用"调和函数"表示的压电介质平面问题的通解*

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摘 要

本文首先从压电介质平面问题基本方程出发，引入一个位移函数，导出其通解，然后利用推广的 Almansi 定理，将通解化为简单形式，即通过三个"调和函数"来表示所有的物理量．其次，推导了楔形体顶端受集中力和点电荷作用时有限形式的解，退化可得半无限平面直线边界受集中力和点电荷的解．

关键词 压电介质 平面问题 通解 楔体

一、引 言

由于电学效应和力学变形的耦合特性，压电材料已在各种传感器和机敏结构中得到广泛的应用．近几年这种材料的研究不断深入并且在压电材料的断裂问题[1,2]、夹杂问题[3,4]和有限元法[5]等方面已取得了一系列重要的进展．对压电材料通解的研究，Wang 和Zheng[6]首先推导了压电材料三维问题的通解．Ding 等[7]针对同一问题，也给出了三组一般解，它们包含了Wang和 Zheng 的工作，并且利用这组通解，给出了半空间表面上作用集中荷载的封闭解．楔形体顶端作用集中力的解是弹性力学中的一个经典问题．Timoshenko 和Goodier[8]针对各向同性问题研究了楔的顶端作用集中力和力偶的解．Lekhniskii[9]给出了各向异性楔顶端作用集中力和力偶的解．丁皓江和李育[10]给出了圆柱型各向异性楔顶端作用集中力和力偶的解．Sosa和Castro[11]研究了压电半平面边界作用集中荷载问题．

本文首先从压电介质平面问题基本方程组出发，引入一个位移函数，导出通解，然后利用推广的 Almansi 定理，将通解化为简单形式，即通过三个"调和的位移函数"来表示所有的物理量，这三个函数各自满足形式相同的类似于调和方程的三个二阶偏微分方程，这类函数我们简称为"调和函数"．其次，推导得了楔形体顶端受集中力和点电荷作用时有限形式的解．

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二、基本方程和通解

按照Sosa和Castro[11,12]给出的二维压电介质的本构方程如下：

$$\left.\begin{aligned}\sigma_1=\sigma_x&=c_{11}\frac{\partial u}{\partial x}+c_{13}\frac{\partial w}{\partial z}+e_{31}\frac{\partial \phi}{\partial z}\\ \sigma_2=\sigma_z&=c_{13}\frac{\partial u}{\partial x}+c_{33}\frac{\partial w}{\partial z}+e_{33}\frac{\partial \phi}{\partial z}\\ \sigma_3=D_z&=e_{31}\frac{\partial u}{\partial x}+e_{33}\frac{\partial w}{\partial z}-\varepsilon_{33}\frac{\partial \phi}{\partial z}\\ \sigma_4=\tau_{xz}&=c_{44}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+e_{15}\frac{\partial \phi}{\partial x}\\ \sigma_5=D_x&=e_{15}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)-\varepsilon_{11}\frac{\partial \phi}{\partial x}\end{aligned}\right\}\tag{2.1}$$

式中σ_{ij}，u_i，D_i和ϕ分别是应力分量、位移分量、电位移分量、电势．

将方程(2.1)代入平衡方程和Gauss方程，得

$$D\begin{Bmatrix}u\\ w\\ \phi\end{Bmatrix}=0\tag{2.2}$$

式中D是微分算子矩阵，即

$$D=\begin{bmatrix}c_{11}\frac{\partial^2}{\partial x^2}+c_{44}\frac{\partial^2}{\partial z^2} & (c_{13}+c_{44})\frac{\partial^2}{\partial x\partial z} & (e_{15}+e_{31})\frac{\partial^2}{\partial x\partial z}\\ (c_{13}+c_{44})\frac{\partial^2}{\partial x\partial z} & c_{44}\frac{\partial^2}{\partial x^2}+c_{33}\frac{\partial^2}{\partial z^2} & e_{15}\frac{\partial^2}{\partial x^2}+e_{33}\frac{\partial^2}{\partial z^2}\\ -(e_{15}+e_{31})\frac{\partial^2}{\partial x\partial z} & -\left(e_{15}\frac{\partial^2}{\partial x^2}+e_{33}\frac{\partial^2}{\partial z^2}\right) & \varepsilon_{11}\frac{\partial^2}{\partial x^2}+\varepsilon_{33}\frac{\partial^2}{\partial z^2}\end{bmatrix}\tag{2.3}$$

其中c_{ij}、ε_{ij}和e_{ij}分别是弹性常数、介电常数和压电常数．计算D的行列式$|D|$得

$$|D|=a\frac{\partial^6}{\partial z^6}+b\frac{\partial^6}{\partial z^4\partial x^2}+c\frac{\partial^6}{\partial z^2\partial x^4}+d\frac{\partial^6}{\partial x^6}\tag{2.4}$$

式中

$$\left.\begin{aligned}a&=c_{44}(e_{33}^2+c_{33}\varepsilon_{33})\\ b&=c_{33}[c_{44}\varepsilon_{11}+(e_{15}+e_{31})^2]+\varepsilon_{33}[c_{11}c_{33}+c_{44}^2-(c_{13}+c_{44})^2]\\ &\quad+e_{33}[2c_{44}e_{15}+c_{11}e_{33}-2(c_{13}+c_{44})(e_{15}+e_{31})]\\ c&=c_{44}[c_{11}\varepsilon_{33}+(e_{15}+e_{31})^2]+\varepsilon_{11}[c_{11}c_{33}+c_{44}^2-(c_{13}+c_{44})^2]\\ &\quad+e_{15}[2c_{11}e_{33}+c_{44}e_{15}-2(c_{13}+c_{44})(e_{15}+e_{31})]\\ d&=c_{11}(e_{15}^2+c_{44}\varepsilon_{11})\end{aligned}\right\}\tag{2.5}$$

再计算$|D|$的代数余子式$\Delta_{ij}(i,\ j=1,\ 2,\ 3)$，例如$i=2$时，有

$$\left.\begin{aligned}\Delta_{21}&=-m_1\frac{\partial^4}{\partial x^3\partial z}-m_2\frac{\partial^4}{\partial x\partial z^3}\\ \Delta_{22}&=c_{11}\varepsilon_{11}\frac{\partial^4}{\partial x^4}+m_3\frac{\partial^4}{\partial x^2\partial z^2}+c_{44}\varepsilon_{33}\frac{\partial^4}{\partial z^4}\\ \Delta_{23}&=c_{11}e_{15}\frac{\partial^4}{\partial x^4}+m_4\frac{\partial^4}{\partial x^2\partial z^2}+c_{44}e_{33}\frac{\partial^4}{\partial z^4}\end{aligned}\right\}\tag{2.6}$$

式中

$$\left.\begin{aligned}m_1&=(c_{13}+c_{44})\varepsilon_{11}+(e_{15}+e_{31})e_{15}\\ m_2&=(c_{13}+c_{44})\varepsilon_{33}+(e_{15}+e_{31})e_{33}\\ m_3&=c_{11}\varepsilon_{33}+c_{44}\varepsilon_{11}+(e_{15}+e_{31})^2\\ m_4&=c_{11}e_{33}-c_{13}(e_{15}+e_{31})-c_{44}e_{31}\end{aligned}\right\}\tag{2.7}$$

引入函数F使满足下列方程

$$\prod_{i=1}^{3}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z_i^2}\right)F=0\tag{2.8}$$

式中$z_i=s_iz(i=1,\ 2,\ 3)$，而s_i^2是下列方程的三个根

$$as^6-bs^4+cs^2-d=0\tag{2.9}$$

则对应方程组(2.1)的一组通解可写成如下形式：

$$u=\Delta_{21}F,\quad w=\Delta_{22}F,\quad \phi=\Delta_{23}F\tag{2.10}$$

将Eubanks和Sternberg[13]的Almansi定理稍加改变，应用于此，则式(2.10)中的F具下列形式：

1．$F=F_1+F_2+F_3$，　　s_i互不相等时　　(2.11a)

2．$F=F_1+F_2+zF_3$，　　$s_1\neq s_2=s_3$时　　(2.11b)

3．$F=F_1+zF_2+z^2F_3$，　　$s_1=s_2=s_2$时　　(2.11c)

式中F_i分别满足下式：

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z_i^2}\right)F_i=0\qquad (i=1,\ 2,\ 3)\tag{2.12}$$

将式(2.11a)代入式(2.10)并利用式(2.12)，并令

$$\psi_i=(m_1-m_2s_i^2)s_i\frac{\partial^3F_i}{\partial z_i^3}\qquad (i=1,\ 2,\ 3)\tag{2.13}$$

则通解(2.10)可写成如下简单形式：

$$u=\sum_{i=1}^{3}\frac{\partial\psi_i}{\partial x},\quad w_j=\sum_{i=1}^{3}\alpha_{ij}\frac{\partial\psi_i}{\partial z_i}\qquad (j=1,\ 2)\tag{2.14}$$

式中引入了记号$w_1=w$，$w_2=\phi$，以及

$$\left.\begin{aligned}\alpha_{i1}&=\frac{c_{11}\varepsilon_{11}-m_3s_i^2+c_{44}\varepsilon_{33}s_i^4}{(m_1-m_2s_i^2)s_i}\\ \alpha_{i2}&=\frac{c_{11}e_{15}-m_4s_i^2+c_{44}e_{33}s_i^4}{(m_1-m_2s_i^2)s_i}\end{aligned}\right\}\quad (i=1,\ 2,\ 3)\tag{2.15}$$

将式(2.11b)和(2.11c)代入式(2.10)并利用式(2.12)，得这两种情形的通解简化式 如下：

当$s_1\neq s_2=s_3$时

引入

$$\left.\begin{aligned}
\psi_1&=(m_1-m_2s_1^2)s_1\frac{\partial^3F_1}{\partial z_1^3}\\
\psi_2&=(m_1-m_2s_2^2)s_2\frac{\partial^3F_2}{\partial z_2^3}+(m_1-3m_2s_2^2)\frac{\partial^2F_3}{\partial z_2^2}\\
\psi_3&=(m_1-m_2s_2^2)\frac{\partial^3F_3}{\partial z_2^3}
\end{aligned}\right\}\tag{2.16}$$

则有

$$\left.\begin{aligned}
u&=\sum_{i=1}^{2}\frac{\partial\psi_i}{\partial x}+z_2\frac{\partial\psi_3}{\partial x}\\
w_j&=\sum_{i=1}^{2}\alpha_{ij}\frac{\partial\psi_i}{\partial z_i}+\alpha_{2j}z_2\frac{\partial\psi_3}{\partial z_2}+\alpha_{4j}\psi_3\qquad(j=1,\ 2)
\end{aligned}\right\}\tag{2.17}$$

式中

$$\left.\begin{aligned}
\alpha_{41}&=\frac{2(2c_{44}\varepsilon_{33}s_2^2-m_3)s_2-(m_1-3m_2s_2^2)\alpha_{21}}{m_1-m_2s_2^2}\\
\alpha_{42}&=\frac{2(2c_{44}e_{33}s_2^2-m_4)s_2-(m_1-3m_2s_2^2)\alpha_{22}}{m_1-m_2s_2^2}
\end{aligned}\right\}\tag{2.18}$$

当$s_1=s_2=s_3$时

引入

$$\left.\begin{aligned}
\psi_1&=(m_1-m_2s_1^2)s_1\frac{\partial^3F_1}{\partial z_1^3}+(m_1-3m_2s_1^2)\frac{\partial^2F_2}{\partial z_1^2}-6m_2s_1\frac{\partial F_3}{\partial z_1}\\
\psi_2&=(m_1-m_2s_1^2)\frac{\partial^3F_2}{\partial z_1^3}+\frac{2(m_1-3m_2s_1^2)}{s_1}\frac{\partial^2F_3}{\partial z_1^2}\\
\psi_2&=\frac{m_1-m_2s_1^2}{s_1}\frac{\partial^2F_3}{\partial z_1^2}
\end{aligned}\right\}\tag{2.19}$$

则有

$$\left.\begin{aligned}
u&=\frac{\partial\psi_1}{\partial x}+z_1\frac{\partial\psi_2}{\partial x}+z_1^2\frac{\partial^2\psi_3}{\partial x\partial z_1}\\
w_j&=\alpha_{1j}\left(\frac{\partial\psi_1}{\partial z_1}+z_1\frac{\partial\psi_2}{\partial z_1}+z_1^2\frac{\partial^2\psi_3}{\partial z_1^2}\right)\\
&\quad+\alpha_{4j}\left(\psi_2+2z_1\frac{\partial\psi_3}{\partial z_1}\right)+\alpha_{5j}\psi_3\qquad(j=1,\ 2)
\end{aligned}\right\}\tag{2.20}$$

式中

$$\left.\begin{aligned}
\alpha_{51}&=\frac{2(3m_1\alpha_{11}s_1+6c_{44}\varepsilon_{33}s_1^2-m_3)s_1-2(m_1-3m_2s_1^2)\alpha_{41}}{m_1-m_2s_1^2}\\
\alpha_{52}&=\frac{2(3m_1\alpha_{12}s_1+6c_{44}e_{33}s_1^2-m_4)s_1-2(m_1-3m_2s_2^2)\alpha_{42}}{m_1-m_2s_1^2}
\end{aligned}\right\}\tag{2.21}$$

按式(2.13)或(2.16)或(2.19)和(2.12)，显然式(2.14)、(2.17)和(2.20)中的ψ_i满足下列方程

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z_i^2}\right)\psi_i=0 \qquad (i=1,\ 2,\ 3) \tag{2.22}$$

三、楔形体顶端作用集中力和点电荷的解

楔体顶角为 2α，取坐标原点在楔顶，z 轴指向楔的内部．楔的侧面为自由边界，集中力 $f_x=T\delta(x)\delta(z)$，$f_z=P\delta(x)\delta(z)$和点电荷 $\rho_f=Q\delta(x)\delta(z)$作用在楔顶．利用叠加原理，下面分两种情形来求解．限于篇幅，以下只给出s_i互不相等时的解．

1）对于$f_x=T\delta(x)\delta(z)$，$f_z=\rho_f=0$情形，取

$$\psi_i=A_i\left(x\ln r_i+z_i\arctan\frac{x}{z_i}-x\right) \qquad (i=1,\ 2,\ 3) \tag{3.1}$$

式中$r_i^2=x^2+z_i^2$，A_i是三个待定常数．

将式(3.1)代入(2.14)，得到相应的位移和电势如下：

$$u=\sum_{i=1}^{3}A_i\ln r_i,\quad w_j=\sum_{i=1}^{3}a_{ij}A_i\arctan\frac{x}{z_i} \qquad (j=1,\ 2) \tag{3.2}$$

将式(3.2)代入本构关系(2.1)，得应力和电位移的表达式如下：

$$\left.\begin{aligned}&\sigma_j=\sum_{i=1}^{3}k_{1j}^{i}A_i\frac{x}{r_i^2} \qquad (j=1,\ 2,\ 3)\\ &\sigma_j=\sum_{i=1}^{3}k_{1j}^{i}A_i\frac{z_i}{r_i^2} \qquad (j=4,\ 5)\end{aligned}\right\} \tag{3.3}$$

式中

$$\left.\begin{aligned}&k_{11}^{i}=c_{11}-c_{13}a_{i1}s_i-e_{31}a_{i2}s_i\\ &k_{12}^{i}=c_{13}-c_{33}a_{i1}s_i-e_{33}a_{i2}s_i\\ &k_{13}^{i}=e_{31}-e_{33}a_{i1}s_i+\varepsilon_{33}a_{i2}s_i \qquad (i=1,\ 2,\ 3)\\ &k_{14}^{i}=c_{44}s_i+c_{44}a_{i1}+e_{15}a_{i2}\\ &k_{15}^{i}=e_{15}s_i+e_{15}a_{i1}-\varepsilon_{11}a_{i2}\end{aligned}\right\} \tag{3.4}$$

由于楔体的侧面自由，所以有如下的边界条件：

当$x/z=\pm\tan\alpha$时，

$$\sigma_x n_x+\tau_{xz}n_z=0 \tag{3.5a}$$

$$\tau_{xz}n_x+\sigma_z n_z=0 \tag{3.5b}$$

$$D_x n_x+D_z n_z=0 \tag{3.5c}$$

在楔体的任意一截面上，由整体平衡条件可得：

$$\int_{-h\tan\alpha}^{h\tan\alpha}\sigma_z dx=0,\quad \int_{-h\tan\alpha}^{h\tan\alpha}\tau_{xz}dx+T=0,\quad \int_{-h\tan\alpha}^{h\tan\alpha}D_z dx=0 \tag{3.6}$$

将式(3.3)代入式(3.5)和(3.6)，由于σ_z和D_z都是x的奇函数，所以边界条件(3.5)的一、三两式自动满足，于是得到：

$$\sum_{i=1}^{3}k_{11}^{i}A_i\frac{l}{\eta_i^2}-\sum_{i=1}^{3}k_{14}^{i}A_i\frac{s_i l}{\eta_i^2}=0 \tag{3.7a}$$

$$\sum_{i=1}^{3} A_i(k_{14}^i\xi_i l+k_{12}^i m)\frac{1}{\eta_i^2}=0 \tag{3.7b}$$

$$\sum_{i=1}^{3} A_i(k_{15}^i\xi_i l+k_{13}^i m)\frac{1}{\eta_i^2}=0 \tag{3.7c}$$

和

$$\sum_{i=1}^{3} A_i k_{14}^i \arctan\frac{1}{\xi_i}+\frac{T}{2}=0 \tag{3.8}$$

式中：

$$\left.\begin{aligned}&\xi=\mathrm{ctan}\alpha,\ \ \xi_i=s_i\xi,\ \ \eta_i^2=1+\xi_i^2\\&l=\cos\alpha,\ \ m=-\sin\alpha\end{aligned}\right\} \tag{3.9}$$

不难验证有 $k_{11}^i-k_{14}^i s_i=0$，因此式(3.7a)也是自动满足。系数A_i ($i=1,\ 2,\ 3$)正好由式(3.7b)、(3.7c)和(3.8)确定。

当$\alpha=\pi/2$时，退化为半无限平面在其表面作用一个平行于x轴的集中力T的问题，确定A_i的方程简化为：

$$\sum_{i=1}^{3} k_{12}^i A_i=0,\ \ \sum_{i=1}^{3} k_{13}^i A_i=0,\ \ \sum_{i=1}^{3} k_{14}^i A_i+\frac{T}{\pi}=0 \tag{3.10}$$

2) 对于$f_z=P\delta(x)\delta(z)$，$\rho_f=Q\delta(x)\delta(z)$和$f_x=0$的情形

此时，整体平衡条件(3.6)应变为：

$$\int_{-h\tan\alpha}^{h\tan\alpha}\sigma_z dx+P=0,\ \ \int_{-h\tan\alpha}^{h\tan\alpha}\tau_{xz}dx=0,\ \ \int_{-h\tan\alpha}^{h\tan\alpha}D_z dx-Q=0 \tag{3.11}$$

取

$$\psi_i=L_i\left(z_i\ln r_i-x\arctan\frac{x}{z_i}-z_i\right)\qquad (i=1,\ 2,\ 3) \tag{3.12}$$

式中L_i是三个待定常数，相应的位移、电势以及应力、电位移为：

$$\left.\begin{aligned}&u=-\sum_{i=1}^{3} L_i\arctan\frac{x}{z_i}\\&w_j=\sum_{i=1}^{3}\alpha_{ij}L_i\ln r_i\qquad (j=1,\ 2)\\&\sigma_j=\sum_{i=1}^{3}k_{2j}^i L_i\frac{z_i}{r_i^2}\qquad (j=1,\ 2,\ 3)\\&\sigma_j=\sum_{i=1}^{3}k_{2j}^i L_i\frac{x}{r_i^2}\qquad (j=4,\ 5)\end{aligned}\right\} \tag{3.13}$$

式中

$$\left.\begin{aligned}&k_{21}^i=-k_{11}^i,\ \ k_{22}^i=-k_{12}^i,\ \ k_{23}^i=-k_{13}^i\\&k_{24}^i=k_{14}^i,\ \ k_{25}^i=k_{15}^i\end{aligned}\right\}\quad (i=1,\ 2,\ 3) \tag{3.14}$$

仿照上节的的作法，由式(3.5)和(3.11)可得确定系数L_i的方程组如下：

$$\sum_{i=1}^{3} L_i(k_{21}^{i}\xi_i l+k_{24}^{i}m)\frac{1}{\eta_i^2}=0,\quad \sum_{i=1}^{3} L_i k_{22}^{i}\arctan\frac{1}{\xi_i}+\frac{P}{2}=0,\quad \sum_{i=1}^{3} L_i k_{23}^{i}\arctan\frac{1}{\xi_i}-\frac{Q}{2}=0 \tag{3.15}$$

当$\alpha=\pi/2$时，退化为半无限平面在其表面作用一个平行于z轴的集中力P和点电荷Q的问题，式(3.15)简化为：

$$\sum_{i=1}^{3} k_{24}^{i}L_i=0,\quad \sum_{i=1}^{3} k_{22}^{i}L_i+\frac{P}{\pi}=0,\quad \sum_{i=1}^{3} k_{23}^{i}L_i-\frac{Q}{\pi}=0 \tag{3.16}$$

四、结　论

本文对于特征根s_i的三种不同情形，给出了简洁而实用的压电介质平面问题的通解(2.14)、(2.17)、(2.20)和(2.22)，因为它们都是用“调和函数”表示的，所以也很容易用复变函数来表示。并求得了楔形体顶端受集中力和点电荷作用时有限形式的解。半无限平面直线边界受集中力和点电荷的解也可以从本文给出的解中退化得到(3.10)和(3.16)。

参 考 文 献

[1] Y. Shindo, E. Ozava and J. P. Nowacki, Singular stress and electric fields of a cracked piezoelectric strip, *International Journal of Applied Electronic Materials*, 1 (1990), 77—87.

[2] Z. Suo, C. M. Kuo, D. M. Barnett and J. R. Willis, Fracture mechanics for piezoelectric ceramics, *Journal of the Mechanics and Physics of Solids*, 40 (1992), 739—765.

[3] M. L. Dunn and M. Taya, An analysis of piezoelectric composite materials containing ellipsoidal inhomogeneities, *Proceedings of the Royal Society of London, Series A*, 443 (1993), 245—287.

[4] T. Y.Chen, Exact Relations of inclusions in piezoelectric media, *International Journal of Engineering Science*, 32 (1994), 265—287.

[5] K. H. Sung, C. Keilers and F. K. Chang, Finite element analysis of composite structures containing distributed piezoceramic sensors and actuators, *AIAA Journal*, 30 (1993), 772—780.

[6] Z. Wang and B. Zheng, The general solution of three-dimensional problem in piezoelectric media, *International Journal of Solids and Structures*, 32 (1995), 105—115.

[7] H. J. Ding, B. Chen and J.Liang, General solutions for coupled equations for piezoelectric media, *International Journal of Solids and Structures*, 33 (1996), 2283—2298.

[8] S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity*, McGraw-Hill (1970).

[9] S. G. Lekhniskii, 《各向异性板》，胡海昌译，科学出版社 (1955).

[10] 丁皓江、李育，圆柱型各向异性弹性力学平面问题，应用力学学报，11 (1994), 11—18.

[11] H. A. Sosa and M. A. Castro, On concentrated loads at the boundary of a piezoelectric half-plane, *Journal of the Mechanics and Physics of Solids*, 42 (1994), 1105—1122.

[12] H. A. Sosa and M. A. Castro, Electroelastic analysis of piezoelectric laminated structures, *Applied Mechanics Review*, 46 (1993), 21—28.

[13] R.A.Eubanks and E. Sternberg, On the axisymmetric problem of elastic theory for a medium with transverse isotropy, *Journal of Rational Mechanics Analysis*, 3 (1954), 89—101.

General Solution of Plane Problem of Piezoelectric Media Expressed by "Harmonic Functions"

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Abstract

First, based on the basic equations of two-dimensional piezoelectroelasticity, a displacement function is introduced and the general solution is then derived. Utilizing the generalized Almansi's theorem, the general solution is so simplified that all physical quantities can be expressed by three "harmonic functions". Second, solutions of problems of a wedge loaded by point forces and point charge at the apex are also obtained in the paper. These solutions can be degenerated to those of problems of point forces and point charge acting on the line boundary of a piezoelectric half-plane.

Key words piezoelectric media, plane problem, general solution, wedge