial x^2}-\left(A_{44}-\frac{8}{h^2}D_{44}+\frac{16}{h^4}F_{44}\right)\frac{\partial^2}{\partial y^2}\\ &+\frac{16}{9h^4}\left[H_{11}\frac{\partial^4}{\partial x^4}+2(H_{12}+2H_{66})\frac{\partial^4}{\partial x^2\partial y^2}+H_{22}\frac{\partial^4}{\partial y^4}\right] \end{aligned}\right\} \tag{2.8}$$

现引入一位移函数 $\Phi(x,y)$，使

$$\left.\begin{aligned} \psi_x&=(L_{35}L_{44}-L_{45}L_{34})\Phi\\ \psi_y&=-(L_{35}L_{34}-L_{45}L_{33})\Phi\\ w_0&=(L_{34}L_{34}-L_{44}L_{33})\Phi \end{aligned}\right\} \tag{2.9}$$

把上式(2.9)代入基本方程(2.7)式，方程组前二式自动满足，只剩下最后一个方程需要求解，它是一个8阶偏微分方程，经过推导得：

$$\left(C_1 \frac{\partial^8}{\partial x^8} + C_2 \frac{\partial^8}{\partial x^6 \partial y^2} + C_3 \frac{\partial^8}{\partial x^4 \partial y^4} + C_4 \frac{\partial^8}{\partial x^2 \partial y^6} + C_5 \frac{\partial^8}{\partial x^8} + C_6 \frac{\partial^6}{\partial x^6} + C_7 \frac{\partial^6}{\partial x^4 \partial y^2} + C_8 \frac{\partial^6}{\partial x^2 \partial y^4} + C_9 \frac{\partial^6}{\partial x^6} + C_{10} \frac{\partial^4}{\partial x^4} + C_{11} \frac{\partial^4}{\partial x^2 \partial y^2} + C_{12} \frac{\partial^4}{\partial y^4} + C_{13} \frac{\partial^2}{\partial x^2} + C_{14} \frac{\partial^2}{\partial y^2} \right) \Phi = q \tag{2.10}$$

其中：$C_1, C_2, \cdots, C_{14}$ 是系数，详见附录1·

将算子的实际式子(2.8)代入(2.9)式，得

$$\left.\begin{aligned} \psi_x &= \left(A_{x1} \frac{\partial^5}{\partial x^5} + A_{x2} \frac{\partial^5}{\partial x^3 \partial y^2} + A_{x3} \frac{\partial^5}{\partial x \partial y^4} + A_{x4} \frac{\partial^3}{\partial x^3} + A_{x5} \frac{\partial^3}{\partial x \partial y^2} + A_{x6} \frac{\partial}{\partial x} \right) \Phi \\ \psi_y &= \left(A_{y1} \frac{\partial^5}{\partial y^5} + A_{y2} \frac{\partial^5}{\partial x^2 \partial y^3} + A_{y3} \frac{\partial^5}{\partial x^4 \partial y} + A_{y4} \frac{\partial^3}{\partial y^3} + A_{y5} \frac{\partial^3}{\partial x^2 \partial y} + A_{y6} \frac{\partial}{\partial y} \right) \Phi \\ w_0 &= \left(A_{w1} \frac{\partial^4}{\partial x^4} + A_{w2} \frac{\partial^4}{\partial x^2 \partial y^2} + A_{w3} \frac{\partial^4}{\partial y^4} + A_{w4} \frac{\partial^2}{\partial x^2} + A_{w5} \frac{\partial^2}{\partial y^2} + A_{w6} \right) \Phi \end{aligned}\right\} \tag{2.11}$$

其中：系数 $A_{x1}, \cdots, A_{x6}, A_{y1}, \cdots, A_{y6}, A_{w1}, \cdots, A_{w6}$ 详见附录1· 由于求解微分方程为8阶，它需要每边提供4个边界条件· 下面将边界条件的广义内力也用位移函数 Φ 来表示·

§3. 边界条件的位移函数 Φ 表示

与边界条件有关的各内力分量用 ψ_x, ψ_y, w_0 表示的表达式已在(2.6)式中列出，将(2.11)式代入，经整理后，可得只有一个位移函数 $\Phi(x, y)$ 表示的表达式：

$$\begin{aligned} M_1 = \Bigg\{ & \left[\left(D_{11} - \frac{4}{3h^2} F_{11} \right) A_{x1} - \frac{4}{3h^2} F_{11} A_{w1} \right] \frac{\partial^6}{\partial x^6} + \left[\left(D_{11} - \frac{4}{3h^2} F_{11} \right) A_{x2} \right. \\ & \left. + \left(D_{12} - \frac{4}{3h^2} F_{12} \right) A_{y3} - \frac{4}{3h^2} (F_{11} A_{w2} + F_{12} A_{w1}) \right] \frac{\partial^6}{\partial x^4 \partial y^2} \\ & + \left[\left(D_{11} - \frac{4}{3h^2} F_{11} \right) A_{x3} + \left(D_{12} - \frac{4}{3h^2} F_{12} \right) A_{y2} - \frac{4}{3h^2} (F_{11} A_{w3} + F_{12} A_{w2}) \right] \frac{\partial^6}{\partial x^2 \partial y^4} \\ & + \left[\left(D_{12} - \frac{4}{3h^2} F_{12} \right) A_{y1} - \frac{4}{3h^2} F_{12} A_{w3} \right] \frac{\partial^6}{\partial y^6} + \left[\left(D_{11} - \frac{4}{3h^2} F_{11} \right) A_{x4} - \frac{4}{3h^2} F_{11} A_{w4} \right] \frac{\partial^4}{\partial x^4} \\ & + \left[\left(D_{11} - \frac{4}{3h^2} F_{11} \right) A_{x5} + \left(D_{12} - \frac{4}{3h^2} F_{12} \right) A_{y5} - \frac{4}{3h^2} (F_{11} A_{w5} + F_{12} A_{w4}) \right] \frac{\partial^4}{\partial x^2 \partial y^2} \\ & + \left[\left(D_{12} - \frac{4}{3h^2} F_{12} \right) A_{y4} - \frac{4}{3h^2} F_{12} A_{w5} \right] \frac{\partial^4}{\partial y^4} + \left[\left(D_{11} - \frac{4}{3h^2} F_{11} \right) A_{x6} - \frac{4}{3h^2} F_{11} A_{w6} \right] \frac{\partial^2}{\partial x^2} \\ & + \left[\left(D_{12} - \frac{4}{3h^2} F_{12} \right) A_{y6} - \frac{4}{3h^2} F_{12} A_{w6} \right] \frac{\partial^2}{\partial y^2} \Bigg\} \Phi(x, y) \\ & \cdots\cdots\cdots \end{aligned} \tag{3.1}$$

通过这样的转换，把求解一个偏微分方程组的问题化为求解一个8阶偏微分方程的问题，未知数只有 $\Phi(x, y)$ 一个，给解析求解带来方便·

§4. 层合板的边界条件表示

根据边界条件的一般形式，可写出矩形板的三种基本形式：

1. 简支边

$$\left.\begin{array}{ll} \text{当 } x = x_1 \text{ 或 } x = x_2 \text{ 时，} & M_1 = P_1 = \psi_y = w = 0 \\ \text{当 } y = y_1 \text{ 或 } y = y_2 \text{ 时，} & M_2 = P_2 = \psi_x = w = 0 \end{array}\right\} \quad (4.1)$$

2. 自由边

$$\left.\begin{array}{ll} \text{当 } x = x_1, \text{ 或 } x = x_2 \text{ 时，} & M_1 = P_1 = 0;\ M_6 - \dfrac{4}{3h^2}P_6 = 0 \\ & Q_1 - \dfrac{4}{h^2}R_1 + 4\dfrac{4}{3h^2}\left(\dfrac{\partial P_6}{\partial y} + \dfrac{\partial P_1}{\partial x}\right) = 0 \\ \text{当 } y = y_1, \text{ 或 } y = y_2 \text{ 时，} & M_2 = P_2 = 0;\ M_6 - \dfrac{4}{3h^2}P_6 = 0 \\ & Q_2 - \dfrac{4}{h^2}R_2 + \dfrac{4}{3h^2}\left(\dfrac{\partial P_6}{\partial x} + \dfrac{\partial P_2}{\partial y}\right) = 0 \end{array}\right\} \quad (4.2)$$

3. 固定边

$$\left.\begin{array}{ll} \text{当 } x = x_1 \text{ 或 } x = x_2 \text{ 时，} & \partial w/\partial x = \psi_x = \psi_y = w = 0 \\ \text{当 } y = y_1 \text{ 或 } y = y_2 \text{ 时，} & \partial w/\partial y = \psi_x = \psi_y = w = 0 \end{array}\right\} \quad (4.3)$$

§5. 四边简支边界条件下层合板的解析解

5.1 位移函数的推导

设板长为 a，宽为 b，简支边条件为：

$$x = 0;\ a; \qquad M_1 = P_1 = \psi_y = w = 0;$$
$$y = 0;\ b; \qquad M_2 = P_2 = \psi_x = w = 0;$$

现取位移函数 $\Phi(x, y)$ 为双三角级数，即

$$\Phi(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin \alpha_m x \sin \beta_n y \quad (5.1)$$

其中： $\alpha_m = m\pi/\alpha$， $\beta_n = n\pi/b$

由 M_1，M_2，P_1，P_2，w，ψ_x，ψ_y 的函数表达式知，$\Phi(x, y)$ 能满足所有的边界条件·

将横向荷重 $q(x, y)$ 也展成双三角级数，如下式：

$$q(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q_{mn} \sin \alpha_m x \sin \beta_n y \quad (5.2)$$

将式代入所求方程(2.10)式中，可得

$$C_1A_{mn}\alpha_m^8 + C_2A_{mn}\alpha_m^6\beta_n^2 + C_3A_{mn}\alpha_m^4\beta_n^4 + C_4A_{mn}\alpha_m^2\beta_n^6 + C_5A_{mn}\beta_n^8 - C_6A_{mn}\alpha_m^6$$
$$- C_7A_{mn}\alpha_m^4\beta_n^2 - C_8A_{mn}\alpha_m^2\beta_n^4 - C_9A_{mn}\beta_n^6 + C_{10}A_{mn}\alpha_m^4 + C_{11}A_{mn}\alpha_m^2\beta_n^2$$
$$+ C_{12}A_{mn}\beta_n^4 - C_{13}A_{mn}\alpha_m^2 - C_{14}A_{mn}\beta_n^2 = q_{mn}$$

从而可得级数系数为：$A_{mn} = q_{mn}/F_{mn}$·

其中：

$$F_{mn} = C_1\alpha_m^8 + C_2\alpha_m^6\beta_n^2 + C_3\alpha_m^4\beta_n^4 + C_4\alpha_m^2\beta_n^6 + C_5\beta_n^8 - C_6\alpha_m^6 - C_7\alpha_m^4\beta_n^2$$
$$- C_8\alpha_m^2\beta_n^4 - C_9\beta_n^6 + C_{10}\alpha_m^4 + C_{11}\alpha_m^2\beta_n^2 + C_{12}\beta_n^4 - C_{13}\alpha_m^2 - C_{14}\beta_n^2$$

5.2 数值例子和比较

设复合材料特性如下：$E_1 = 25\times10^6$psi(172.35GPa)；$E_2 = 1\times10^6$psi(6.894GPa)；$G_{12} =$

$G_{13} = 0.5 \times 10^6$psi(3.447GPa); $G_{23} = 0.2 \times 10^6$psi(1.379GPa); $\nu_{12} = \nu_{13} = 0.25$•

引入无量纲符号:

$$w = w\left(a/2, b/2\right. E_2 \times 10^2 h^3/(q_0 a^4)$$

现考虑层合板弯曲问题的三个算例:

例1 正方形叠层板(0°/90°/0°)铺设, 各层等厚度, 受正弦分布荷载 $q(x, y) = q_0\sin(\pi x/a)\sin(\pi y/b)$ 作用, 计算结果及对比见表1所示:

表1 (0°/90°/0°)正方形层合板受正弦荷载作用, 板中心点 w 值比较

a/h	Reddy[8]	本文解	FSDT		
			$K_1^2 = K_2^2 = 1$	$K_1^2 = K_2^2 = 5/6$	$K_1^2 = K_2^2 = 3/4$
4	1.9218	1.9218	1.5681	1.7763	1.9122
10	0.7125	0.7125	0.6306	0.6693	0.6949
100	0.4342	0.4342	0.4333	0.4337	0.4340

由比较可看出; 本文结果与Reddy理论传统方法相吻合, 比FSDT理论好•

例2 矩形叠层板 $b/a = 3$, 其余条件和例1相同, 计算结果及对比见表2:

表2 矩形板受正弦荷重作用中心点处挠度 w 值比较($b/a = 3$)

a/h	Reddy[8]	本文解	Pagano[3]	FSDT($K_1^2 = K_2^2 = 5/6$)
4	2.6411	2.6411	2.82	2.3626
10	0.8622	0.8622	0.919	0.803
20	0.5937	0.5937	0.610	0.5784
100	0.5070	0.5070	0.508	0.5064

从比较中可看出, 本文结果和Reddy理论传统方法一样, 比FSDT理论好•

例3 正方形板受均布荷载重作用, 其中, $q_{mn} = 16q_0/\pi^2 mn$, 只取奇数项, 计算结果及对比见表3(表中 N 代表叠加项数):

表3 正方形层合板受均布荷重作用中心点挠度 w 值比较

a/h	Reddy[8]		本文解		FSDT($K_1^2 = K_2^2 = 5/6$)	
	$N = 9$	$N = 29$	$N = 9$	$N = 29$	$N = 9$	$N = 29$
2	7.7681	7.7661	7.7681	7.7661	7.7170	7.0666
4	2.9103	2.9091	2.9103	2.9091	2.5623	2.5697
10	1.0903	1.0900	1.0903	1.0900	1.0244	1.0220
20	0.7661	0.7660	0.7661	0.7660	0.7574	0.7573
50	0.6839	0.6838	0.6839	0.6838	0.6808	0.6807
100	0.6705	0.6705	0.6705	0.6705	0.6679	0.6697

从比较中可看出, 本文结果和Reddy理论传统方法吻合得很好•

§ 6. 一对边简支, 另一对边任意的 Levy 型解

6.1 位移函数的推导

设板的形式如图 1 所示, 长为 a, 宽为 b, 边界条件为:

$$x = 0, a; \qquad M_1 = P_1 = \psi_y = w = 0$$

$$y = 0, b; \qquad \text{任意}$$

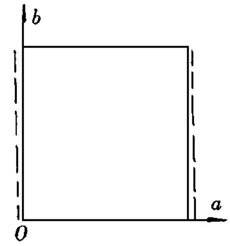

图 1

取位移函数 $\Phi(x, y)$ 为下示的单三角级数形式:

$$\Phi(x, y) = \sum_{m=1}^{\infty} \phi_m(y) \sin \alpha_m x \tag{a}$$

由 M_1, P_1, w, ψ_y 的函数表达式知, 该位移函数满足 $x = 0$, a 的边界条件, 所要求的是 $\phi_m(y)$, 它是 y 的函数• 把横向荷载也展成单三角级数:

$$q(x, y) = \sum_{m=1}^{\infty} Q_m(y) \sin \alpha_m(x) \tag{b}$$

由于控制方程(2.10)是一个 8 阶偏微分方程, 把(a)、(b)代入(2.10)式, 可得:

$$\begin{aligned} &C_1 \alpha_m^8 \phi_m(y) - C_2 \alpha_m^6 \phi_m^{(2)}(y) + C_3 \alpha_m^4 \phi_m^{(4)}(y) - C_4 \alpha_m^2 \phi_m^{(6)}(y) + C_5 \phi_m^{(8)}(y) \\ &- C_6 \alpha_m^6 \phi_m(y) + C_7 \alpha_m^4 \phi_m^{(2)}(y) - C_8 \alpha_m^2 \phi_m^{(4)}(y) + C_9 \phi_m^{(6)}(y) + C_{10} \alpha_m^4 \phi_m(y) \\ &- C_{11} \alpha_m^2 \phi_m^{(2)}(y) + C_{12} \phi_m^{(4)}(y) - C_{13} \alpha_m^2 \phi_m(y) + C_{14} \phi_m^{(2)}(y) = Q_m \end{aligned} \tag{6.1}$$

整理后, 可得:

$$a_{m1} \phi_m^{(8)}(y) + a_{m2} \phi_m^{(6)}(y) + a_{m3} \phi_m^{(4)}(y) + a_{m4} \phi_m^{(2)}(y) + a_{m5} \phi_m(y) = Q_m \tag{6.1$'$}$$

其中:

$$a_{m1} = C_5;\ a_{m2} = C_9 - C_4 \alpha_m^2;\ a_{m3} = C_3 \alpha_m^4 - C_8 \alpha_m^2 + C_{12}$$

$$a_{m4} = - C_2 \alpha_m^6 + C_7 \alpha_m^4 - C_{11} \alpha_m^2 + C_{14};\ a_{m5} = C_1 \alpha_m^8 - C_8 \alpha_m^6 + C_{10} \alpha_m^4 - C_{13} \alpha_m^2$$

(6.1)$'$ 式为一非齐次 8 阶常微分方程, 其特征方程为:

$$a_{m1} \lambda_m^8 + a_{m2} \lambda_m^6 + a_{m3} \lambda_m^4 + a_{m4} \lambda_m^2 + a_{m5} = 0 \tag{6.1$''$}$$

令 $x_m = \lambda_m^2$ 代入上式得

$$a_{m1} x_m^4 + a_{m2} x_m^3 + a_{m3} x_m^2 + a_{m4} x_m + a_{m5} = 0$$

左右同除以 a_{m1}, 可简化为:

$$x_m^4 + b_{m1} x_m^3 + b_{m2} x_m^2 + b_{m3} x_m + b_{m4} = 0$$

它的根与下面两个方程的 4 个根完全相同•

$$x_m^2 + \left(b_{m1} \pm \sqrt{8y + b_{m1} - 4b_{m2}} \right) \frac{x_m}{2} + \left(y + \frac{b_{m1} y + b_{m3}}{\sqrt{8y + b_{m1} - 4b_{m2}}} \right) = 0$$

$$x_m^2 - \left(b_{m1} + \sqrt{8y + b_{m1} - 4b_{m2}} \right) \frac{x_m}{2} + \left(y - \frac{b_{m1} y + b_{m3}}{\sqrt{8y + b_{m1} - 4b_{m2}}} \right) = 0$$

式中 y 是三次方程

$$8y^3 - 4b_{m2} y^2 + (2b_{m1} b_{m3} - 8b_{m4}) y + b_{m4}(4b_{m2} - b_{m1}^2) - b_{m3}^2 = 0 \tag{c}$$

的任一实根•

这样一来, 可求得 λ_m 的 8 个根, 则其齐次方程的解为:

$$\phi_m(y) = C_{m1} e^{\lambda_{m1} y} + C_{m2} e^{\lambda_{m2} y} + C_{m3} e^{\lambda_{m3} y} + C_{m4} e^{\lambda_{m4} y} + C_{m5} e^{\lambda_{m5} y}$$

$$+ C_{m6}e^{\lambda_{m6}y} + C_{m7}e^{\lambda_{m7}y} + C_{m8}e^{\lambda_{m8}y}$$

设非齐次方程的特解为 $Q^*_m(y)$, 因此可得原微分方程的通解为:

$$\phi_m(y) = C_{m1}e^{\lambda_{m1}y} + C_{m2}e^{\lambda_{m2}y} + C_{m3}e^{\lambda_{m3}y} + C_{m4}e^{\lambda_{m4}y} + C_{m5}e^{\lambda_{m5}y} + C_{m6}e^{\lambda_{m6}y} + C_{m7}e^{\lambda_{m7}y} + C_{m8}e^{\lambda_{m8}y} + Q^*_m(y) \tag{6.2}$$

其中: C_{m1}, C_{m2}, …, C_{m8} 是8个需要确定的系数, 可利用剩下的两个边界条件来决定• 由于每边有 4 个条件加起来共有 8 个条件, 正好可求得这 8 个系数• 用单三角级数表示的位移函数 $\Phi(x, y)$ 总可以写成下列形式:

$$\Phi(x, y) = \sum_{m=1}^{\infty}\left[C_{m1}e^{\lambda_{m1}y} + C_{m2}e^{\lambda_{m2}y} + C_{m3}e^{\lambda_{m3}y} + C_{m4}e^{\lambda_{m4}y} + C_{m5}e^{\lambda_{m5}y} + C_{m6}e^{\lambda_{m6}y} + C_{m7}e^{\lambda_{m7}y} + C_{m8}e^{\lambda_{m8}y} + Q^*_m(y)\right]\sin\alpha_m x \tag{6.3}$$

6.2　数值计算和结果比较

为了检验上述方法的可靠性, 作了如下的例题计算, 所考虑的层合板是(0°/ 90°/ 0°) 铺设, 每层厚度相同• 设复合材料性质如下:

表 4　　(0°/90°/0°)层合板在各种边界条件下中心点处的挠度值 w

a/b	h/a	SS 本文解 $N=9$	SS 本文解 $N=29$	SS Khdeir[5]	CC 本文解 $N=9$	CC 本文解 $N=29$	CC Khdeir[5]
3	0.2	46.39	46.34	46.33	26.85	26.81	26.80
	0.14	96.63	96.53	96.52	47.69	47.59	47.57
4	0.2	21.67	21.62	21.61	13.06	13.01	13.01
	0.14	41.58	41.48	41.46	22.54	22.44	22.42
5	0.2	12.20	12.16	12.15	7.24	7.37	7.36
	0.14	22.05	21.95	21.83	12.80	12.71	12.69

a/b	h/a	FF 本文解 $N=9$	FF 本文解 $N=29$	FF Khdeir[5]	CF 本文解 $N=9$	CF 本文解 $N=29$	CF Khdeir[5]
3	0.2	438.2	437.8	437.7	104.21	104.17	104.16
	0.14	934.3	933.5	933.3	217.9	217.0	216.9
4	0.2	435.9	435.4	435.4	55.95	55.90	55.89
	0.14	936.2	935.3	935.1	112.58	112.49	112.47
5	0.2	436.2	435.9	435.7	32.31	32.27	32.26
	0.14	937.1	936.2	936.0	62.83	62.75	62.73

a/b	h/a	SC 本文解 $N=9$	SC 本文解 $N=29$	SC Khdeir[5]	SF 本文解 $N=9$	SF 本文解 $N=29$	SF Khdeir[5]
3	0.2	35.22	35.17	35.17	237.1	236.6	236.5
	0.14	66.20	66.10	66.08	512.4	511.6	511.4
4	0.2	16.95	16.91	16.90	191.76	191.73	191.72
	0.14	30.32	30.22	30.20	415.0	414.20	414.0
5	0.2	9.61	9.57	9.56	157.89	157.83	157.83
	0.14	16.83	16.73	16.72	340.7	339.8	339.6

$E_1 = 19.2 \times 10^6 \text{psi}(132.36\text{GPa})$; $E_2 = 1.56 \times 10^6 \text{psi}(10.75\text{GPa})$

$G_{12} = G_{13} = 0.82 \times 10^6 \text{psi}(5.65\text{GPa})$;

$G_{23} = 0.523 \times 10^6 \text{psi}(3.60\text{GPa})$; $\nu_{12} = \nu_{13} = 0.24$

$\bar{w} = C[w(a/2, b/2)/q^0] 10^6$; $a = 200 \text{in}(5.08\text{m})$

受均布荷载 $q^0 = \sum_{m=1,3,5}^{\infty} \frac{2}{a\pi m} \sin \alpha_m x$

计算结果与比较见表 4•

表中的 S 表示简支边, C 表示固定边, F 表示自由边• N 表示项数• 由表中比较可看出:

(1) 用本文方法算得的结果 ($N = 29$) 和文献[5] 用状态空间法所得的结果非常接近, 有较好的精度•

(2) 由表中所列数字可看出, 在相同的边长比和厚跨比条件下(例如 $a/b = 3$, $h/a = 0.2$) 边界约束较强的板的中心挠度一般是较小的, 即 C C 情况最小, F F 情况最大, 这点和一般的物理概念是符合的•

§7. 结 束 语

(1) 本文的特点是引入一个位移函数 $\Phi(x, y)$ 使 Reddy 高阶理论的对称正交复合层合板的微分方程组转化为用 $\Phi(x, y)$ 表示的一个高阶微分方程, 得到了四边简支情况下的 Navier 解, 和一对边简支另一对边任意情况下的 Levy 型解•

(2) 在上述解的基础上利用叠加原理可得到各种边界条件下对称层合板的解析解•

(3) 对四边简支板本文举了三个例子• 在第一个例子中, 本文得出的结果与 Reddy 理论挠度相吻合, 比 FSDT 理论的结果要好• 在第二个例子中, 无论 $a/h = 4, 10, 20, 100$, 本文得出的结果和 Reddy 理论传统方法完全一致, 比 Pagano 的弹性理论解要小, 比 FSDT 结果大• 在第三个例子中, 无论 $a/h = 2, 4, 10, \cdots, 100$, 当取项数相同时($N = 9$, 或 $N = 29$), 所得结果和 Reddy 理论传统方法一致, 这就说明了本文所提供的计算方法是正确的•

(4) 对一对边简支, 另一对边任意的板, 本文列举了层合板在各种边界条件下的数值例子• 共有 6 种情况(SS, SC, SF, CC, CF, FF) 不论边长比为 3, 4, 5, 厚长比 $h/a = 0.2$ 或 0.14, 其计算结果(当 $N = 29$ 时) 都和 Khdeir 等人[5] 的解相接近, 有较高的精度•

(5) 本文的程序均用 MS_FORTRAN5 编制, 可在微机上完成•

附录 1

经过推导高阶理论的控制方程为:

$$\left(C_1 \frac{\partial^8}{\partial x^8} + C_2 \frac{\partial^8}{\partial x^6 \partial y^2} + C_3 \frac{\partial^8}{\partial x^4 \partial y^4} + C_4 \frac{\partial^8}{\partial x^2 \partial y^6} + C_5 \frac{\partial^8}{\partial y^8} + C_6 \frac{\partial^6}{\partial x^6} + C_7 \frac{\partial^6}{\partial x^4 \partial y^2} + C_8 \frac{\partial^6}{\partial x^2 \partial y^4} + C_9 \frac{\partial^6}{\partial y^6} + C_{10} \frac{\partial^4}{\partial x^4} + C_{11} \frac{\partial^4}{\partial x^2 \partial y^2} + C_{12} \frac{\partial^4}{\partial y^4} + C_{13} \frac{\partial^2}{\partial x^2} + C_{14} \frac{\partial^2}{\partial y^2} \right) \Phi = q \quad (2.10)$$

其中

$$C_1 = -\frac{4}{3h^2}\left(F_{11} - \frac{4}{3h^2} H_{11} \right) A_{x1} + \frac{16}{9h^4} H_{11} A_{w1}$$

$$C_2 = -\frac{4}{3h^2}\left\{ \left(F_{11} - \frac{4}{3h^2} H_{11} \right) A_{x2} + \left[F_{12} + 2F_{66} - \frac{4}{3h^2}(H_{12} + 2H_{66}) \right] (A_{x1} + A_{y3}) \right.$$

$$+\frac{16}{9h^4}\left[H_{11}A_{w2}+2(H_{12}+2H_{66})A_{w1}\right]$$

$$C_3=-\frac{4}{3h^2}\left\{\left(F_{11}-\frac{4}{3h^2}H_{11}\right)A_{x3}+\left[F_{12}+2F_{66}-\frac{4}{3h^2}(H_{12}+2H_{66})\right](A_{x2}+A_{y2})+\left(F_{22}-\frac{4}{3h^2}H_{22}\right)A_{y3}\right\}+\frac{16}{9h^4}\left[H_{11}A_{w3}+H_{22}A_{w1}+2(H_{12}+2H_{66})A_{w2}\right]$$

$$C_4=-\frac{4}{3h^2}\left\{\left[F_{12}+2F_{66}-\frac{4}{3h^2}(F_{12}+2F_{66})\right](A_{x3}+A_{y1})+\left(F_{22}-\frac{4}{3h^2}H_{22}\right)A_{y2}\right\}+\frac{16}{9h^4}\left[H_{22}A_{w2}+2(H_{12}+2H_{66})A_{w3}\right]$$

$$C_5=-\frac{4}{3h^2}\left(F_{22}-\frac{4}{3h^2}H_{22}\right)A_{y1}+\frac{16}{9h^4}H_{22}A_{w3}$$

$$C_6=-\left(A_{55}-\frac{8}{h^2}D_{55}+\frac{16}{h^4}F_{55}\right)(A_{x1}+A_{w2})-\frac{4}{3h^2}\left(F_{11}-\frac{4}{3h^2}H_{11}\right)A_{x4}+\frac{16}{9h^4}H_{11}A_{w4}$$

$$C_7=-\left(A_{55}-\frac{8}{h^2}D_{55}+\frac{16}{h^4}F_{55}\right)(A_{x1}+A_{w2})-\left(A_{44}-\frac{8}{h^2}D_{44}-\frac{16}{h^4}F_{44}\right)(A_{y3}+A_{w1})+\frac{16}{9h^2}\left[H_{11}A_{w5}+2(H_{12}+H_{66})A_{w4}\right]-\frac{4}{3h^2}\left\{\left(F_{11}-\frac{4}{3h^2}H_{11}\right)A_{x5}+\left[F_{12}+2F_{66}-\frac{4}{3h^2}(H_{12}+2H_{66})\right](A_{x4}+A_{y5})\right\}$$

$$C_8=-\left(A_{55}-\frac{8}{h^2}D_{55}+\frac{16}{h^2}F_{55}\right)(A_{x3}+A_{w3})-\left(A_{44}-\frac{8}{h^2}D_{44}+\frac{16}{h^4}F_{44}\right)(A_{y2}+A_{w2})+\frac{16}{9h^4}\left[H_{22}A_{w4}+2(H_{12}+2H_{66})A_{w5}\right]-\frac{4}{3h^2}\left\{\left(F_{22}-\frac{4}{3h^2}H_{22}\right)A_{y5}+\left[F_{12}+2F_{66}-\frac{4}{3h^2}(H_{12}+2H_{66})\right](A_{x5}+A_{y4})\right\}$$

$$C_9=-\left(A_{44}-\frac{8}{h^2}D_{44}+\frac{16}{h^4}F_{44}\right)(A_{y1}+A_{w3})-\frac{4}{3h^2}\left(F_{22}-\frac{4}{3h^2}H_{22}\right)A_{y4}+\frac{16}{9h^4}H_{22}A_{w5}$$

$$C_{10}=-\left(A_{55}-\frac{8}{h^2}D_{55}+\frac{16}{h^4}F_{55}\right)(A_{x4}+A_{w4})+\frac{-4}{3h^2}\left(F_{11}-\frac{4}{3h^2}H_{11}\right)A_{x6}+\frac{16}{9h^4}H_{11}A_{w6}$$

$$C_{11}=-\left(A_{55}-\frac{8}{h^2}D_{55}+\frac{16}{h^4}F_{55}\right)(A_{x5}+A_{w5})-\left(A_{44}-\frac{8}{h^2}D_{44}+\frac{16}{h^4}F_{44}\right)(A_{y5}+A_{w4})+\frac{16}{9h^4}2(H_{12}+2H_{66})A_{w6}-\frac{4}{3h^2}\left[F_{12}+2F_{66}-\frac{4}{3h^2}(H_{12}+2H_{66})\right](A_{x6}+A_{y6})$$

$$C_{12}=-\left(A_{44}-\frac{8}{h^2}D_{44}+\frac{16}{h^4}F_{44}\right)(A_{y4}+A_{w5})-\frac{4}{3h^2}\left(F_{22}-\frac{4}{3h^2}H_{22}\right)A_{y6}+\frac{16}{9h^4}H_{22}A_{w6}$$

$$C_{13}=-\left(A_{44}-\frac{8}{h^2}D_{44}+\frac{16}{h^4}F_{44}\right)(A_{x6}+A_{w6})$$

$$C_{14}=-\left(A_{55}-\frac{8}{h^2}D_{55}+\frac{16}{h^4}F_{55}\right)(A_{y6}+A_{w6})$$

将算子的实际式子(2.8)代入(2.9)式, 得:

$$\left.\begin{aligned}\phi_x&=\left(A_{x1}\frac{\partial^5}{\partial x^5}+A_{x2}\frac{\partial^5}{\partial x^3\partial y^2}+A_{x3}\frac{\partial^5}{\partial x\partial y^4}+A_{x4}\frac{\partial^3}{\partial x^3}+A_{x5}\frac{\partial^3}{\partial x\partial y^2}+A_{x6}\frac{\partial}{\partial x}\right)\Phi\\ \phi_y&=\left(A_{y1}\frac{\partial^5}{\partial y^5}+A_{y2}\frac{\partial^5}{\partial x^2\partial y^3}+A_{y3}\frac{\partial^5}{\partial x^4\partial y}+A_{y4}\frac{\partial^3}{\partial y^3}+A_{y5}\frac{\partial^3}{\partial x^2\partial y}+A_{y6}\frac{\partial}{\partial y}\right)\Phi\\ w_0&=\left(A_{w1}\frac{\partial^4}{\partial x^4}+A_{w2}\frac{\partial^4}{\partial x^2\partial y^2}+A_{w3}\frac{\partial^4}{\partial y^4}+A_{w4}\frac{\partial^2}{\partial x^2}+A_{w5}\frac{\partial^2}{\partial y^2}+A_{w6}\right)\Phi\end{aligned}\right\}\qquad(2.11)$$

其中

$$A_{x1}=-\frac{4}{3h^2}\left\{\left(D_{66}-\frac{8}{3h^2}F_{66}+\frac{16}{9h^4}H_{66}\right)\left(F_{11}-\frac{4}{3h^2}H_{11}\right)\right.$$

$$A_{x2}=-\frac{4}{3h^2}\left\{\left(D_{66}-\frac{8}{3h^2}F_{66}+\frac{16}{9h^4}H_{66}\right)\left[F_{12}+2F_{66}-\frac{4}{3h^2}(H_{12}+2H_{66})\right]+\left(F_{11}-\frac{4}{3h^2}H_{11}\right)\left(D_{22}-\frac{8}{3h^2}F_{22}+\frac{16}{9h^4}H_{22}\right)\left[F_{12}+2F_{66}-\frac{4}{3h^2}(H_{12}+2H_{66})\right]\right.$$

$$\cdot\left[D_{12}+D_{66}-\frac{8}{3h^2}(F_{12}+F_{66})+\frac{16}{9h^2}(H_{12}+H_{66})\right]\Bigg\}$$

$$A_{x3}=-\frac{4}{3h^2}\left\{\left(F_{22}-\frac{4}{3h^2}H_{22}\right)\left[D_{12}+D_{66}-\frac{8}{3h^2}(F_{12}+F_{66})+\frac{16}{9h^2}(H_{12}+H_{66})\right]\right.$$

$$\left.-\left(D_{22}-\frac{8}{3h^2}F_{22}+\frac{16}{9h^4}H_{22}\right)\left[F_{12}+2F_{66}-\frac{4}{3h^2}(H_{12}+2H_{66})\right]\right\}$$

$$A_{x4}=-\left(D_{66}-\frac{8}{3h^2}F_{66}+\frac{16}{9h^4}H_{66}\right)\left(A_{55}-\frac{8}{h^2}D_{55}+\frac{16}{h^4}F_{55}\right)$$

$$-\frac{4}{3h^2}\left(F_{11}-\frac{4}{3h^2}H_{11}\right)\left(A_{44}-\frac{8}{h^2}D_{44}+\frac{16}{h^4}F_{44}\right)$$

$$A_{x5}=-\left(D_{22}-\frac{8}{3h^2}F_{22}+\frac{16}{9h^4}H_{22}\right)\left(A_{55}-\frac{8}{h^2}D_{55}+\frac{16}{h^4}F_{55}\right)$$

$$+\frac{4}{3h^2}\left(A_{44}-\frac{8}{h^2}D_{44}+\frac{16}{h^4}F_{44}\right)\left[F_{12}+2F_{66}-\frac{4}{3h^2}(H_{12}+2H_{66})\right]$$

$$+\left(A_{44}-\frac{8}{h^2}D_{44}+\frac{16}{h^4}F_{55}\right)\left[D_{12}+D_{66}-\frac{8}{3h^2}F_{12}+F_{66}+\frac{16}{9h^4}(H_{12}+H_{66})\right]$$

$$A_{x6}=\left(A_{55}-\frac{8}{h^4}D_{55}+\frac{16}{h^2}F_{55}\right)\left(A_{44}-\frac{8}{h^2}D_{44}+\frac{16}{h^4}F_{44}\right)$$

$$A_{y1}=-\frac{4}{3h^2}\left(D_{66}-\frac{8}{3h^2}F_{66}+\frac{16}{9h^4}H_{66}\right)\left(F_{22}-\frac{4}{3h^2}H_{22}\right)$$

$$A_{y2}=-\frac{4}{3h^2}\left\{\left(D_{66}-\frac{8}{3h^2}F_{66}+\frac{16}{9h^4}H_{66}\right)\left[F_{12}+2F_{66}-\frac{4}{3h^2}(H_{12}+2H_{66})\right]\right.$$

$$+\left(F_{22}-\frac{4}{3h^2}H_{22}\right)\left(D_{11}-\frac{8}{3h^2}F_{11}+\frac{16}{9h^4}H_{11}\right)-\left[F_{12}+2F_{66}-\frac{4}{3h^2}(H_{12}+2H_{66})\right].$$

$$\left.\cdot\left[D_{12}+D_{66}-\frac{8}{3h^2}(F_{12}+F_{66})+\frac{16}{9h^4}(H_{12}+H_{66})\right]\right\}$$

$$A_{y3}=-\frac{4}{3h^2}\left\{\left(F_{11}-\frac{4}{3h^2}H_{11}\right)+\left[D_{12}+D_{66}-\frac{8}{3h^2}(F_{12}+F_{66})+\frac{16}{9h^4}(H_{12}+H_{66})\right]\right.$$

$$\left.-\left(D_{11}-\frac{8}{3h^2}F_{11}+\frac{16}{9h^4}H_{11}\right)\left[F_{12}+2F_{66}-\frac{4}{3h^2}(H_{12}+2H_{66})\right]\right\}$$

$$A_{y4}=-\left(D_{66}-\frac{8}{3h^2}F_{66}+\frac{16}{9h^4}H_{66}\right)\left(A_{44}-\frac{8}{h^2}D_{44}+\frac{16}{h^4}F_{44}\right)$$

$$-\frac{4}{3h^2}\left(-H_{22}\frac{4}{3h^2}+F_{22}\right)\left(A_{55}-\frac{8}{h^2}D_{55}+\frac{16}{h^4}F_{55}\right)$$

$$A_{y5}=-\left(D_{11}-\frac{8}{3h^2}F_{11}+\frac{16}{9h^4}H_{11}\right)\left(A_{44}-\frac{8}{h^2}D_{44}+\frac{16}{h^4}F_{44}\right)$$

$$+\frac{4}{3h^2}\left(A_{55}-\frac{8}{h^2}D_{55}+\frac{16}{h^4}F_{55}\right)\left[F_{12}+2F_{66}-\frac{4}{3h^2}(H_{12}+2H_{66})\right]$$

$$+\left(A_{55}-\frac{8}{h^2}D_{55}+\frac{16}{h^4}F_{55}\right)\left[D_{12}+D_{66}-\frac{8}{3h^2}(F_{12}+F_{66})+\frac{16}{9h^4}(H_{12}+H_{66})\right]$$

$$A_{y6}=\left(A_{55}-\frac{8}{h^2}D_{55}+\frac{16}{h^4}F_{55}\right)\left(A_{44}-\frac{8}{h^2}D_{44}+\frac{16}{h^4}F_{44}\right)$$

$$A_{w1}=-\left(D_{66}-\frac{8}{3h^2}F_{66}+\frac{16}{9h^4}F_{66}\right)\left(D_{11}-\frac{8}{3h^2}F_{11}+\frac{16}{9h^4}H_{11}\right)$$

$$A_{w2}=\left[D_{12}+D_{66}-\frac{8}{3h^2}(F_{12}+F_{66})+\frac{16}{9h^4}(H_{12}+H_{66})\right]^2-\left(D_{22}-\frac{8}{3h^2}F_{22}+\frac{16}{9h^4}H_{22}\right)$$

$$\cdot\left(D_{11}-\frac{8}{3h^2}F_{11}+\frac{16}{9h^4}H_{11}\right)-\left(D_{66}-\frac{8}{3h^2}F_{66}+\frac{16}{9h^4}H_{66}\right)^2$$

$$A_{w3}=-\left(D_{22}-\frac{8}{3h^2}F_{22}+\frac{16}{9h^4}H_{22}\right)\left(D_{66}-\frac{8}{3h^2}F_{66}+\frac{16}{9h^4}H_{66}\right)$$

$$A_{w4}=\left(D_{11}-\frac{8}{3h^2}F_{11}+\frac{16}{9h^4}H_{11}\right)\left(A_{44}-\frac{8}{h^2}D_{44}+\frac{16}{h^4}F_{44}\right)$$

$$+\left(A_{55}-\frac{8}{h^2}D_{55}+\frac{16}{h^4}F_{55}\right)\left(D_{66}-\frac{8}{3h^2}F_{66}+\frac{16}{9h^4}H_{66}\right)$$

$$A_{w5}=\left(A_{44}-\frac{8}{h^2}D_{44}+\frac{16}{h^4}F_{44}\right)\left(D_{66}-\frac{8}{3h^2}F_{66}+\frac{16}{9h^4}H_{66}\right)$$
$$+\left(A_{55}-\frac{8}{h^2}D_{55}+\frac{16}{h^4}F_{55}\right)\left(D_{22}-\frac{8}{3h^2}F_{22}+\frac{16}{9h^4}H_{22}\right)$$
$$A_{w6}=\left(A_{44}-\frac{8}{h^2}D_{44}+\frac{16}{h^4}F_{44}\right)\left(A_{55}-\frac{8}{h^2}D_{55}+\frac{16}{h^4}F_{55}\right)$$

参 考 文 献

1 J. M. Whitney and A. W. Leissa, Analysis of heterogoneous anisotropic plates, *J. Appl. Mech.*, **36**(2) (1969), 261—266.

2 J. M. Whitney and N. J. Pagano, Shear deformation in heterogeneous anisotropic plates, *J. Appl. Mech.*, **37**(4) (1970), 1031—1036.

3 N. J. Pagano, Exact solutions for rectangular bidirectional and sandwich plates, *Journal of Composite Material*, **4**(4) (1970), 20—34.

4 J. N. Reddy, A. A. Khdeir and L. Librescu, The Levy type solutions for symmetric rectangular composite plates using the first_order shear deformation theory, *J. Appl. Mech.*, **54**(3) (1987), 740—742.

5 A. A. Khdeir, J. N. Reddy and L. Librescu, Analysical solution of refined deformation theory for rectangular composite plates, *Internat. J. Solids Structures*, **23**(10) (1987), 1447—1463.

6 Ye Kaiyuan and Deng Liangbo, Free vibration of composite laminated plates clamped at four edges, *Chinese Science Bulletin*, **34**(1) (1989), 25—30.

7 夏传友、闻立洲, 各种边界条件对称正交复合材料层板自由振动的解析解, 复合材料学报, **8**(4) (1991), 88—99.

8 J. N. Reddy, A simple higher_order theory for laminated composite plates, *J. Appl. Mech.*, **51**(4) (1984), 745—752.

An Analytical Solution of Rectangular Laminated Plates by Higher_Order Theory

Fan Yeli Lin Fangyong

(Department of Engineering Mechanics, South China University of Technology, Guangzhou 510641, P. R. China)

Abstract

On the basis of the Reddy's higher_order theory of composites, this paper introduces a displacement function Φ into it and transforms its three differential equations for symmetric cross_ply composites into only one eight_order differential equation generated by the displacement_function. When a proper Φ is chosen, both solutions are obtained, namely, the Navier_type solution of simply supported rectangular laminated plates and the Levy_type solution with the boundary condition where two opposite edges are simply supported and remains are arbitrary. The numerical examples show that the present results concide well with the existing results in the references, thus validating that the present solving method is reliable. The higher_order theory of Reddy is simpler in calculation but has higher precision than the first_order shear deformation theory because the former has fewer unknows than the latter and requires no shear coefficients.

Key words laminated plate, higher_order theory, analytical solution