应用数学和力学，第17卷 第8期（1996年8月）
Applied Mathematics and Mechanics

应用数学和力学编委会编
重庆出版社出版

扁薄球壳非对称大变形问题

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（1995年6月16日收到）

摘 要

本文用修正迭代法研究了扁球壳非对称大变形问题，求得了在线性液体载荷作用下的扁球壳变形的二次近似解析解并绘出了摄动点的挠度与载荷的特征曲线族。应用本文方法还可对其他板壳的非轴对称大变形问题进行讨论。本文通过算例对平板及不同初挠度的扁球壳大挠度变形进行了讨论。

关键词 大位移 非轴对称 迭代法

一、引 言

扁球壳非对称问题的研究在理论方面及应用方面都有着重要意义。在工程中常能遇到此类问题，如两端用扁球壳封头的平置柱壳油罐、水罐等就是非轴对称问题。在工程中计算此问题的强度和变形时，一般常用线性分析并加大安全系数来处理该问题，这种方法对研究结构中的实际应力分布，误差较大，并在制造设备时所用钢材偏多，造成不必要的浪费。在理论研究方面，据我们目前掌握的资料分析，关于扁球壳非轴对称非线性问题的文章资料，在我们所能查阅到的各种中外文期刊中尚无发现。王新志教授等人最近几年在研究圆薄板非对称非线性问题方面做了许多工作[1～3]，而扁球壳的非轴对称非线性大变形问题在数学方面难度更大。扁球壳作为压力容器的一类部件由于其加工制造的难度较小，在工程中使用非常普遍。随着在压力容器设计中常规设计法逐渐要被应力分类设计法所代替的发展趋势，要求对扁球壳非轴对称非线性大变形问题进行较精确的应力及变形计算，这也是本文研究的实际意义所在。

本文利用Fourier级数将非线性偏微分方程组划为非线性常微分方程组，采用修正迭代法求解，得到了在线性液体载荷作用下的扁球壳变形的二次近似解析解，通过算例对平板及不同初挠度的扁球壳大挠度变形进行了讨论。

二、基本方程及边界条件

1．几何关系

扁球壳为旋转壳，与θ坐标无关（见图1），其初挠度为

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$$\overline{w}=\frac{R^2}{2R_0}\left(1-\frac{r^2}{R^2}\right) \quad (2.1)$$

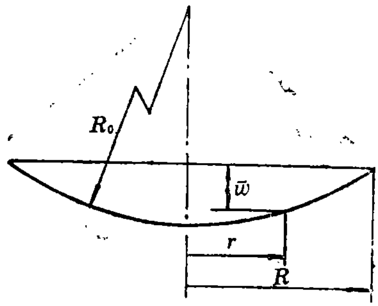


图1 结构图

中面变形几何方程为[4]

$$\varepsilon_r=\frac{\partial u_r}{\partial r}+\frac{\partial \overline{w}}{\partial r}\frac{\partial w}{\partial r}+\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 \quad (2.2)$$

$$\varepsilon_\theta=\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}+\frac{1}{2r^2}\left(\frac{\partial w}{\partial\theta}\right)^2 \quad (2.3)$$

$$\gamma_{r\theta}=\frac{1}{r}\left(\frac{\partial u_r}{\partial\theta}+r\frac{\partial u_\theta}{\partial r}-u_\theta+\frac{\partial\overline{w}}{\partial r}\frac{\partial w}{\partial\theta}+\frac{\partial w}{\partial r}\frac{\partial w}{\partial\theta}\right) \quad (2.4)$$

2．物理关系

$$\sigma_r=\frac{E}{1-\mu^2}(\varepsilon_r+\mu\varepsilon_\theta) \quad (2.5)$$

$$\sigma_\theta=\frac{E}{1-\mu^2}(\varepsilon_\theta+\mu\varepsilon_r) \quad (2.6)$$

$$\tau_{r\theta}=\frac{E}{2(1+\mu)}\gamma_{r\theta} \quad (2.7)$$

其中，E——板的弹性模量，μ——泊松比，$\sigma_r,\sigma_\theta,\tau_{r\theta}$——板的径向、环向和剪切应力。

3．平衡方程

略去体力的平衡方程为[5~7]

$$\frac{\partial\sigma_r}{\partial r}+\frac{\sigma_r-\sigma_\theta}{r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}=0 \quad (2.8)$$

$$\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+2\frac{\tau_{r\theta}}{r}=0 \quad (2.9)$$

$$D\Delta_1^2(w)=\overline{Q}+h\frac{\partial^2 w_t}{\partial r^2}\sigma_r+2h\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w_t}{\partial\theta}\right)\tau_{r\theta}+h\left(\frac{1}{r}\frac{\partial w_t}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w_t}{\partial\theta^2}\right)\sigma_\theta \quad (2.10)$$

其中，h——板壳厚度；$\overline{Q}$——外载集度．

$$w_t=\overline{w}+w,\quad D=\frac{Eh^3}{12(1-\mu^2)},\quad \Delta_1=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

将几何方程和物理方程代入平衡方程，整理后得

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}-\frac{u}{x}+\frac{1-\mu}{2x}\frac{\partial^2 u}{\partial\theta^2}+\frac{1}{2}\left[(1+\mu)\frac{\partial^2 v}{\partial x\partial\theta}-\frac{3-\mu}{x}\frac{\partial v}{\partial\theta}\right]=L_1(W) \quad (2.11)$$

$$x\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial x}-\frac{v}{x}+\frac{2}{1-\mu}\frac{1}{x}\frac{\partial^2 v}{\partial\theta^2}+\frac{1}{1-\mu}\left[(1+\mu)\frac{\partial^2 u}{\partial x\partial\theta}+\frac{3-\mu}{x}\frac{\partial u}{\partial\theta}\right]=L_2(W)$$

(2.12)

$$\Delta^2(W)=q+L_3(u,v,W) \quad (2.13)$$

其中

无量纲量为

$$x=\frac{r}{R},\ u=\frac{Ru_r}{h^2},\ v=\frac{Ru_\theta}{h^2},\ W=\frac{w}{h},\ \overline{W}=\frac{\overline{w}}{h},\ q=\frac{R^4}{Dh}\overline{Q}$$

而

$$L_1(W)=-x\frac{\partial}{\partial x}\left[\frac{\partial\overline{W}}{\partial x}\frac{\partial W}{\partial x}+\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2+\frac{\mu}{2}\left(\frac{1}{x}\frac{\partial W}{\partial\theta}\right)^2\right]$$
$$-(1-\mu)\frac{\partial\overline{W}}{\partial x}\frac{\partial W}{\partial x}-\frac{1-\mu}{2}\left[\left(\frac{\partial W}{\partial x}\right)^2-\left(\frac{1}{x}\frac{\partial W}{\partial\theta}\right)^2\right.$$
$$\left.+\frac{\partial}{\partial\theta}\left(\frac{1}{x}\frac{\partial\overline{W}}{\partial x}\frac{\partial W}{\partial\theta}+\frac{1}{x}\frac{\partial W}{\partial x}\frac{\partial W}{\partial\theta}\right)\right]$$

$$L_2(W)=-x\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial\overline{W}}{\partial x}\frac{\partial W}{\partial\theta}+\frac{1}{x}\frac{\partial W}{\partial x}\frac{\partial W}{\partial\theta}\right)$$
$$-\frac{2}{x}\left(\frac{\partial\overline{W}}{\partial x}\frac{\partial W}{\partial\theta}+\frac{\partial W}{\partial x}\frac{\partial W}{\partial\theta}\right)-\frac{2\mu}{1-\mu}\frac{\partial\overline{W}}{\partial x}\frac{\partial^2 W}{\partial x\partial\theta}$$
$$-\frac{1}{1-\mu}\frac{\partial}{\partial\theta}\left[\mu\left(\frac{\partial W}{\partial x}\right)^2+\left(\frac{1}{x}\frac{\partial W}{\partial\theta}\right)^2\right]$$

$$L_3(u,v,W)=12\left\{\left(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2\overline{W}}{\partial x^2}\right)\left[\frac{\partial u}{\partial x}+\mu\frac{u}{x}+\frac{\mu}{x}\frac{\partial v}{\partial\theta}+\frac{\partial\overline{W}}{\partial x}\frac{\partial W}{\partial x}+\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2\right.\right.$$
$$\left.+\frac{\mu}{2}\left(\frac{1}{x}\frac{\partial W}{\partial\theta}\right)^2\right]+(1-\mu)\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial W}{\partial\theta}\right)\left[\frac{1}{x}\frac{\partial u}{\partial\theta}+\frac{\partial v}{\partial x}-\frac{v}{x}\right.$$
$$\left.+\frac{1}{x}\left(\frac{\partial\overline{W}}{\partial x}\frac{\partial W}{\partial\theta}+\frac{\partial W}{\partial x}\frac{\partial W}{\partial\theta}\right)\right]+\left(\frac{1}{x}\frac{\partial W}{\partial x}+\frac{1}{x}\frac{\partial\overline{W}}{\partial x}\right.$$
$$\left.+\frac{1}{x^2}\frac{\partial^2 W}{\partial\theta^2}\right)\left[\mu\frac{\partial u}{\partial x}+\frac{u}{x}+\frac{1}{x}\frac{\partial v}{\partial\theta}+\mu\frac{\partial\overline{W}}{\partial x}\frac{\partial W}{\partial x}\right.$$
$$\left.\left.+\frac{\mu}{2}\left(\frac{\partial W}{\partial x}\right)^2+\frac{1}{2}\left(\frac{1}{x}\frac{\partial W}{\partial\theta}\right)^2\right]\right\}$$

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{1}{x}\frac{\partial}{\partial x}+\frac{1}{x^2}\frac{\partial^2}{\partial\theta^2}$$

4. 边界条件

为了便于说明求解方法，在这里只讨论周边固支的情况，此时有

$$x\to 0,\ W,u,v,\frac{\partial W}{\partial x},\ \frac{\partial u}{\partial x},\ \frac{\partial v}{\partial x}\ \text{有限}$$

$$x=1,\ W=u=v=0,\ \partial W/\partial x=0$$

三、问 题 的 求 解

取

$$u(x,\theta)=\sum_{k=-\infty}^{+\infty}[u_{rk}(x)+iu_{ik}(x)]e^{ik\theta} \tag{3.1}$$

$$v(x,\theta)=\sum_{k=-\infty}^{+\infty}[v_{rk}(x)+iv_{ik}(x)]e^{ik\theta} \tag{3.2}$$

$$W(x,\theta)=\sum_{k=-\infty}^{+\infty}[W_{rk}(x)+iW_{ik}(x)]e^{ik\theta} \tag{3.3}$$

$$q(x,\theta)=\sum_{k=-\infty}^{+\infty}[q_{rk}(x)+iq_{ik}(x)]e^{ik\theta} \tag{3.4}$$

其中$u_{rk}(x)$，$u_{ik}(x)$，$v_{rk}(x)$，$v_{ik}(x)$，$W_{rk}(x)$，$W_{ik}(x)$，$q_{rk}(x)$，$q_{ik}(x)$ 均为x的实函数。

由式(2.1)得

$$\overline{W}=\frac{\overline{w}}{h}=\frac{R^2}{2R_0h}(1-x^2)$$

令

$$C=R^2/R_0h$$

则有

$$\overline{W}=C(1-x^2)/2 \tag{3.5}$$

将式(3.1)～(3.5)代入式(2.11)、(2.12)和(2.13)中，并利用$e^{ik\theta}$的正交性质及复数的性质，经整理得

$$xu''_{rk}+u'_{rk}-\frac{1}{x}u_{rk}-(1-\mu)\frac{k^2}{2x}u_{rk}+\frac{1}{2}\left[-(1+\mu)kv'_{ik}+\frac{3-\mu}{x}kv_{ik}\right]$$
$$=L_{1rk}(W) \tag{3.6}$$

$$xv''_{ik}+v'_{ik}-\frac{1}{x}\left(1+\frac{2k^2}{1-\mu}\right)v_{ik}+\frac{1+\mu}{1-\mu}ku'_{rk}+\frac{3-\mu}{1-\mu}\frac{k}{x}u_{rk}=L_{2ik}(W) \tag{3.7}$$

$$xu''_{ik}+u'_{ik}-\frac{1}{x}u_{ik}-(1-\mu)\frac{k^2}{2x}u_{ik}+\frac{1}{2}\left[(1+\mu)kv'_{rk}-\frac{3-\mu}{x}kv_{rk}\right]$$
$$=L_{1ik}(W) \tag{3.8}$$

$$xv''_{rk}+v'_{rk}-\frac{1}{x}\left(1+\frac{2k^2}{1-\mu}\right)v_{rk}-\frac{1+\mu}{1-\mu}ku'_{ik}-\frac{3-\mu}{1-\mu}\frac{k}{x}u_{ik}=L_{2rk}(W) \tag{3.9}$$

$$H_k(W_{rk})=q_{rk}+L_{3rk}(u,v,W) \tag{3.10}$$

$$H_k(W_{ik})=q_{ik}+L_{3ik}(u,v,W) \tag{3.11}$$

其中

$$H_k=\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}-\frac{k^2}{x^2}\right)^2,\ '=\frac{d}{dx},\ ''=\frac{d^2}{dx^2},\qquad (k=0,\pm1,\pm2,\cdots)$$

$$L_{1rk}(W)=(2-\mu)CxW'_{rk}+Cx^2W''_{rk}+\frac{1-\mu}{2}Ck^2W_{rk}$$

$$+\sum_{m=-\infty}^{+\infty}\left\{-x(W'_{rm}W''_{r(k-m)}-W'_{im}W''_{i(k-m)})+\frac{1+\mu}{2}\frac{1}{x^2}m(k\right.$$

$$-m)(-W_{rm}W_{r(k-m)}+W_{im}W_{i(k-m)})-\frac{1-\mu}{2}(W'_{rm}W'_{r(k-m)}$$

$$-W'_{im}W'_{i(k-m)})+\frac{m}{2x}[(1+\mu)(k-m)$$

$$\left.+m(1-\mu)](W_{rm}W'_{r(k-m)}-W_{im}W'_{i(k-m)})\right\}$$

$$L_{2ik}(W)=\left(1+\frac{2\mu}{1-\mu}\right)CxkW'_{rk}+2CkW_{rk}+\sum_{m=-\infty}^{+\infty}\Big[-\frac{1}{x}m(W_{rm}W'_{r(k-m)}$$

$$-W_{im}W'_{i(k-m)})-m(W_{rm}W''_{r(k-m)}-W_{im}W''_{i(k-m)})$$

$$-\left(1+\frac{2\mu}{1-\mu}\right)m(W'_{rm}W'_{r(k-m)}-W'_{im}W'_{i(k-m)})$$

$$+\frac{2}{1-\mu}\ \frac{1}{x^2}m^2(k-m)(W_{rm}W_{r(k-m)}-W_{im}W_{i(k-m)})\Big]$$

$$L_{1ik}(W)=(2-\mu)CxW'_{ik}+Cx^2W''_{ik}+\frac{1-\mu}{2}Ck^2W_{ik}+\sum_{m=-\infty}^{+\infty}\Big\{-x(W'_{rm}W''_{i(k-m)}$$

$$+W'_{im}W''_{r(k-m)})-\frac{1+\mu}{2}\ \frac{1}{x^2}m(k-m)(W_{rm}W_{i(k-m)}$$

$$+W_{im}W_{r(k-m)})-\frac{1-\mu}{2}(W'_{rm}W'_{i(k-m)}+W'_{im}W'_{r(k-m)})$$

$$+\frac{m}{2x}[(k-m)(1+\mu)+m(1-\mu)](W_{rm}W'_{i(k-m)}+W_{im}W'_{r(k-m)})\Big\}$$

$$L_{2rk}(W)=-\left(1+\frac{2\mu}{1-\mu}\right)CxkW'_{ik}-2CkW_{ik}$$

$$+\sum_{m=-\infty}^{+\infty}\Big[\frac{1}{x}m(W_{rm}W'_{i(k-m)}+W_{im}W'_{r(k-m)})+m(W_{rm}W''_{i(k-m)}$$

$$+W_{im}W''_{r(k-m)})+\left(1+\frac{2\mu}{1-\mu}\right)m(W'_{rm}W'_{i(k-m)}+W'_{im}W'_{r(k-m)})$$

$$-\frac{2}{1-\mu}\ \frac{1}{x^2}m^2(k-m)(W_{rm}W_{i(k-m)}+W_{im}W_{r(k-m)})\Big]$$

$$L_{3rk}(u,v,W)=-12(1+\mu)C\left(u'_{rk}+\frac{u_{rk}}{x}-\frac{k}{x}v_{ik}-CxW'_{rk}\right)$$

$$+12\sum_{m=-\infty}^{+\infty}\Big\{(W''_{rm}u'_{r(k-m)}-W''_{im}u'_{i(k-m)})+\frac{\mu}{x}(W''_{rm}u_{r(k-m)}$$

$$-W''_{im}u_{i(k-m)})-\frac{\mu}{x}(k-m)(W''_{rm}v_{i(k-m)}+W''_{im}v_{r(k-m)})$$

$$-Cx(W'_{rm}W''_{r(k-m)}-W'_{im}W''_{i(k-m)})+\frac{1}{2}\Big(W''_{rm}+\frac{\mu}{x}W'_{rm}$$

$$-\frac{\mu}{x^2}m^2W_{rm}\Big)\sum_{l=-\infty}^{+\infty}(W'_{rl}W'_{r(k-m-l)}-W'_{il}W'_{i(k-m-l)})$$

$$-\frac{1}{2}\Big(W''_{im}+\frac{\mu}{x}W'_{im}-\frac{\mu}{x^2}m^2W_{im}\Big)\sum_{l=-\infty}^{+\infty}(W'_{rl}W'_{i(k-m-l)}$$

$$+W'_{il}W''_{r(k-m-l)})+\frac{\mu}{2x^2}\Big(W''_{rm}-\frac{1-2\mu}{\mu x}W'_{rm}$$

$$-\frac{1}{\mu x^2}m^2W_{rm}\Big)\sum_{l=-\infty}^{+\infty}l(k-m-l)(-W_{rl}W_{r(k-m-l)}$$

$$+W_{il}W_{i(k-m-l)})+\frac{\mu}{2x^2}\Big(W''_{im}-\frac{1-2\mu}{\mu x}W'_{im}$$

$$-\frac{1}{\mu x^2}m^2W_{im}\Big)\sum_{l=-\infty}^{+\infty}l(k-m-l)(W_{rl}W_{i(k-m-l)}+W_{il}W_{r(k-m-l)})$$

$$-\frac{1+3\mu}{2}C(W'_{rm}W'_{r(k-m)}-W'_{im}W'_{i(k-m)})$$

$$+\frac{1-3\mu}{2}\ \frac{C}{x^2}m(k-m)(-W_{rm}W_{r(k-m)}+W_{im}W_{i(k-m)})$$

$$+\frac{1}{x^2}[m(k-m)(1-\mu)-1](-W'_{rm}u_{r(k-m)}+W'_{im}u_{i(k-m)})$$

$$-\frac{1-\mu}{x}m(W'_{rm}v'_{i(k-m)}+W'_{im}v'_{r(k-m)})$$

$$+\frac{1}{x^2}[m(1-\mu)-(k-m)](W'_{rm}v_{i(k-m)}+W'_{im}v_{r(k-m)})$$

$$+\frac{mC}{x}[(k-m)(1-\mu)+m\mu](W_{rm}W'_{r(k-m)}-W_{im}W'_{i(k-m)})$$

$$-\frac{1-\mu}{x^2}mW'_{rm}\sum_{l=-\infty}^{+\infty}l(W_{rl}W'_{r(k-m-l)}-W_{il}W'_{i(k-m-l)})$$

$$+\frac{1-\mu}{x^2}mW'_{im}\sum_{l=-\infty}^{+\infty}l(W_{rl}W'_{i(k-m-l)}+W_{il}W'_{r(k-m-l)})$$

$$+\frac{m}{x^3}[(k-m)(1-\mu)-m](W_{rm}u_{r(k-m)}-W_{im}u_{i(k-m)})$$

$$+\frac{1-\mu}{x^2}m(W_{rm}v'_{i(k-m)}+W_{im}v'_{r(k-m)})$$

$$-\frac{m}{x^3}[(1-\mu)-m(k-m)](W_{rm}v_{i(k-m)}+W_{im}v_{r(k-m)})$$

$$+\frac{\mu}{x}(W'_{rm}u'_{r(k-m)}-W'_{im}u'_{i(k-m)})$$

$$-\frac{\mu}{x^2}m^2(W_{rm}u'_{r(k-m)}-W_{im}u'_{i(k-m)})\Big\}$$

$$L_{3ik}(u,v,W)=-12(1+\mu)C\Big(u'_{ik}+\frac{u_{ik}}{x}+\frac{k}{x}v_{rk}-CxW'_{ik}\Big)$$

$$+12\sum_{m=-\infty}^{+\infty}\Big\{(W''_{rm}u'_{i(k-m)}+W''_{im}u'_{r(k-m)})+\frac{\mu}{x}(W''_{rm}u_{i(k-m)}$$

$$+W''_{im}u_{r(k-m)})+\frac{\mu}{x}(k-m)(W''_{rm}v_{r(k-m)}-W''_{im}v_{i(k-m)})$$

$$-Cx(W'_{rm}W''_{i(k-m)}+W'_{im}W''_{r(k-m)})+\frac{1}{2}\Big(W''_{rm}+\frac{\mu}{x}W'_{rm}$$

$$-\frac{\mu}{x^2}m^2W_{rm}\Big)\sum_{l=-\infty}^{+\infty}(W'_{rl}W'_{i(k-m-l)}+W'_{il}W'_{r(k-m-l)})+\frac{1}{2}\Big(W''_{im}$$

$$+\frac{\mu}{x}W'_{im}-\frac{\mu}{x^2}m^2W_{im}\Big)\sum_{l=-\infty}^{+\infty}\Big(W'_{rl}W'_{r(k-m-l)}-W'_{il}W'_{i(k-m-l)}\Big)$$

$$-\frac{\mu}{2x^2}\Big(W''_{rm}-\frac{1-2\mu}{\mu x}W'_{rm}-\frac{1}{\mu x^2}m^2W_{rm}\Big)\sum_{l=-\infty}^{+\infty}l(k-m$$

$$-l)(W_{rl}W_{i(k-m-l)}+W_{il}W_{r(k-m-l)})+\frac{\mu}{2x^2}\Big(W''_{im}-\frac{1-2\mu}{\mu x}W'_{im}$$

$$-\frac{1}{\mu x^2}m^2W_{im}\Big)\sum_{l=-\infty}^{+\infty}l(k-m-l)(-W_{rl}W_{r(k-m-l)}$$

$$+W_{il}W_{i(k-m-l)})-\frac{1+3\mu}{2}C(W'_{rm}W'_{i(k-m)}+W'_{im}W'_{r(k-m)})$$

$$-\frac{1-3\mu}{2}\frac{C}{x^2}m(k-m)(W_{rm}W_{i(k-m)}+W_{im}W_{r(k-m)})$$

$$-\frac{1}{x^2}[m(k-m)(1-\mu)-1](W'_{rm}u_{i(k-m)}+W'_{im}u_{r(k-m)})$$

$$+\frac{1-\mu}{x}m(W'_{rm}v'_{r(k-m)}-W'_{im}v'_{i(k-m)})$$

$$-\frac{1}{x^2}[m(1-\mu)-(k-m)](W'_{rm}v_{r(k-m)}-W'_{im}v_{i(k-m)})$$

$$+\frac{mC}{x}[(k-m)(1-\mu)+m\mu](W_{rm}W'_{i(k-m)}+W_{im}W'_{r(k-m)})$$

$$-\frac{1-\mu}{x^2}mW'_{rm}\sum_{l=-\infty}^{+\infty}l(W_{rl}W'_{i(k-m-l)}+W_{il}W'_{r(k-m-l)})$$

$$-\frac{1-\mu}{x^2}mW'_{im}\sum_{l=-\infty}^{+\infty}l(W_{rl}W'_{r(k-m-l)}-W_{il}W'_{i(k-m-l)})$$

$$+\frac{m}{x^3}[(k-m)(1-\mu)-m](W_{rm}u_{i(k-m)}+W_{im}u_{r(k-m)})$$

$$-\frac{1-\mu}{x^2}m(W_{rm}v'_{r(k-m)}-W_{im}v'_{i(k-m)})$$

$$+\frac{m}{x^3}[(1-\mu)-m(k-m)](W_{rm}v_{r(k-m)}-W_{im}v_{i(k-m)})$$

$$+\frac{\mu}{x}(W'_{rm}u'_{i(k-m)}+W'_{im}u'_{r(k-m)})$$

$$-\frac{\mu}{x^2}m^2(W_{rm}u'_{i(k-m)}+W_{im}u'_{r(k-m)})\Big\}$$

边界条件

$x=1$时，$u_{rk}=u_{ik}=v_{rk}=v_{ik}=0$　(3.12a,b,c,d)

$W_{rk}=W_{ik}=W'_{rk}=W'_{ik}=0$　(3.13a,b,c,d)

$x=0$时，$u_{rk},u_{ik},v_{rk},v_{ik},W_{rk},W_{ik},W'_{rk},W'_{ik}$，有限　(3.14a～h)

式(3.6)～(3.14)构成了问题的基本方程。

采用修正迭代法即可求得该边值问题的各次近似解析解。修正迭代法的解题步骤为：

① 令$L_{3rk}(u,v,W)=0$及$L_{3ik}(u,v,W)=0$，利用式(3.10)和(3.11)求出一次近似解析解$W_{rk}^{①}$，$W_{ik}^{①}$；

② 令$W^{①}(x_0,\theta_0)=y_0$，则可得载荷与y_0间的关系，将此关系代入到求得的W近似解中；

③ 将$W_{rk}^{①}$及$W_{ik}^{①}$代入式(3.6)～(3.9)的右端算子中，求解方程组即可得一次近似解$u_{rk}^{①},u_{ik}^{①},v_{rk}^{①},v_{ik}^{①}$；

④ 将一次近似解析解代入式(3.10)和(3.11)右端算子$L_{3rk}(u,v,W)$和$L_{3ik}(u,v,W)$中，对式(3.10)和(3.11)求解左端算子中的未知量W_{rk}和W_{ik}，所得结果即为二次近似解析解$W_{rk}^{②}$和$W_{ik}^{②}$；

若n次近似解析解$W_{rk}^{(n)}$，$W_{ik}^{(n)}$已知，将②步骤中的各量上标用(n)代替，再进行②、③和④步骤即可求出$n+1$次近似解析解$W_{rk}^{(n+1)}$和$W_{ik}^{(n+1)}$。

四、算　例

设有一圆周边半径为R，厚度为h，曲面的曲率半径为$R_0(R_0\gg R)$的浅球壳，$\mu=0.3$周边固定夹紧，其凹面承受的法向载荷为

$$\bar{Q}(r,\theta)=\gamma_j R\left(1-\frac{r}{R}\sin\theta\right)$$

其中γ_j为某种液体的比重。

无量纲化后的载荷为

$$q=\frac{R^4}{Dh}\bar{Q}=A(1-x\sin\theta)$$

其中　$A=\gamma_j R^5/Dh$

将q展为Fourier级数

$$q(x,\theta)=\sum_{k=-\infty}^{+\infty}(q_{rk}+iq_{ik})e^{ik\theta}$$

可得
$$q_{rk}=\begin{cases}A & (k=0)\\ 0 & (k=\pm1,\pm2,\cdots)\end{cases} \tag{4.1}$$

$$q_{ik}=\begin{cases}-Ax/2, & (k=-1)\\ Ax/2, & (k=1)\\ 0, & (k=0,\pm2,\pm3,\cdots)\end{cases} \tag{4.2}$$

1．**W的一次近似解的边值问题**

$$H_k(W^{①}_{rk})=q_{rk} \tag{4.3}$$

$$H_k(W^{①}_{ik})=q_{ik} \tag{4.4}$$

边界条件

$x=1$时，$W^{①}_{rk}=W^{①}_{ik}=W^{①\prime}_{rk}=W^{①\prime}_{ik}=0$

$x=0$时，$W^{①}_{rk}$，$W^{①}_{ik}$，$W^{①\prime}_{rk}$，$W^{①\prime}_{ik}$有限

其中$k=0,\pm1,\pm2,\cdots$．求解上述边值问题，可得

$$W^{①}_{r0}=\frac{A}{64}(x+1)^2(x-1)^2,\quad W^{①}_{rk}=0\qquad(k=\pm1,\pm2,\cdots) \tag{4.5}$$

$$W^{①}_{i1}=-W^{①}_{i-1}=\frac{A}{384}x(x+1)^2(x-1)^2,\quad W^{①}_{ik}=0\qquad(k=0,\pm2,\pm3,\cdots) \tag{4.6}$$

由式(3.3)得

$$W①(x,\theta)=W^{①}_{r0}-2W^{①}_{i1}\sin\theta=\frac{A}{384}(x+1)^2(x-1)^2(6-2x\sin\theta) \tag{4.7}$$

令
$$y=W①(x_0,\ \theta_0)=\frac{A}{384}(x_0+1)^2(x_0-1)^2(6-2x_0\sin\theta_0)$$

取
$$W_0=\frac{A}{384}=\frac{y_0}{6(x_0+1)^2(x_0-1)^2(1-(x_0/3)\sin\theta_0)} \tag{4.8}$$

式中(x_0,θ_0)为给定点，它满足$0\leqslant x_0<1,0\leqslant\theta_0<2\pi,W①(x_0,\theta_0)\neq0$

将式(4.8)代入式(4.5)和(4.6)，可得

$$W^{①}_{r0}=6W_0(x+1)^2(x-1)^2 \tag{4.9}$$

$$W^{①}_{i1}=W_0x(x+1)^2(x-1)^2 \tag{4.10}$$

2．**u，v的一次近似解的边值问题**

边界条件为

$x=1$时，$u_{rk}=u_{ik}=v_{rk}=v_{ik}=0$

$x=0$时，u_{rk}，u_{ik}，v_{rk}，v_{ik}有限

(i) $k=0$

由式(3.6)～(3.9)，右端算子为

$$L_{1r0}(W①)=Cx^2W_{r0}^{①''}+(2-\mu)CxW_{r0}^{①'}-xW_{r0}^{①'}W_{r0}^{①''}-2xW_{t1}^{①'}W_{t1}^{①''}$$

$$+\frac{1+\mu}{x^2}(W_{t1}^{①})^2-\frac{1-\mu}{2}(W_{r0}^{①'})^2-(1-\mu)(W_{t1}^{①'})^2-\frac{2\mu}{x}W_{t1}^{①}W_{t1}^{①'} \tag{4.11}$$

$$L_{1t0}(W①)=L_{2r0}(W①)=L_{2t0}(W①)=0$$

对应的方程为

$$xu_{r0}^{①''}+u_{r0}^{①'}-\frac{1}{x}u_{r0}^{①}=L_{1r0}(W①) \tag{4.12}$$

$$u_{t0}^{①}=0,\quad v_{r0}^{①}=0,\quad v_{t0}^{①}=0$$

将式(4.9)和(4.10)代入，再由边界条件解得

$$u_{r0}^{①}=W_0^2(b_{2v1}x^9+b_{2v2}x^7+b_{2v3}x^5+b_{2v4}x^3+b_{2v5}x)+W_0(b_{1v1}x^5+b_{1v2}x^3+b_{1v3}x) \tag{4.13}$$

其中

$b_{2v1}=-2.74$， $b_{2v2}=-31.7333$， $b_{2v3}=103.667$， $b_{2v4}=-93.2$

$b_{2v5}=24.0067$， $b_{1v1}=4.7C$， $b_{1v2}=-8.1C$， $b_{1v3}=3.4C$

(ii) $k=1$

$$L_{1r1}(W①)=0,\quad L_{2t1}(W①)=0$$

$$L_{1t1}(W①)=Cx^2W_{t1}^{①''}+(2-\mu)CxW_{t1}^{①'}+\frac{1-\mu}{2}CW_{t1}^{①}-xW_{r0}^{①'}W_{t1}^{①''}$$

$$-xW_{t1}^{①'}W_{r0}^{①''}-(1-\mu)W_{r0}^{①'}W_{t1}^{①'}+\frac{1-\mu}{2}\frac{1}{x}W_{t1}^{①}W_{r0}^{①'} \tag{4.14}$$

$$L_{2r1}(W①)=-\left(1+\frac{2\mu}{1-\mu}\right)CxW_{t1}^{①'}-2CW_{t1}^{①}+\frac{1}{x}W_{t1}^{①}W_{r0}^{①'}+W_{t1}^{①}W_{r0}^{①''}$$

$$+\left(1+\frac{2\mu}{1-\mu}\right)W_{t1}^{①}W_{r0}^{①'} \tag{4.15}$$

由式(3.6)～(3.9)、(4.9)和(4.10)式及边界条件可求得

$$u_{r1}^{①}=0,\quad v_{t1}^{①}=0$$

$$u_{t1}^{①}=W_0^2(b_{2v6}x^8+b_{2v7}x^6+b_{2v8}x^4+b_{2v9}x^2+b_{2v10})+W_0(b_{1v6}x^6+b_{1v7}x^4+b_{1v8}x^2\ln x+b_{1v9}x^2+b_{1v10}) \tag{4.16}$$

$$v_{r1}^{①}=W_0^2(b_{2v12}x^8+b_{2v13}x^6+b_{2v14}x^4+b_{2v15}x^2+b_{2v16})+W_0(b_{1v12}x^6+b_{1v13}x^4+b_{1v14}x^2\ln x+b_{1v15}x^2+b_{1v16}) \tag{4.17}$$

其中

$b_{2v6}=-14.66$， $b_{2v7}=42.9$， $b_{2v8}=-41.0$， $b_{2v9}=12.2096$

$b_{2v10}=0.55037$， $b_{1v6}=0.82995C$， $b_{1v7}=-1.54687C$， $b_{1v8}=0.00625C$

$b_{1v9}=0.68639C$， $b_{1v10}=0.03054C$

$b_{2v12}=0.74$， $b_{2v13}=-2.7$， $b_{2v14}=3.4$， $b_{2v15}=-0.88963$

$b_{2v16}=-0.55037$， $b_{1v12}=0.03620C$， $b_{1v13}=-0.19063C$， $b_{1v14}=0.33125C$

$b_{1v15}=0.18496C$， $b_{1v16}=-0.03054C$

(iii) $k=2$

$$L_{1r2}(W①)=xW_{i1}^{①\prime}W_{i1}^{①\prime\prime}+\frac{1+\mu}{2}\frac{1}{x^2}(W_{i1}^{①})^2+\frac{1-\mu}{2}(W_{i1}^{①\prime})^2-\frac{1}{x}W_{i1}^{①}W_{i1}^{①\prime} \quad (4.18)$$

$$L_{2i2}(W①)=\frac{1}{x}W_{i1}^{①}W_{i1}^{①\prime}+W_{i1}^{①}W_{i1}^{①\prime\prime}+\left(1+\frac{2\mu}{1-\mu}\right)(W_{i1}^{①\prime})^2$$

$$-\frac{2}{1-\mu}\frac{1}{x^2}(W_{i1}^{①})^2 \quad (4.19)$$

$L_{1i2}(W①)=0$，$L_{2r2}(W①)=0$

由式(3.6)～(3.9)、(4.9)和(4.10)及边界条件可求得

$u_{i2}^{①}=0$，$v_{r2}^{①}=0$

$$u_{r2}^{①}=W_0^2(b_{2v18}x^9+b_{2v19}x^7+b_{2v20}x^5+b_{2v21}x^3+b_{2v22}x) \quad (4.20)$$

$$u_{i2}^{①}=W_0^2(b_{2v23}x^9+b_{2v24}x^7+b_{2v25}x^5+b_{2v26}x^3+b_{2v27}x) \quad (2.21)$$

其中

$b_{2v18}=1.35$，　$b_{2v19}=-4.13333$，$b_{2v20}=4.36667$，　$b_{2v21}=-1.79037$

$b_{2v22}=0.20704$，　$b_{2v23}=0.19$，　$b_{2v24}=-0.78333$，$b_{2v25}=1.23333$

$b_{2v26}=-0.84704$，$b_{2v27}=0.20704$

(iv) 其他的u_{rk}，u_{ik}，v_{rk}和v_{ik}

由$W①$的性质决定了，当$k=\pm3$，±4，±5,…时

$u_{rk}=u_{ik}=v_{rk}=v_{ik}=0$

3. W的二次近似解的边值问题

由式(3.10)和(3.11)得

$$H_k(W_{rk}^{②})=q_{rk}+L_{3rk}(u①,\ v①,\ W①) \quad (4.22)$$

$$H_k(W_{ik}^{②})=q_{ik}+L_{3ik}(u①,\ v①,\ W①) \quad (4.23)$$

令$Q=A$，由式(4.1)和(4.2)得

$$q_{rk}=\begin{cases}Q & (k=0)\\ 0 & (k=\pm1,\pm2,\pm3,\cdots)\end{cases} \quad (4.24)$$

$$q_{ik}=\begin{cases}-xQ/2 & (k=-1)\\ xQ/2 & (k=1)\\ 0 & (k=0,\pm2,\pm3,\cdots)\end{cases} \quad (4.25)$$

边界条件

$$x=1\text{时},\quad W_{rk}^{②}=W_{ik}^{②}=W_{rk}^{②\prime}=W_{ik}^{②\prime}=0 \quad (4.26\text{a,b,c,d})$$

$$x=0\text{时},\quad W_{rk}^{②},\ W_{ik}^{②},\ W_{rk}^{②\prime},W_{ik}^{②\prime}\text{有限} \quad (4.27\text{a,b,c,d})$$

为方便起见，将算子L_{3rk}，L_{3ik}的具体表达式中各参数的右上角标①省略.

(i) $k=0$

$L_{3i0}(u①,v①,W①)=0$

$$L_{3r0}(u①,v①,W①)=-12(1+\mu)C\left(u'_{r0}+\frac{u_{r0}}{x}-CxW'_{r0}\right)+12\left\{(W''_{r0}u'_{r0}+2W''_{i1}u'_{i1})\right.$$

$$+\frac{\mu}{x}(W''_{r0}u_{r0}+2W''_{t1}u_{t1})-Cx(W'_{r0}W''_{r0}+2W'_{t1}W''_{t1})$$

$$+\frac{1}{2}\left[W''_{r0}+\frac{\mu}{x}W'_{r0}-(1+3\mu)C\right][(W'_{r0})^2+2(W'_{t1})^2]$$

$$+2\left(W''_{t1}+\frac{\mu}{x}W'_{t1}-\frac{\mu}{x^2}W_{t1}\right)W'_{r0}W'_{t1}+\frac{\mu}{x^2}\left(W''_{r0}-\frac{1-2\mu}{\mu x}W'_{r0}\right.$$

$$\left.+\frac{1-3\mu}{\mu}C\right)W^{2}_{t1}+\frac{1}{x^2}[W'_{r0}u_{r0}+2(2-\mu)W'_{t1}u_{t1}]$$

$$-2(1-2\mu)\frac{C}{x}W_{t1}W'_{t1}+2\frac{1-\mu}{x^2}W'_{t1}W_{t1}W'_{r0}-2\frac{2-\mu}{x^3}W_{t1}u_{t1}$$

$$+\frac{\mu}{x}(W'_{r0}u'_{r0}+2W'_{t1}u'_{t1})-2\frac{\mu}{x^2}W_{t1}u'_{t1}\Big\} \tag{4.28}$$

由式(4.22)～(4.27)解得

$$W^{②}_{t0}=0$$

$$\begin{aligned}W^{②}_{r0}=&W_0^3(d_{3r1}x^{14}+d_{3r2}x^{12}+d_{3r3}x^{10}+d_{3r4}x^8+d_{3r5}x^6+d_{3r6}x^4+d_{3r9}x^2+d_{3r10})\\&+W_0^2(d_{2r1}x^{12}+d_{2r2}x^{10}+d_{2r3}x^8\ln x+d_{2r4}x^8+d_{2r5}x^6\ln x+d_{2r6}x^6\\&+d_{2r7}x^4+d_{2r9}x^2+d_{2r10})+W_0(d_{1r1}x^8+d_{1r2}x^6+d_{1r3}x^4+d_{1r9}x^2+d_{1r10})\\&+\frac{Q}{64}(x+1)^2(x-1)^2\end{aligned} \tag{4.29}$$

其中

$d_{3r1}=-0.05110$, $d_{3r2}=-0.7252$, $d_{3r3}=8.7804$, $d_{3r4}=-39.6712$
$d_{3r5}=130.694$, $d_{3r6}=-295.278$, $d_{3r9}=317.966$, $d_{3r10}=-121.715$
$d_{2r1}=0.01360C$, $d_{2r2}=0.31111C$, $d_{2r3}=36.855C$, $d_{2r4}=-24.6086C$
$d_{2r5}=-19.4133C$, $d_{2r6}=32.8202C$, $d_{2r7}=-51.9707C$. $d_{2r9}=93.5573C$
$d_{2r10}=-50.1229C$, $d_{1r1}=-0.02844C^2$, $d_{1r2}=0.2275C^2$, $d_{1r3}=-1.6575C^2$
$d_{1r9}=2.74625C^2$, $d_{1r10}=-1.28781C^2$

(ii) $k=1$

$$L_{3r1}(u^{①},v^{①},W^{①})=0$$

$$L_{3t1}(u^{①},v^{①},W^{①})=-12(1+\mu)C\left(u'_{t1}+\frac{u_{t1}}{x}+\frac{v_{r1}}{x}-CxW'_{t1}\right)$$

$$+12\Big\{[W''_{r0}u'_{t1}+W''_{t1}(u'_{r0}-u'_{r2})]+\frac{\mu}{x}[W''_{r0}u_{t1}+W''_{t1}(u_{r0}-u_{r2})]$$

$$+\frac{\mu}{x}(W''_{r0}v_{r1}+2W''_{t1}v_{t2})-Cx(W'_{r0}W''_{t1}+W'_{t1}W''_{r0})$$

$$+\left[W''_{r0}+\frac{\mu}{x}W'_{r0}-(1+3\mu)C\right]W'_{r0}W'_{t1}$$

$$+\frac{1}{2}\left(W''_{t1}+\frac{\mu}{x}W'_{t1}-\frac{\mu}{x^2}W_{t1}\right)[(W'_{r0})^2+3(W'_{t1})^2]$$

$$+\frac{\mu}{2x^2}\left(W''_{t1}-\frac{1-2\mu}{\mu x}W'_{t1}-\frac{1}{\mu x^2}W_{t1}\right)W^2_{t1}$$

$$+\frac{1}{x^2}[W'_{r0}u_{t1}+W_{t1}u_{r0}-(3-2\mu)W'_{t1}u_{r2}]-\frac{1-\mu}{x}W'_{t1}v'_{t2}$$

$$+\frac{1}{x^2}[W'_{r0}v_{r1}+(3-\mu)W'_{t1}v_{t2}]+\frac{\mu C}{x}W_{t1}W'_{r0}$$

$$+\frac{1-\mu}{x^2}W_{t1}(W'_{t1})^2+\frac{1}{x^3}[-W_{t1}u_{r0}+(3-2\mu)W_{t1}u_{r2}]$$

$$+\frac{1-\mu}{x^2}W_{t1}v'_{t2}-\frac{3-\mu}{x^3}W_{t1}v_{t2}+\frac{\mu}{x}[W'_{r0}u'_{t1}+W'_{t1}(u'_{r0}-u'_{r2})]$$

$$-\frac{\mu}{x^2}W_{t1}(u'_{r0}-u'_{r2}) \qquad (4.30)$$

由式(4.22)～(4.27)解得

$$W^{②}_{r1}=0$$

$$\begin{aligned}W^{②}_{t1}=&W^3_0(d_{3\nu11}x^{15}+d_{3\nu12}x^{13}+d_{3\nu13}x^{11}+d_{3\nu14}x^9+d_{3\nu15}x^7+d_{3\nu16}x^6+d_{3\nu17}x^5\\&+d_{3\nu18}x^4+d_{3\nu19}x^3\ln x+d_{3\nu22}x^3+d_{3\nu23}x)+W^2_0(d_{2\nu11}x^{11}+d_{2\nu12}x^9\\&+d_{2\nu13}x^7\ln x+d_{2\nu14}x^7+d_{2\nu15}x^5\ln x+d_{2\nu16}x^5+d_{2\nu17}x^3\ln x+d_{2\nu22}x^3\\&+d_{2\nu23}x)+W_0(d_{1\nu11}x^9+d_{1\nu12}x^7+d_{1\nu13}x^5\ln x+d_{1\nu14}x^5+d_{1\nu15}x^3\ln x\\&+d_{1\nu22}x^3+d_{1\nu23}x)+\frac{Q}{384}x(x+1)^2(x-1)^2\end{aligned} \qquad (4.31)$$

其中

$d_{3\nu11}=-0.00487$, $d_{3\nu12}=-0.77529$, $d_{3\nu13}=4.55813$, $d_{3\nu14}=-14.5369$

$d_{3\nu15}=33.9960$, $d_{3\nu16}=0.44479$, $d_{3\nu17}=-55.6053$, $d_{3\nu18}=-4.39087$

$d_{3\nu19}=0$, $d_{3\nu22}=54.7395$, $d_{3\nu23}=-18.4253$, $d_{2\nu11}=0.11106C$

$d_{2\nu12}=-0.87697C$, $d_{2\nu13}=98.28C$, $d_{2\nu14}=-66.0203C$, $d_{2\nu15}=-43.68C$

$d_{2\nu16}=43.1479C$, $d_{2\nu17}=0$, $d_{2\nu22}=87.4175C$, $d_{2\nu23}=-63.7793C$

$d_{1\nu11}=-0.00344C^2$, $d_{1\nu12}=0.02607C^2$, $d_{1\nu13}=-1.82C^2$, $d_{1\nu14}=2.02174C^2$

$d_{1\nu15}=0$, $d_{1\nu22}=-3.19794C^2$, $d_{1\nu23}=1.15357C^2$

(iii) $k=2$

$$L_{3t2}(u①,v①,W①)=0$$

$$L_{3r2}(u①,v①,W①)=-12(1+\mu)C\left(u'_{r2}+\frac{u_{r2}}{x}-\frac{2v_{t2}}{x}\right)+12\left\{(W''_{r0}u'_{r2}-W''_{t1}u'_{t1})\right.$$

$$+\frac{\mu}{x}(W''_{r0}u_{r2}-W''_{t1}u_{t1})-\frac{\mu}{x}(2W''_{r0}v_{t2}+W''_{t1}v_{r1})$$

$$+CxW'_{t1}W''_{t1}-\frac{1}{2}\left[W''_{r0}+\frac{\mu}{x}W'_{r0}-(1+3\mu)C\right](W'_{t1})^2$$

$$-\left(W''_{t1}+\frac{\mu}{x}W'_{t1}-\frac{\mu}{x^2}W_{t1}\right)W'_{r0}W'_{t1}+\frac{\mu}{2x^2}\left(W''_{r0}\right.$$

$$\left.-\frac{1-2\mu}{\mu x}W'_{r0}+\frac{1-3\mu}{\mu}C\right)W^2_{t1}+\frac{1}{x^2}(W'_{r0}u_{r2}-\mu W'_{t1}u_{t1})$$

$$-\frac{1-\mu}{x}W'_{t1}v'_{r1}-\frac{1}{x^2}(2W'_{r0}v_{t2}+\mu W'_{t1}v_{r1})-\frac{C}{x}W_{t1}W'_{t1}$$

$$+\frac{1-\mu}{x^2}W'_{t1}W_{t1}W'_{r0}+\frac{\mu}{x^3}W_{t1}u_{t1}+\frac{1-\mu}{x^2}W_{t1}v'_{r1}+\frac{\mu}{x^3}W_{t1}v_{r1}$$

$$+\frac{\mu}{x}(W'_{r0}u'_{r2}-W'_{t1}u'_{t1})+\frac{\mu}{x^2}W_{t1}u'_{t1}\Big\} \tag{4.32}$$

由式(4.22)～(4.27)可解得

$$W^{②}_{t2}=0$$

$$W^{②}_{r2}=W_0^3(d_{3r24}x^{14}+d_{3r25}x^{12}+d_{3r26}x^{10}+d_{3r27}x^8+d_{3r28}x^6+d_{3r29}x^5$$
$$+d_{3r30}x^4\ln x+d_{3r33}x^4+d_{3r31}x^3+d_{3r34}x^2)+W_0^2(d_{2r24}x^{12}$$
$$+d_{2r25}x^{10}+d_{2r26}x^8\ln x+d_{2r27}x^8+d_{2r28}x^6\ln x+d_{2r29}x^6$$
$$+d_{2r30}x^4\ln x+d_{2r33}x^4+d_{2r31}x^3+d_{2r34}x^2) \tag{4.33}$$

式中

$d_{3r24}=0.05915$, $d_{3r25}=1.72678$, $d_{3r26}=-4.39217$, $d_{3r27}=4.49000$
$d_{3r28}=-2.04407$, $d_{3r29}=-6.50077$, $d_{3r30}=0$, $d_{3r31}=-31.1353$
$d_{3r33}=24.5168$, $d_{3r34}=13.2796$, $d_{2r24}=-0.00953C$, $d_{2r25}=-0.01441C$
$d_{2r26}=-29.12C$, $d_{2r27}=18.6327C$, $d_{2r28}=16.38C$, $d_{2r29}=-16.8579C$
$d_{2r30}=0$, $d_{2r31}=-1.29730C$, $d_{2r33}=-15.0584C$, $d_{2r34}=14.6048C$

(iv) $k=3$

$$L_{3r3}(u^{①},v^{①},W^{①})=0$$

$$L_{3t3}(u^{①},v^{①},W^{①})=12\Big\{W''_{t1}u'_{r2}+\frac{\mu}{x}W''_{t1}u_{r2}-\frac{2\mu}{x}W''_{t1}v_{t2}-\frac{1}{2}\Big(W''_{t1}+\frac{\mu}{x}W'_{t1}$$

$$-\frac{\mu}{x^2}W_{t1}\Big)(W'_{t1})^2+\frac{\mu}{2x^2}\Big(W''_{t1}-\frac{1-2\mu}{\mu x}W'_{t1}-\frac{1}{\mu x^2}W_{t1}\Big)W^2_{t1}$$

$$-\frac{1-2\mu}{x^2}W'_{t1}u_{r2}-\frac{1-\mu}{x}W'_{t1}v'_{t2}-\frac{1+\mu}{x^2}W'_{t1}v_{t2}+\frac{1-\mu}{x^2}W_{t1}(W'_{t1})^2$$

$$+\frac{1-2\mu}{x^3}W_{t1}u_{r2}+\frac{1-\mu}{x^2}W_{t1}v'_{t2}+\frac{1+\mu}{x^3}W_{t1}v_{t2}+\frac{\mu}{x}W'_{t1}u'_{r2}$$

$$-\frac{\mu}{x^2}W_{t1}u'_{r2}\Big\} \tag{4.34}$$

由式(4.22)～(4.27)可解得

$$W^{②}_{r3}=0$$

$$W^{②}_{t3}=W_0^3(d_{3r36}x^{15}+d_{3r37}x^{13}+d_{3r38}x^{11}+d_{3r39}x^9+d_{3r40}x^7+d_{3r41}x^6$$
$$+d_{3r42}x^5\ln x+d_{3r43}x^5+d_{3r44}x^4+d_{3r45}x^3) \tag{4.35}$$

式中

$d_{3r36}=0.00329$, $d_{3r37}=0.39595$, $d_{3r38}=-1.20052$, $d_{3r39}=1.53935$
$d_{3r40}=-1.14111$, $d_{3r41}=-0.34848$, $d_{3r42}=0$, $d_{3r43}=0.75557$
$d_{3r44}=0.46778$, $d_{3r45}=-0.47184$

(v) $k=\pm4,\pm5,\pm6,\cdots$

因

$$L_{3rk}(u①,v①,W①)=0,\ L_{4ik}(u①,v①,W①)=0$$

$$q_{rk}=0,\ q_{ik}=0$$

故有

$$W_{rk}^{②}=W_{ik}^{②}=0 \qquad (k=\pm4,\pm5,\pm6,\cdots)$$

将式(4.29)、(4.31)、(4.33)和(4.35)代入式(3.3)，注意到 $W_{ik}=-W_{i-k}$，$W_{rk}=W_{r-k}(k=0,\pm1,\pm2,\cdots)$，经整理可得$W$的二次近似解析解

$$W②(x,\theta)=W_{r0}^{②}-2W_{i1}^{②}\sin\theta+2W_{r2}^{②}\cos2\theta-2W_{i3}^{②}\sin3\theta \tag{4.36}$$

4. 特征曲线

取材料的泊松比$\mu=0.3$，设给定点(x_0,θ_0)的挠度为y_0，由式(4.8)知

$$W_0=y_0/[6(x_0+1)^2(x_0-1)^2(1-(x_0/3)\sin\theta_0)] \tag{4.37}$$

将式(4.37)代入式(4.36)并令$W②(x_0,\theta_0)=y_0$，整理可得

$$Q=\left[y_0-W_{r0}^{②}(x_0)+2W_{i1}^{②}(x_0)\sin\theta_0+\frac{Q}{64}(x_0+1)^2(x_0-1)^2\left(1-\frac{x_0}{3}\sin\theta_0\right)-2W_{r2}^{②}(x_0)\cos2\theta_0+2W_{i3}^{②}(x_0)\sin3\theta_0\right]\Big/\left[\frac{1}{64}(x_0+1)^2(x_0-1)^2\left(1-\frac{x_0}{3}\sin\theta_0\right)\right] \tag{4.38}$$

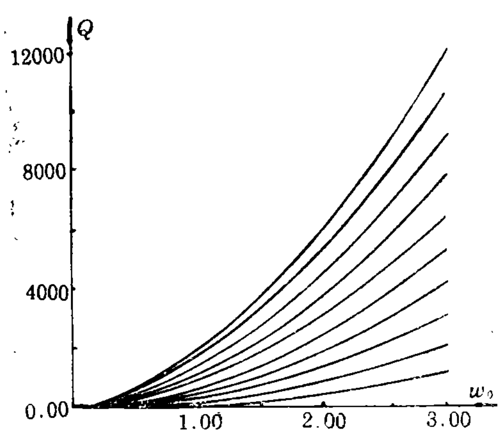
由式(4.29)、(4.31)和(4.36)可看出，式(4.38)右端中不含Q项。当 x_0，θ_0一经确定，Q只是y_0与C的函数。

由式(4.24)和(4.25)可知

$$Q=A=\gamma_j R^5/Dh$$

当C值给定时，利用式(4.38)可得载荷与点$(x_0,\ \theta_0)$的挠度关系的特征曲线。

$x_0=(\sqrt{41}-6)/5$，$\theta_0=-\pi/2$时的特征曲线如图2所示，这里点$(x_0,\ \theta_0)$对应于一次近似结果（即线性结果）中挠度最大的点。



注：图中曲线族自下而上依次为$C=0$，1，2，3，4，5，6，7，8，9.

图2 载荷挠度特征曲线图

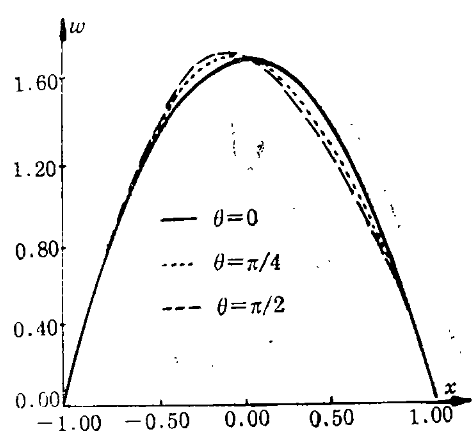


图3 剖面挠度曲线图

5．二次近似结果的变形图

将式(4.37)代入式(4.29)、(4.31)、(4.33)和(4.35)中，可知式(4.38)右端y_0的最高次幂为三次．若给定x_0，θ_0且Q已知，可利用牛顿切线法等数值解法求出y_0，但运算较为复杂．为了方便说明问题，我们给定点(x_0，θ_0)及其y_0值，将式(4.38)和(4.37)代入式(4.36)，可得到各点的挠度值$W^{②}(x, \theta)$($0\leqslant x\leqslant 1$，$0\leqslant\theta\leqslant 2\pi$)．为了清楚地看明结构内各点的变形情况，我们取出$\theta=0$，$\pi/4$，$\pi/2$三个剖面，并绘出相应的挠度曲线图，如图3所示．

五、结　论

1．本文给出了扁球壳的二次近似解析解．通过式(4.7)及图2可以看出，一次近似解析解y_0与载荷呈线性关系，而二次近似解的y_0与Q的关系为非线性的．从图2可知，由非线性分析所得的结构的承载能力高于线性的．随着变形的增大，这种差距更明显．非线性分析较接近真实情况，故在结构有较大变形时应采用非线性分析方法．在对有扁球壳部件的压力容器进行承载能力的评估或优化设计时，用二次近似解析解进行计算分析，有很重要的实用价值．

2．在运算中我们发现，非轴对称非线性问题的最大挠度点随载荷大小的变化而变化，当变形较大时，该现象较明显，这与轴对称问题有较大差异[8]．

3．在绘制图4时我们注意到，摄动参数点选取的是否合适将直接影响到分析结果的好坏．摄动参数点的位置可根据整个变形图的情况加以确定与修正，一般在不理想区域附近选摄动点较为适宜．

4．由图2可看出，初挠度较大时，扁球壳的承载能力较高．

5．本文所采用的计算方法具有较普遍的适用性，可用于其他类型的浅薄壳非轴对称非线性问题．其结果可为有关工程的优化设计提供理论依据．

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Non-Symmetrical Large Deformation of a Shallow Thin Spherical Shell

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Abstract

By the modified iteration method, in this paper, nonsymmetrical large deflection of a shallow spherical shell is discussed. We solve the second-order approximate analytical solution of the deflection of a shallow spherical shell subjected to linear liquid loads, and portray the characteristic curves of load-deflection on a perturbing point. With this paper's method, the similar questions of other kind of shell can be discussed. Through the examples, we discuss the large deflection of a plane and shallow spherical shells with different initial deflections.

Key words large deformation, non-symmetrical, iteration method

ISSN 1000-0887
CODEN YSHLEM

应用数学和力学

Applied Mathematics and Mechanics

（力学类、数学类、物理学类核心期刊）

交通部重庆交通学院主办
重 庆 出 版 社 出 版

第17卷　1996年　第8期

ISSN 1000-0887

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