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在一集中载荷作用下斜边自由的三角形板弯曲

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摘　　要

本文应用功的互等定理研究了在一集中载荷作用下斜边自由两边固定的三角形板弯曲问题．该法简单、通用．

关键词　功的互等定理　集中载荷　三角形板

一、引　　言

研究弹性薄板的弯曲问题具有重要的实际意义．世界上很多国家的学者和专家在这方面曾经做过大量工作．然而，对三角形板在各种支承形式和荷重作用下的问题很少有人探讨．文献[1]将功的互等定理引入矩形板的弯曲问题．文献[2]应用源象法和叠加原理，研究了简支等边三角形板和简支等腰直角三角形板的弯曲问题．现在，本文应用功的互等定理，在文献[4～6]基础上，进一步研究在该板上任意一点作用一集中载荷斜边自由两边固定的直角三角形板的弯曲问题．

二、斜边自由两边固定的三角形板弯曲

1．基本系统的必须参量

为计算弯曲的三角形板，取在一流动坐标$(\xi,\ \eta)$点处有一单位集中力作用的四边简支矩形板的一半为基本系统，如图1所示，而取斜边自由两边固定、在任意一点$(x_0,\ y_0)$处有一集中载荷作用的同一直角三角形板为实际系统，如图2所示．

在计算三角形板的弯曲时，将要用到单位载荷基本系统的相应关系．

$$M_n=M_x\cos^2\alpha+M_y\sin^2\alpha-2M_{xy}\sin\alpha\cos\alpha$$

$$V_n=V_x\cos\alpha+V_y\sin\alpha$$

其中　$\sin\alpha=a/l$，$\cos\alpha=b/l$

根据上面的公式和文献[4]中的式(2.4.4)和(2.4.5)，我们有

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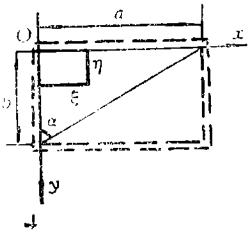

图 1

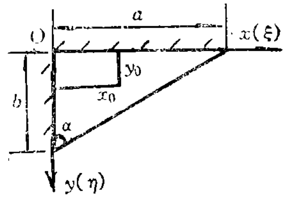

图 2

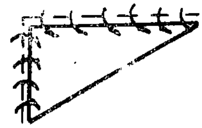

图 3

$$M_{1n}=\frac{a^2}{\pi^3 b^2 l^2}\sum_{m=1}^{\infty}\left\{\left[(1+\mu)l^2-(a^2-b^2)(1-\mu)\left(\beta_m \mathrm{cth}\beta_m-\frac{\beta_m\eta}{b}\,\mathrm{cth}\frac{\beta_m\eta}{b}\right)\right]\right.$$

$$\cdot\mathrm{sh}\frac{\beta_m x}{a}\,\sin\frac{m\pi x}{a}+(a^2-b^2)(1-\mu)\frac{\beta_m x}{a}\mathrm{ch}\frac{\beta_m x}{a}\,\sin\frac{m\pi x}{a}$$

$$+2(1-\mu)ab\left[\left(\beta_m \mathrm{cth}\beta_m-\frac{\beta_m\eta}{b}\,\mathrm{cth}\frac{\beta_m\eta}{b}\right)\mathrm{ch}\frac{\beta_m x}{a}\cdot\cos\frac{m\pi x}{a}\right.$$

$$\left.\left.-\frac{\beta_m x}{a}\mathrm{sh}\frac{\beta_m x}{a}\cos\frac{m\pi x}{a}\right]\right\}\frac{\beta_m^2}{m^2\mathrm{sh}\beta_m}\cdot\mathrm{sh}\frac{\beta_m\eta}{b}\sin\frac{m\pi\xi}{a}\qquad\left(0\leqslant x\leqslant\frac{a(b-\eta)}{b}\right)\qquad(2.1)$$

$$M_{1n}=\frac{a^2}{\pi^3 b^2 l^2}\sum_{m=1}^{\infty}\left\{\left[(1+\mu)l^2-(a^2-b^2)(1-\mu)\left(\beta_m \mathrm{cth}\beta_m-\frac{\beta_m(b-\eta)}{b}\right.\right.\right.$$

$$\left.\left.\cdot\mathrm{cth}\frac{\beta_m(b-\eta)}{b}\right)\right]\mathrm{sh}\frac{\beta_m(a-x)}{a}\cdot\sin\frac{m\pi x}{a}+(a^2-b^2)(1-\mu)\frac{\beta_m(a-x)}{a}$$

$$\cdot\mathrm{ch}\frac{\beta_m(a-x)}{a}\sin\frac{m\pi x}{a}-2(1-\mu)ab\left[\left(\beta_m \mathrm{cth}\beta_m-\frac{\beta_m(b-\eta)}{b}\,\mathrm{cth}\frac{\beta_m(b-\eta)}{b}\right)\right.$$

$$\left.\left.\cdot\mathrm{ch}\frac{\beta_m(a-x)}{a}\cos\frac{m\pi x}{a}-\frac{\beta_m(a-x)}{a}\mathrm{sh}\frac{\beta_m(a-x)}{a}\cos\frac{m\pi x}{a}\right]\right\}$$

$$\cdot\frac{\beta_m^2}{m^3\mathrm{sh}\beta_m}\mathrm{sh}\frac{\beta_m(b-\eta)}{b}\sin\frac{m\pi\xi}{a}\qquad\left(\frac{a(b-\eta)}{b}\leqslant x\leqslant a\right)\qquad(2.2)$$

$$V_{1n}=\frac{a^2}{\pi^3 b^3 l}\sum_{m=1}^{\infty}\left\{b\left[(3-\mu)-(1-\mu)\left(\beta_m \mathrm{cth}\beta_m-\frac{\beta_m\eta}{b}\,\mathrm{cth}\frac{\beta_m\eta}{b}\right)\right]\mathrm{sh}\frac{\beta_m x}{a}\cos\frac{m\pi x}{a}\right.$$

$$+b(1-\mu)\frac{\beta_m x}{a}\,\mathrm{ch}\,\frac{\beta_m x}{a}\cos\frac{m\pi x}{a}-a\left[2+(1-\mu)\left(\beta_m \mathrm{cth}\beta_m-\frac{\beta_m\eta}{b}\,\mathrm{cth}\frac{\beta_m\eta}{b}\right)\right]$$

$$\left.\cdot\mathrm{ch}\frac{\beta_m x}{a}\sin\frac{m\pi x}{a}-a(1-\mu)\frac{\beta_m x}{a}\mathrm{sh}\frac{\beta_m x}{a}\sin\frac{m\pi x}{a}\right\}\frac{\beta_m^3}{m^3\mathrm{sh}\beta_m}\mathrm{sh}\frac{\beta_m\eta}{b}\sin\frac{m\pi\xi}{a}$$

$$\left(0\leqslant x\leqslant\frac{a(b-\eta)}{b}\right)\qquad(2.3)$$

$$V_{1n}=\frac{a^2}{\pi^3 b^3 l}\sum_{m=1}^{\infty}\left\{b\left[(3-\mu)-(1-\mu)\left(\beta_m \mathrm{cth}\beta_m-\frac{\beta_m(b-\eta)}{b}\,\mathrm{cth}\,\frac{\beta_m(b-\eta)}{b}\right)\right]\right.$$

$$\cdot\operatorname{sh}\frac{\beta_m(a-x)}{a}\cos\frac{m\pi x}{a}+b(1-\mu)\frac{\beta_m(a-x)}{a}\operatorname{ch}\frac{\beta_m(a-x)}{a}\cos\frac{m\pi x}{a}$$

$$+a\left[2+(1-\mu)\left(\beta_m\operatorname{cth}\beta_m-\frac{\beta_m(b-\eta)}{b}\operatorname{cth}\frac{\beta_m(b-\eta)}{b}\right)\right]\operatorname{ch}\frac{\beta_m(a-x)}{a}\sin\frac{m\pi x}{a}$$

$$-a(1-\mu)\frac{\beta_m(a-x)}{a}\operatorname{sh}\frac{\beta_m(a-x)}{a}\sin\frac{m\pi x}{a}\Big\}\frac{\beta_m^3}{m^3\operatorname{sh}\beta_m}\operatorname{sh}\frac{\beta_m(b-\eta)}{b}\sin\frac{m\pi\xi}{a}$$

$$\left(\frac{a(b-\eta)}{b}\leqslant x\leqslant a\right) \qquad (2.4)$$

2．斜边自由两边固定的三角形板挠曲面方程

对于实际系统图2，解除固定边的弯曲约束，这一约束被分布弯矩

$$M(x)_{y=0}=\sum_{m=1}^{\infty}E_m\sin\frac{m\pi x}{a} \qquad (2.5)$$

$$M(y)_{x=0}=\sum_{m=1}^{\infty}A_n\sin\frac{n\pi y}{b} \qquad (2.6)$$

所代替，如图 3 所示。假设自由边的挠度和转角方程分别为

$$W_n=\sum_{m=1}^{\infty}a_m\sin\left(1-\frac{x}{a}\right)m\pi \qquad (2.7)$$

$$\theta_n=\sum_{m=1}^{\infty}b_m\cos\left(1-\frac{x}{a}\right)m\pi \qquad (2.8)$$

并且在图 1 所示基本系统与图 3 所示实际系统之间应用功的互等定理，则可得到做为实际系统的三角形板的挠曲面方程为如下形式

$$W(\xi,\ \eta)=PW_1(x_0,\ y_0;\ \xi,\ \eta)-\frac{l}{a}\int_0^{\frac{a(b-\eta)}{b}}V_{1n}W_ndx-\frac{l}{a}\int_{\frac{a(b-\eta)}{b}}^{a}V_{1n}W_ndx$$

$$-\frac{l}{a}\int_0^{\frac{a(b-\eta)}{b}}M_{1n}\theta_ndx-\frac{l}{a}\int_{\frac{a(b-\eta)}{b}}^{a}M_{1n}\theta_ndx$$

$$+\int_0^a\left(\frac{\partial W_1}{\partial y}\right)_{y=0}\sum_{m=1}^{\infty}E_m\sin\frac{m\pi x}{a}dx+\int_0^b\left(\frac{\partial W_1}{\partial x}\right)_{x=0}\sum_{n=1}^{\infty}A_n\sin\frac{n\pi y}{b}dy \qquad (2.9)$$

令 $W_1'=PW_1(x_0,\ y_0;\ \xi,\ \eta)$

$$W_2=-\frac{l}{a}\int_0^{\frac{a(b-\eta)}{b}}V_{1n}W_ndx-\frac{l}{a}\int_{\frac{a(b-\eta)}{b}}^{a}V_{1n}W_ndx$$

$$W_3=-\frac{l}{a}\int_0^{\frac{a(b-\eta)}{b}}M_{1n}\theta_ndx-\frac{l}{a}\int_{\frac{a(b-\eta)}{b}}^{a}M_{1n}\theta_ndx$$

$$W_4=\int_0^a\left(\frac{\partial W_1}{\partial y}\right)_{y=0}\sum_{m=1}^{\infty}E_m\sin\frac{m\pi x}{a}dx$$

$$W_5=\int_0^b\left(\frac{\partial W_1}{\partial x}\right)_{x=0}\sum_{n=1}^{\infty}A_n\sin\frac{n\pi y}{b}dy$$

于是我们得到

$$W(\xi,\ \eta)=W_1'+W_2+W_3+W_4+W_5 \tag{2.10}$$

注意到文献[4]中式(2.4.4)，则有

$$W_1'=\frac{Pa^2}{\pi^3D}\sum_{m=1}^{\infty}\left[1+\beta_m\mathrm{cth}\beta_m-\frac{\beta_m(b-y_0)}{b}\mathrm{cth}\frac{\beta_m(b-y_0)}{b}-\frac{\beta_m\eta}{b}\mathrm{cth}\frac{\beta_m\eta}{b}\right]$$

$$\cdot\frac{1}{m^3\mathrm{sh}\beta_m}\mathrm{sh}\frac{\beta_m\eta}{b}\mathrm{sh}\frac{\beta_m(b-y_0)}{b}\sin\frac{m\pi x_0}{a}\sin\frac{m\pi\xi}{a}\qquad(\eta\leqslant y_0) \tag{2.11}$$

当$\eta\geqslant y_0$时，在应用式(2.11)时，$b-y_0$必须以y_0代替，η以$b-\eta$代替.

由式(2.3)、(2.4)和(2.7)，可得

$$W_2=-\frac{a}{\pi}\sum_{m=1}^{\infty}\left\{\left[\frac{8a^4+2a^2b^2(4-\mu)+b^4}{a(4a^2+b^2)^2}\mathrm{sh}\beta_m+\frac{a(1-\mu)\beta_m}{4a^2+b^2}\mathrm{ch}\beta_m\right]\sin\frac{2m\pi\eta}{b}\right.$$

$$-\frac{(1-\mu)}{b}\mathrm{sh}\frac{\beta_m\eta}{b}\cdot\mathrm{sh}\frac{\beta_m(b-\eta)}{b}\left[1+\frac{b^2(4a^2-b^2)}{(4a^2+b^2)^2}\cos\frac{2m\pi\eta}{b}\right]$$

$$+\frac{1-\mu}{2b}\left(1-\frac{b^2}{4a^2+b^2}\cos\frac{2m\pi\eta}{b}\right)\left(2\beta_m\mathrm{sh}\frac{\beta_m\eta}{b}\cdot\mathrm{ch}\frac{\beta_m(b-\eta)}{b}\right.$$

$$\left.+\frac{\beta_m\eta}{b}\mathrm{sh}\frac{\beta_m(b-2\eta)}{b}-\mathrm{sh}\beta_m\frac{\beta_m\eta}{b}\right)+\frac{a(1-\mu)}{4a^2+b^2}\sin\frac{2m\pi\eta}{b}$$

$$\cdot\left[\frac{4b^2}{4a^2+b^2}\mathrm{ch}\frac{\beta_m\eta}{b}\mathrm{sh}\frac{\beta_m(b-\eta)}{b}-\beta_m\mathrm{ch}\frac{\beta_m(b-2\eta)}{b}\right.$$

$$\left.\left.-2\frac{\beta_m\eta}{b}\mathrm{sh}\frac{\beta_m\eta}{b}\mathrm{sh}\frac{\beta_m(b-\eta)}{b}\right]\right\}\frac{a_m\cos m\pi}{m\mathrm{sh}\beta_m}\sin\frac{m\pi\xi}{a} \tag{2.12}$$

由式(2.1)、(2.2)和(2.8)，可得

$$W_3=\frac{ab}{\pi^2l}\sum_{m=1}^{\infty}\frac{4a^4+a^2b^2(4+\mu)+b^4}{(4a^2+b^2)^2}\frac{b_m\cos m\pi}{m^2}\sin\frac{2m\pi\eta}{b}\sin\frac{m\pi\xi}{a} \tag{2.13}$$

注意到文献[4]中式(2.4.5)和(2.4.3)，可得

$$W_4=\frac{a^2}{2\pi^2D}\sum_{m=1}^{\infty}\frac{E_m}{m^2}\left[-\frac{\beta_m}{\mathrm{sh}^2\beta_m}\mathrm{sh}\frac{\beta_m\eta}{b}\right.$$

$$\left.+\mathrm{cth}\beta_m\frac{\beta_m\eta}{b}\mathrm{ch}\frac{\beta_m\eta}{b}-\frac{\beta_m\eta}{b}\mathrm{sh}\frac{\beta_m\eta}{b}\right]\sin\frac{m\eta\xi}{a} \tag{2.14}$$

$$W_5=\frac{b^2}{2\pi^2D}\sum_{n=1}^{\infty}\frac{A_n}{n^2}\left[-\frac{\alpha_n}{\mathrm{sh}^2\alpha_n}\mathrm{sh}\frac{\alpha_n\xi}{a}\right.$$

$$\left.+\mathrm{cth}\alpha_n\frac{\alpha_n\xi}{a}\mathrm{ch}\frac{\alpha_n\xi}{a}\cdot\frac{\alpha_n\xi}{a}\cdot\mathrm{sh}\frac{\alpha_n\xi}{a}\right]\sin\frac{n\pi\eta}{b} \tag{2.5}$$

这里 $\beta_m = m\pi b/a$, $\alpha_n = n\pi a/b$

将式(2.11)～(2.15)代入式(2.10)，我们便得到三角形板的挠曲面的一般方程.

三、满足边界条件

当集中载荷不作用在板的边缘上时，挠曲面方程(2.10)必须满足下述边界条件

$$\frac{\partial W}{\partial \eta}\Big|_{\eta=0}=0 \tag{3.1}$$

$$\frac{\partial W}{\partial \xi}\Big|_{\xi=0}=0 \tag{3.2}$$

$$M_n(\xi,\ \eta)\Big|_{\xi=a(b-\eta)/b}=M_\xi\cos^2\alpha+M_\eta\sin^2\alpha-2M_{\xi\eta}\sin\alpha\cos\alpha\Big|_{\xi=a(b-\eta)/b}=0 \tag{3.3}$$

$$V_n(\xi,\ \eta)\Big|_{\xi=a(b-\eta)/b}=V_\xi\cos\alpha+V_\eta\sin\alpha\Big|_{\xi=a(b-\eta)/b}=0 \tag{3.4}$$

为了满足这些边界条件，必须做一系列相应的计算，并且将一部分相应量表达式的双曲线函数展成三角级数。经整理，最后得到

$$\frac{Pb^2\pi^2}{a\beta_m}\left[\mathrm{cth}\beta_m-\frac{b-y_0}{b}\mathrm{cth}\frac{\beta_m(b-y_0)}{b}\right]\mathrm{sh}\frac{\beta_m(b-y_0)}{b}\sin\frac{m\pi x_0}{a}$$

$$+\frac{\pi^2 b}{2}\left(\frac{\mathrm{ch}\beta_m}{\beta_m}-\frac{1}{\mathrm{sh}\beta_m}\right)E_m+\frac{2a^2}{b}\sum_{n=1}^{\infty}\frac{m\,\mathrm{sh}\beta_m A_n}{n^3\left(\frac{a^2}{b^2}+\frac{m^2}{n^2}\right)^2}-\frac{2\pi^3 D}{a(4a^2+b^2)}$$

$$\cdot\left[\frac{8a^4\mu+6a^2b^2+b^4}{4a^2+b^2}\mathrm{sh}\beta_m+2a^2(1-\mu)\beta_m\mathrm{ch}\beta_m\right]\frac{m\cos m\pi a_m}{\beta_m}$$

$$+\frac{2b\pi^2 D[4a^4+a^2b^2(4+\mu)+b^4]}{I(4a^2+b^2)^2}\cdot\frac{\cos m\pi\,\mathrm{sh}\beta_m b_m}{\beta_m}=0 \qquad (m=1,\ 2,\ 3,\ \cdots) \tag{3.5}$$

$$\frac{Pa}{\pi D}\left[\mathrm{cth}\alpha_n-\frac{a-x_0}{a}\mathrm{cth}\frac{\alpha_n(a-x_0)}{a}\right]\frac{1}{n\,\mathrm{sh}\alpha_n}\mathrm{sh}\frac{\alpha_n(a-x_0)}{a}\sin\frac{n\pi y_0}{b}$$

$$+\frac{a}{2D}\left(\frac{1}{\alpha_n}\mathrm{cth}\alpha_n-\frac{1}{\mathrm{sh}^2\alpha_n}\right)A_n+\frac{2b^2}{a\pi^2 D}\sum_{m=1}^{\infty}\frac{nE_m}{m^3\left(\frac{b^2}{a^2}+\frac{n^2}{m^2}\right)^2}+\sum_{m=1}^{\infty}\frac{(1-\mu)a_m\cos m\pi}{2\mathrm{sh}\beta_m}$$

$$\cdot\left\{\frac{b^2n\,\mathrm{ch}\beta_m(1-\cos n\pi)}{\pi(4a^2+b^2)^2G_1^2}[(4a^2-b^2)n^2-4(12a^2+b^2)m^2]+\frac{n\pi}{(4\beta_m^2+n^2\pi^2)^2}\right.$$

$$\cdot[\beta_m\mathrm{sh}\beta_m(4\beta_m^2+n^2\pi^2)-\mathrm{ch}\beta_m(1-\cos n\pi)(8\beta_m^2+n^2\pi^2)]-\frac{b^2n\beta_m\mathrm{sh}\beta_m}{\pi(4a^2+b^2)G_1}$$

$$-\frac{1}{b}\left[\left(1+\frac{b}{n\pi}\cos n\pi\right)\beta_m\mathrm{sh}\beta_m-\mathrm{ch}\beta_m\right]-\frac{b^2n\pi\beta_m\mathrm{sh}\beta_m}{(4a^2+b^2)I}[4\beta_m^2-(20m^2-n^2)\pi^2]$$

$$-\frac{b^2n\pi\,\mathrm{ch}\beta_m(1-\cos n\pi)}{(4a^2+b^2)^2I}[4a^2(4\beta_m^2-G_1\pi^2)-b^2(4\beta_m^2-(36m^2-n^2)\pi^2)]$$

$$+\frac{4b^2n\pi\,\mathrm{ch}\beta_m(1-\cos n\pi)}{(4a^2+b^2)I^2}[\beta_m^2(16\beta_m^4-8\beta_m^2\pi^2G_1-(12m^2+n^2)G_1\pi^4)$$

$$-2m^2\pi^2(48\beta_m^4+8\beta_m^2\pi^2(4m^2+n^2)-G_1^2\pi^4)]\}=0 \qquad (n=1,\ 2,\ 3,\ \cdots) \tag{3.6}$$

$$M_{n1}+M_{n2}+M_{n3}+M_{n4}+M_{n5}=0 \qquad (n=1,\ 2,\ 3,\ \cdots) \tag{3.7}$$

$$V_{n1}+V_{n2}+V_{n3}+V_{n4}+V_{n5}=0 \qquad (n=1,\ 2,\ 3,\ \cdots) \tag{3.8}$$

$$M_{n1}=-\frac{2Pb\pi n}{l^2}\sum_{m=1}^{\infty}\Big\{(1-\mu)abn^2\pi\,\mathrm{sh}\beta_m\cdot C+(\mathrm{ch}\beta_m\cdot C-1)\Big[(1+\mu)(a^2+b^2)m$$
$$-(1-\mu)abn^2\pi\Big(\mathrm{cth}\beta_m-\frac{b-y_0}{b}\,\mathrm{cth}\frac{\beta_m(b-y_0)}{b}\Big)\Big]-\frac{(1-\mu)m(\mathrm{ch}\beta_m\cdot C-1)}{H}$$
$$\cdot[a^2(3\beta_m^4+2\beta_m^2\pi^2(m^2+n^2)-\pi^4E^2)-b^2(\beta_m^4+2\beta_m^2\pi^2G+\pi^4(5m^4-2m^2n^2$$
$$-3n^4))]\Big\}\frac{\beta_m\cos m\pi}{Hm\,\mathrm{sh}\beta_m}\mathrm{sh}\frac{\beta_m(b-y_0)}{b}\sin\frac{m\pi x_0}{a} \qquad (\eta\leqslant y_0) \tag{3.9a}$$

$$M_{n1}=-\frac{2Pb\pi n}{l^2}\sum_{m=1}^{\infty}\Big\{-(1-\mu)abn^2\pi\,\mathrm{sh}\beta_m+(\mathrm{ch}\beta_m-C)\Big[(1+\mu)(a^2+b^2)m$$
$$-(1-\mu)abn^2\pi\Big(\mathrm{cth}\beta_m-\frac{y_0}{b}\,\mathrm{cth}\frac{\beta_m y_0}{b}\Big)\Big]-\frac{(1-\mu)m(\mathrm{ch}\beta_m-C)}{H}[a^2(3\beta_m^4$$
$$+2\beta_m^2\pi^2(m^2+n^2)-\pi^4E^2)-b^2(\beta_m^4+2\beta_m^2\pi^2G+\pi^4(5m^4-2m^2n^2$$
$$-3n^4))]\Big\}\frac{\beta_m\cos m\pi}{Hm\,\mathrm{sh}\beta_m}\mathrm{sh}\frac{\beta_m(b-y_0)}{b}\sin\frac{m\pi x_0}{a} \qquad (\eta\geqslant y_0) \tag{3.9b}$$

$$M_{n2}=\frac{\pi^3D(1-\mu)n}{a(4a^2+b^2)l^2}\sum_{m=1}^{\infty}\Big\{\frac{(1+C)\mathrm{sh}\beta_m}{2\pi(4a^2+b^2)A}[2m\pi\beta_m(16a^6(1+2\mu)+a^2b^4(4+23\mu)$$
$$+4a^4b^2(11+7\mu)+3b^6)-ab(32a^4+22a^2b^2-b^4)(4\beta_m^2-\pi^2E)(1-\mu)]-\frac{(1+C)\mathrm{sh}\beta_m}{2\pi(4a^2+b^2)B}$$
$$\cdot[6m\pi\beta_m(16a^6+12a^4b^2(1+\mu)+a^2b^4(4+5\mu)+b^6)+ab^3(2a^2+b^2)(4\beta_m^4-\pi^2F)(1-\mu)]$$
$$+\frac{C\beta_m\mathrm{ch}\beta_m}{2\pi bB}[6b^5m\pi\beta_m-a(8a^4+10a^2b^2\mu+2b^4)(4\beta_m^2-\pi^2F)]$$
$$+\frac{b^2(4a^2-b^2)mC\beta_m^2\mathrm{ch}\beta_m}{A}+\frac{m\,\mathrm{ch}\beta_m}{2A}[(4a^4-a^2b^2(12-17\mu)-b^4(1-2\mu))E\pi^2$$
$$+2\beta_m^2(4a^4(9-2\mu)+a^2b^2(28+3\mu)+8b^4\mu)]-\frac{m\,\mathrm{ch}\beta_m}{2B}[(4a^4-a^2b^2(4-9\mu)$$
$$-b^4(1-2\mu))F\pi^2+2\beta_m^2(28a^4+a^2b^2(8+27\mu)+2b^4(1+4\mu))]$$
$$+\frac{m\beta_m\mathrm{sh}\beta_m(1+C)}{2B^2}[3(12a^4+15a^2b^2\mu-b^4(1-4\mu))(48\beta_m^4+8\beta_m^2\pi^2(9m^2+n^2)$$
$$-\pi^2F^2)+4(4a^4+a^2b^2(4+\mu)+b^4(3-2\mu))((4\beta_m^2-F\pi^2)^2-36m^2\pi^4F)]$$
$$-\frac{m\beta_m\mathrm{sh}\beta_m(1+C)}{2A^2}[(4a^4(11-2\mu)+a^2b^2(8+37\mu)-3b^4(1-4\mu))(48\beta_m^4$$
$$+8\beta_m^2\pi^2(m^2+n^2)-\pi^4E^2)-4(4a^4-a^2b^2(12-17\mu)-b^4(1-2\mu))((4\mu_m^2-E\pi^2)^2$$
$$-4m^2\pi^4E)]+\frac{4m\beta_m\mathrm{sh}\beta_m(1-C)}{\pi^4E^2F^2}[(4a^4(1+2\mu)+3a^2b^2(4+\mu)+3b^4)F^2$$
$$-3(4a^4-a^2b^2(4-9\mu)+b^4)E^2]+\frac{4\mathrm{sh}\beta_m(1-C)}{ab(4a^2+b^2)(1-\mu)EF\pi^3}[a^2b^2(4a^4(1-\mu^2)$$

$$+3a^2b^2(3-4\mu+\mu^2)+b^4(1-\mu))n^2-(32a^8+4a^6b^2(13+12\mu-9\mu^2)$$

$$+3a^4b^4(17+4\mu-7\mu^2)+a^2b^6(14-3\mu^2)+b^8)m^2]+\frac{4a\beta_m\mathrm{ch}\beta_m}{6\pi^3EF}$$

$$\cdot[a^2b^2(1-\mu)n^2-(4a^4+5a^2b^2+b^4)m^2]\Big\}\frac{m}{\mathrm{sh}\beta_m}(a_m) \tag{3.10}$$

$$M_{n3}=\frac{Dn[(2a^2+b^2)^2+a^2b^2\mu]}{2a\pi l^3(4a^2+b^2)^2}\sum_{m=1}^{\infty}(1-C)\Big\{\frac{1}{E}[4a^4+a^2b^2(4+\mu)+b^4]$$

$$-\frac{1}{F}[4a^4-a^2b^2(4-9\mu)+b^4]\Big\}(b_m) \tag{3.11}$$

$$M_{n4}=\frac{b\pi^2n}{l^2}\sum_{m=1}^{\infty}\Big\{-bC\mathrm{sh}\beta_m[(1+\mu)b_m\beta_m+a\pi(n^2+m^2-(n^2-m^2)\mu)]$$

$$+b(\mathrm{ch}\beta_m\cdot C-1)\Big[((1+\mu)b_m\beta_m+a\pi(n^2+m^2-(n^2-m^2)\mu))\mathrm{cth}\beta_m$$

$$-(1-\mu)a\pi n^2\frac{\beta_m}{\mathrm{sh}^2\beta_m}\Big]+\frac{(1-\mu)m\beta_m(\mathrm{ch}\beta_m-C)}{H\mathrm{sh}\beta_m}[a^2(3\beta_m^4+2\beta_m^2\pi^2(m^2+n^2)-E^2\pi^4)$$

$$-b^2(\beta_m^4+2\beta_m^2\pi^2(3m^2-n^2)+\pi^4(5m^4-2m^2n^2-3n^4))]\Big\}\frac{\cos m\pi}{H}(E_m) \tag{3.12}$$

$$M_{n5}=\frac{1}{2al^2}\sum_{n=1}^{\infty}\frac{n\pi A_n}{(\alpha_n^2+4n^2\pi^2)^2}\Big\{2b^2\mathrm{cth}\alpha_n(\mathrm{ch}\alpha_n-1)\Big[b^2(\alpha_n^2-4n^2\pi^2-\mu(3\alpha_n^2+4n^2\pi^2))$$

$$-2a^2(\alpha_n^2+4n^2\pi^2)+\frac{2b^2(1-\mu)\alpha_n(\alpha_n^2+4n^2\pi^2)}{\mathrm{sh}2\alpha_n}\Big]+2n^2\pi^2\mathrm{sh}\alpha_n[2(1+\mu)$$

$$\cdot(a^4+2b^4)+a^2b^2(7+11\mu)]+a^2n\pi(\mathrm{ch}\alpha_n+2ab(1-\mu))(\alpha_n^2+4n^2\pi^2)\Big\} \tag{3.13}$$

$$V_{n1}=\frac{Pb\pi n}{al}\sum_{m=1}^{\infty}\Big\{-(1-\mu)b\beta_m\mathrm{cth}\beta_m(\beta_m^2-G\pi^2)-b\Big[\Big(3-\mu-(1-\mu)\Big(\beta_m\mathrm{cth}\beta_m$$

$$-\frac{\beta_m(b-y_0)}{b}\mathrm{cth}\frac{\beta_m(b-y_0)}{b}\Big)\Big)(\beta_m^2+n^2\pi^2)+m^2\pi^2\Big(1+\mu+3(1-\mu)\Big(\beta_m\mathrm{cth}\beta_m$$

$$-\frac{\beta_m(b-y_0)}{b}\mathrm{cth}\frac{\beta_m(b-y_0)}{b}\Big)\Big)\Big]+\frac{2(1-\mu)\beta_m}{H}[b\beta_m((\beta_m^2-E\pi^2)^2-4\pi^4m^2E)$$

$$-am\pi(3\beta_m^4+2\beta_m^2\pi^2(m^2+n^2)-\pi^4E^2)]\Big\}\frac{\cos n\pi}{H}\mathrm{sh}\frac{\beta_m(b-y_0)}{b}\sin\frac{m\pi x_0}{a}$$

$$(\eta\leqslant y_0) \tag{3.14a}$$

$$V_{n1}=\frac{Pb\pi n}{al}\sum_{m=1}^{\infty}\Big\{(1-\mu)b\beta_m\mathrm{cth}\beta_m(\beta_m^2-G\pi^2)-b\Big[\Big(3-\mu-(1-\mu)\Big(\beta_m\mathrm{cth}\beta_m$$

$$-\frac{\beta_my_0}{b}\mathrm{cth}\frac{\beta_my_0}{b}\Big)\Big)(\beta_m^2+n^2\pi^2)+m^2\pi^2(1+\mu+3(1-\mu)\Big(\beta_m\mathrm{cth}\beta_m$$

$$-\frac{\beta_my_0}{b}\mathrm{cth}\frac{\beta_my_0}{b}\Big)\Big)\Big]-\frac{2(1-\mu)\beta_m}{H}[b\beta_m((\beta_m^2-E\pi^2)^2-4\pi^4m^2E)-am\pi(3\beta_m^4$$

$$+2\beta_m^2\pi^2(m^2+n^2)-\pi^4E^2)]\Big\}\frac{\cos m\pi}{H}\mathrm{sh}\frac{\beta_m y_0}{b}\sin\frac{m\pi x_0}{a}\qquad(\eta\geqslant y_0)\tag{3.14b}$$

$$V_{n2}=\frac{\pi^3D(1-\mu)n}{a^2b(4a^2+b^2)l}\sum_{m=1}^{\infty}\Big\{\frac{2\beta_m\mathrm{sh}\beta_m}{\pi^2EF}[a^2b^2(1-\mu)n^2-(8a^4+a^2b^2(5-7\mu)-b^4)m^2]$$

$$+\frac{2\mathrm{ch}\beta_m(1-C)}{(4a^2+b^2)\pi^2EF}[(64a^6+4a^4b^2(9-5\mu)+a^2b^4(15-19\mu)-b^6)m^2-a^2b^2(4a^2$$

$$+3b^2)(1-\mu)n^2]-\frac{\beta_m\mathrm{sh}\beta_m}{4B}[(40a^4-2a^2b^2(6-5\mu)-b^4(7-4\mu))(4\beta_m^2+n^2\pi^2)$$

$$-3m^2n^2(56a^4+2a^2b^2(46+7\mu)+b^4(3+20\mu))]+\frac{\beta_m\mathrm{sh}\beta_m}{4A}[(40a^4-2a^2b^2(10-9\mu)$$

$$-b^4(7-4\mu))(4\beta_m^2+n^2\pi^2)+m^2\pi^2(24a^4-2a^2b^2(54-65\mu)-b^4(33+28\mu))]$$

$$+\frac{m^2\mathrm{ch}\beta_m(1-C)}{\pi^2E^2F^2}[3(8a^4-2a^2b^2(2-\mu)-b^4)E^2-(8a^4+6a^2b^2(2-\mu)+b^4)F^2]$$

$$+\frac{\mathrm{ch}\beta_m(1-C)}{4(4a^2+b^2)B}[3m^2\pi^2(192a^6+16a^4b^2(20+3\mu)+4a^2b^4(25+12\mu)+b^6(37-20\mu))$$

$$-(64a^6+16a^4b^2\mu-4a^2b^4(5-4\mu)-b^6)(4\beta_m^2+n^2\pi^2)]+\frac{\mathrm{ch}\beta_m(1-C)}{4(4a^2+b^2)A}[(64a^6$$

$$+16a^4b^2(2-\mu)+4a^2b^4(1-2\mu)-b^6)(4\beta_m^2+n^2\pi^2)-m^2\pi^2(64a^6+16a^4b^2(70-9\mu)$$

$$+4a^2b^4(57-18\mu)+b^6(55+4\mu))]-\frac{\mathrm{ch}\beta_m(1-C)}{A^2}[\beta_m^2(40a^4-2a^2b^2(10-9\mu)$$

$$-b^4(7-4\mu))(16\beta_m^4-8\beta_m^2\pi^2E-(3m^2+n^2)E\pi^4)+m^2\pi^2(8a^4-2a^2b^2(8+7\mu)$$

$$-b^4(5+3\mu))(48\beta_m^4+8\beta_m^2\pi^2(m^2+n^2)-E^2\pi^4]+\frac{\mathrm{ch}\beta_m(1-C)}{B^2}[\beta_m^2(40a^4$$

$$-2a^2b^2(6-5\mu)-b^4(7-4\mu))(16\beta_m^4-8\beta_m^2\pi^2F-(27m^2+n^2)F\pi^4)+3m^2\pi^2(8a^4$$

$$-2a^2b^2(8-\mu)-b^4(3+\mu))(48\beta_m^4+8\beta_m^2\pi^2(9m^2+n^2)-F^2\pi^4)]\Big\}\frac{m^2}{\mathrm{sh}\beta_m}(a_m)\tag{3.15}$$

$$V_{n3}=0\tag{3.16}$$

$$V_{n4}=\frac{b\pi^2n}{al}\sum_{m=1}^{\infty}\Big\{b[(2-\mu)(\beta_m^2+n^2\pi^2)-(1-2\mu)m^2\pi^2]+\frac{(1-\mu)\beta_m}{H}[am\pi(3\beta_m^4$$

$$+2\beta_m^2\pi^2(m^2+n^2)-\pi^4E^2)-b\beta_m(\beta_m^4-2\beta_m^2\pi^2E-(3m^2+n^2)E\pi^4)]\Big\}\frac{m\cos m\pi}{H}(E_m)\tag{3.17}$$

$$V_{n5}=\frac{\pi^3}{bl}\sum_{n=1}^{\infty}\frac{n^3A_n}{\alpha_n(\alpha_n^2+4n^2\pi^2)^2}\{(1-\mu)(b^2\alpha_n^2-a^2n^2\pi^2)(3\mathrm{ch}\alpha_n-\alpha_n\mathrm{cth}\alpha_n)$$

$$-(a^2+4b^2-2b^2\mu)\alpha_n^2-4n^2\pi^2(2a^2+2b^2-a^2\mu)\}\tag{3.18}$$

其中 $A=(4\beta_m^2+(m^2+n^2)\pi^2)^2-4m^2n^2\pi^4$，$C=\cos m\pi\cos n\pi$

$B=(4\beta_m^2+(9m^2+n^2)\pi^2)^2-36m^2n^2\pi^4$，$E=m^2-n^2$

$H=(\beta_m^2+(m^2+n^2)\pi^2)^2-4m^2n^2\pi^4$，$F=9m^2-n^2$

$I=(4\beta_m^2+(4m^2+n^2)\pi^2)^2-16m^2n^2\pi^4$, $G_1=4m^2-n^2$, $G=3m^2-n^2$

至此，我们得到一无穷线性联立方程组(3.5)～(3.8)。据这些方程，能够计算出系数a_m，b_m，E_m和A_n，进而能够分别算出挠度，内矩分量，内力分量和内应力分量。

对于给定的具体问题，作用于三角形板上的集中载荷P的作用点(x_0, y_0)是已知的，因而问题是可解的。当集中载荷作用于该板自由边缘上任意一点时，上述边界条件仍然适用。

四、结　论

由以上的推导过程可以看出，由于本文以已被求解的受单位集中载荷的简支矩形板的一半为一基本系统，因而计算过程比较简单。本法还可用于求解在复杂载荷作用下具有复杂边界条件的三角形板的弯曲问题。它是求解三角形板弯曲问题的有效方法。

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The Bending of Set-Squares with One Free Oblique Edge under a Concentrated Load

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Abstract

In this paper, the reciprocal theorem is applied to research on the bending problem of set-squares with one free oblique edge and two clamped edges under a concentrated load acting at any of its points. This method is simpler and general.

Key words the reciprocal theorem, concentrated Load, set-squares