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非线性四阶常微分方程三点边值问题解的存在性

高永馨[1]

（林宗池推荐，1994年5月30日收到，1995年12月4日收到修改稿）

摘要

本文利用文献[1]、[2]的方法，讨论了非线性四阶常微分方程

$$y^{(4)}=f(t,y,y',y'',y''') \tag{*}$$

满足如下条件

$$\left.\begin{array}{l} g(y(a),y'(a),y''(a),y'''(a))=0,\ \ h(y(b),y''(b))=0 \\ y'(b)=b_1,\ \ k(y(c),y'(c),y''(c),y'''(c))=0 \end{array}\right\} \tag{**}$$

的三点边值问题解的存在性。

对于非线性四阶常微子方程

$$y^{(4)}=f(t,y,y',y'',y''')$$

的边值问题，到目前已经有了一系列的研究。而本文所讨论的方程(*)，具非线性边界条件(**)的边值问题解的存在性，则是上述工作未曾涉及的。本文主要定理的推论，方程(*)之具如下形式的边界条件

$$a_0y(a)+a_1y'(a)+a_2y''(a)+a_3y'''(a)=y_0,\quad b_0y(b)+b_2y''(b)=y_1$$
$$y'(b)=y_2,\ \ c_0y(c)+c_1y'(c)+c_2y''(c)+c_3y'''(c)=y_3$$

的边值问题解的存在性，以前的工作也未涉及。

关键词 非线性四阶微分方程 三点边值问题 解的存在性

一、准备知识

全文总假设

(H_1) 函数$f(t,y,z,w,\eta)$在区域$\Omega=\{(t,y,z,w,\eta):a\leqslant t\leqslant c,-\infty<y,z,w,\eta<+\infty\}$上连续。

(H_2) 方程(*)初值问题的解可延至$[a,c]$或在其最大存在区间上无界。

仿照[3]给出下面的定义：

定义 若函数$\phi(t)\in C^4[a,c]$，使得

$$\phi^{(4)}(t)\leqslant f(t,\phi(t),\phi'(t),\phi''(t),\phi'''(t)),\qquad a\leqslant t\leqslant c$$

则称$\phi(t)$为方程(*)于$[a,c]$上的上解；若函数$\psi(t)\in C^4[a,c]$，使得

1 东北电力学院基础部，吉林市 132012

$$\psi^{(4)}(t)\geqslant f(t,\psi(t),\psi'(t),\psi''(t),\psi'''(t)),\qquad a\leqslant t\leqslant c$$

则称$\psi(t)$为方程(*)于$[a,c]$上的下解。

利用Kamke定理[4,p.14]并仿文[1]引理4的证明，便不难得到下面的引理：

引理1 假设条件(H_1)、(H_2)成立，若函数列$f_n(t,y,z,w,\eta)$ $(n=1,2,\cdots)$在Ω上连续，且在Ω的任一有界闭子集上一致收敛于$f(t,y,z,w,\eta)$；方程

$$y^{(4)}=f_n(t,y,y',y'',y''')$$

的解$y_n(t)$及其$y_n'(t)$，$y_n''(t)$ $(n=1,2,\cdots)$所构成的序列皆于$[a,c]$上存在并且一致有界，则方程(*)在$[a,c]$上必有解$y(t)$，并且$\{y_n(t)\}$存在子序列$\{y_{n(1)}(t)\}$能使

$$\lim_{n(1)\to\infty} y_{n(1)}^{(i)}(t)=y^{(i)}(t)\qquad (i=0,1,2,3)$$

在$[a,c]$上一致地成立。

二、三点边值问题

为叙述方便起见，用(A_i) $(i=1,2,\cdots,5)$表示下述各条件：

(A_1) 函数$f(t,y,z,w,\eta)$对于固定的t,w,η在$[a,c]$上关于y不增，在$[a,b]$上关于z不减，在$[b,c]$上关于z不增。

(A_2) 方程(*)在$[a,c]$上存在上解$\phi(t)$和下解$\psi(t)$，满足

$$\psi(t)\leqslant\phi(t),\quad t\in[a,c];\qquad \phi'(t)\leqslant\psi'(t),\quad t\in[a,b]$$

$$\psi'(t)\leqslant\phi'(t),\quad t\in[b,c];\qquad \psi''(t)\leqslant\phi''(t),\quad t\in[a,c]$$

(A_3) 函数$g(x,y,z,w)$在R^4上连续，对于固定的z，关于x，w不增，关于y不减，并且

$$g(\psi(a),\psi'(a),\psi''(a),\psi'''(a))\leqslant 0\leqslant g(\phi(a),\phi'(a),\phi''(a),\phi'''(a))$$

(A_4) 函数$h(x,y)$在R^2上连续，对于固定的x，关于y不减，并且

$$h(\phi(b),\phi''(b))\leqslant 0\leqslant h(\psi(b),\psi''(b))$$

(A_5) 函数$K(x,y,z,w)$在R^4上连续，对固定的z，关于x,y不增，关于w不减，并且

$$K(\psi(c),\psi'(c),\psi''(c),\psi'''(c))\leqslant 0\leqslant K(\phi(c),\phi'(c),\phi''(c),\phi'''(c))$$

引理2 假设条件(H_1)成立，若存在$M>0$，使得对任意的$(t,y,z,w,\eta)\in\Omega$，有$|f(t,y,z,w,\eta)|\leqslant M$，则边值问题

$$y^{(4)}=f(t,y,y',y'',y''') \tag{*}$$

$$y''(a)=a_2,\ y(b)=b_0,\ y'(b)=b_1,\ y''(c)=c_2 \tag{2.1}$$

有解。

证明 对边值问题

$$(\mathrm{I})\quad \begin{cases} y^{(4)}=f(t,y,y',y'',y''') \\ y''(a)=a_2,\ y(b)=b_0,\ y'(b)=b_1,\ y''(c)=c_2 \end{cases}$$

作变换$t=b+x$，设$y(t)=z(x)$，则$z(x)$满足

$$(\mathrm{II})\quad \begin{cases} z^{(4)}(x)=f(b+x,z(x),z'(x),z''(x),z'''(x)) \\ z''(\alpha)=a_2,z(0)=b_0,z'(0)=b_1,z''(r)=c_2 \end{cases} \tag{2.2}$$

其中，$\alpha=a-b$，$r=c-b$。显然边值问题(Ⅰ)与(Ⅱ)等价。

另一方面，若边值问题

$$(\text{Ⅲ})\begin{cases} z^{(4)}(x)=f(b+x,z(x)+p(x),z'(x)+p'(x),z''(x)+p''(x),\\ \qquad z'''(x)+p'''(x)) \\ z''(a)=0,z(0)=0,z'(0)=0,z''(r)=0 \end{cases}$$

其中$p(x)$是满足(2.2)的三次多项式．有解$w(x)$，则易知$z(x)=w(x)+p(x)$ 即为边值问题(Ⅱ)的解，从而边值问题(Ⅰ)有解．因此，证明边值问题(Ⅰ)解的存在性，只需证明边值问题(Ⅲ)的存在性．

事实上，容易求出边值问题

$$\begin{cases} z^{(4)}=0 \\ z''(a)=0,\ z(0)=0,\ z'(0)=0,\ z''(r)=0 \end{cases}$$

的Green函数为：

(i) 对于参数$s\in[a,0]$

$$G(x,s)=\begin{cases} \dfrac{s^2(s-3x)}{6}+\dfrac{x^2(3a-x)(r-s)}{6(r-a)}, & a\leqslant x\leqslant s\leqslant 0 \\ \dfrac{s^2(s-3x)}{6}+\dfrac{x^2(3a-x)(r-s)}{6(r-a)}+\dfrac{(x-s)^3}{6}, & a\leqslant s<x\leqslant 0 \end{cases}$$

(ii) 对于参数$s\in[0,r]$

$$G(x,s)=\begin{cases} \dfrac{x^2(3a-x)(r-s)}{6(r-a)}, & 0\leqslant x\leqslant s\leqslant r \\ \dfrac{x^2(3a-x)(r-s)}{6(r-a)}+\dfrac{(x-s)^3}{6}, & 0\leqslant s<x\leqslant r \end{cases}$$

鉴于当$x\leqslant 0$时

$$\begin{aligned} z(x)=&\int_a^0 G(x,s)f(s,z(s)+p(s),z'(s)+p'(s),z''(s)+p''(s),z'''(s)+p'''(s))ds \\ &+\int_0^r \frac{x^2(3a-x)(r-s)}{6(r-a)}f(s,z(s)+p(s),z'(s)+p'(s),z''(s) \\ &+p''(s),z'''(s)+p'''(s))ds \end{aligned}\tag{2.3}$$

当$x>0$时

$$\begin{aligned} z(x)=&\int_a^0\left[\frac{s^2(s-3x)}{6}+\frac{x^2(3a-x)(r-s)}{6(r-a)}+\frac{(x-s)^3}{6}\right]f(s,z(s) \\ &+p(s),z'(s)+p'(s),z''(s)+p''(s),z'''(s)+p'''(s))ds \\ &+\int_0^r G(x,s)f(s,z(s)+p(s),z'(s)+p'(s),z''(s) \\ &+p''(s),z'''(s)+p'''(s))ds \end{aligned}\tag{2.4}$$

下面应用 Schauder 不动点定理，便不难证明边值问题(Ⅲ)有解，从而，边值问题(Ⅰ)有解．

引理3 假设条件(H_1),(H_2),(A_1),(A_2)成立，并且有条件

$$\psi''(a)\leqslant a_2\leqslant\phi''(a),\psi(b)\leqslant b_0\leqslant\phi(b),\psi''(c)\leqslant c_2\leqslant\phi''(c),\psi'(b)=b_1=\phi'(b)$$

也成立，则边值问题

$$y^{(4)}=f(t,y,y',y'',y''') \tag{*}$$

$$y''(a)=a_2,\ y(b)=b_0,\ y'(b)=b_1,\ y''(c)=c_2 \tag{2.1}$$

有解$y(t)$，且满足：

$$\left.\begin{array}{llll}\psi(t)\leqslant y(t)\leqslant\phi(t), & t\in[a,c]; & \phi'(t)\leqslant y'(t)\leqslant\psi'(t), & t\in[a,b]\\ \psi'(t)\leqslant y'(t)\leqslant\phi'(t), & t\in[b,c]; & \psi''(t)\leqslant y''(t)\leqslant\phi''(t), & t\in[a,c]\end{array}\right\}\tag{2.5}$$

证明 对于$n=1,2,\cdots$, $(t,y,z,w,\eta)\in\Omega$, 定义函数

$$f_n(t,y,z,w,\eta)=\begin{cases}f(t,y,z,w,n), & \eta>n\\ f(t,y,z,w,\eta), & |\eta|\leqslant n\\ f(t,y,z,w,-n), & \eta<-n\end{cases}$$

$$G_n(t,y,z,w,\eta)=\begin{cases}f_n(t,y,z,\phi''(t),\eta)+\dfrac{w-\phi''(t)}{1+w-\phi''(t)}, & w>\phi''(t)\\ f_n(t,y,z,w,\eta), & \psi''(t)\leqslant w\leqslant\phi''(t)\\ f_n(t,y,z,\psi''(t),\eta)-\dfrac{\psi''(t)-w}{1+\psi''(t)-w}, & w<\psi''(t)\end{cases}$$

$$H_n(t,y,z,w,\eta)=\begin{cases}G_n(t,y,\phi'(t),w,\eta), & \begin{cases}t\in[a,b] & y'<\phi'(t)\\ t\in[b,c] & y'>\phi'(t)\end{cases}\\ G_n(t,y,\psi'(t),w,\eta), & \begin{cases}t\in[a,b] & y'>\psi'(t)\\ t\in[b,c] & y'<\psi'(t)\end{cases}\\ G_n(t,y,z,w,\eta), & \text{其它情况}\end{cases}$$

$$F_n(t,y,z,w,\eta)=\begin{cases}H_n(t,\phi(t),z,w,\eta), & y>\phi(t)\\ H_n(t,y,z,w,\eta), & \psi(t)\leqslant y\leqslant\phi(t)\\ H_n(t,\psi(t),z,w,\eta), & y<\psi(t)\end{cases}$$

易见，对任意$n\in N$, $F_n(t,y,z,w,\eta)$在$[a,c]\times R^4$上有界，据引理2知，边值问题

$$y^{(4)}=f(t,y,y',y''y,''')$$

$$y''(a)=a_2,\ y(b)=b_0,\ y'(b)=b_1,\ y''(c)=c_2$$

有解$y_n(t)(n=1,2,\cdots)$。

若记

$$N=\max\{\max_{[a,c]}|\phi'''(t)|,\max_{[a,c]}|\psi'''(t)|\}$$

则不难证明，当$n\geqslant N$时，

$$\psi''(b)\leqslant y_n''(b)\leqslant\phi''(b)\tag{2.6}$$

事实上，若(2.6)式不成立，不妨设其左端不等式不成立（至于右端不等式不成立的情形，证明类似），即存在$n=n_0\geqslant N$，使得$y_{n_0}''(b)<\psi''(b)$。因为在$t=a$，c两点反向不等式成立，故$y_{n_0}''(t)-\psi''(t)$在(a,c)内某点t_0处取到负的最小值，且满足若$t_0<b$,则$y_{n_0}'(t_0)>\psi'(t_0)$，或者，若$t_0>b$，则$y_{n_0}'(t_0)<\psi'(t_0)$。易知，在$y_{n_0}''(t)-\psi''(t)$的最小值点t_0处有

$$y_{n_0}''(t_0)<\psi''(t_0),\ y_{n_0}'''(t_0)=\psi'''(t_0),\ y_{n_0}^{(4)}(t_0)\geqslant\psi^{(4)}(t_0)\tag{2.7}$$

另一方面，由F_{n_0}的定义及f的单调性假设及(2.7)式可得

$$\begin{aligned}y_{n_0}^{(4)}(t_0)-\psi^{(4)}(t_0)&\leqslant F_{n_0}(t_0,y_{n_0},y_{n_0}',y_{n_0}'',y_{n_0}''')-f(t_0,\psi(t_0),\psi'(t_0),\psi''(t_0),\psi'''(t_0))\\&\leqslant f_{n_0}(t_0,\psi,\psi',\psi'',\psi''')-\frac{\psi'''(t_0)-y_{n_0}''}{1+\psi''(t_0)-y_{n_0}''}-f(t_0,\psi,\psi',\psi'',\psi''')\\&\leqslant-\frac{\psi''(t_0)-y_{n_0}''}{1+\psi''(t_0)-y_{n_0}''}<0\end{aligned}$$

这与(2.7)相矛盾，于是(2.6)式成立。

类似于上面的证明，由于对任意$n\in N$，有$\psi''(b)\leqslant y_n''(b)\leqslant\phi''(b)$，可证得

$$\psi''(t)\leqslant y_n''(t)\leqslant\phi''(t),\quad t\in[a,b] \text{ 及 } \psi''(t)\leqslant y_n''(t)\leqslant\phi''(t),\quad t\in[b,c] \tag{2.8}$$

由(2.8)式及假设易知有

$$\psi(t)\leqslant y_n(t)\leqslant\phi(t),\quad t\in[a,c];\quad \phi'(t)\leqslant y_n'(t)\leqslant\psi'(t),\quad t\in[a,b]$$

$$\psi'(t)\leqslant y_n'(t)\leqslant\phi'(t),\quad t\in[b,c]$$

从而，$y=y_n(t)$ $(n=1,2,\cdots)$是方程$y^{(4)}=f_n(t,y,y',y'',y''')$满足(2.1) 的解。根据引理1，便不难完成本引理的证明。

定理1 假设条件(H_1)，(H_2)，(A_1)，(A_2)，(A_3)成立，若有

$$\psi(b)\leqslant b_0\leqslant\phi(b),\quad \psi''(c)\leqslant c_2\leqslant\phi''(c),\quad \psi'(b)=b_1=\phi'(b)$$

成立，则边值问题

$$y^{(4)}=f(t,y,y',y'',y''') \tag{*}$$

$$g(y(a),y'(a),y''(a),y'''(a))=0,\quad y(b)=b_0,\quad y'(b)=b_1,\quad y''(c)=c_2 \tag{2.9}$$

有解$y(t)$，且满足(2.5)式。

证明 对于满足$\psi''(a)\leqslant s\leqslant\phi''(a)$的任意$s$，根据引理3知，边值问题

$$y^{(4)}=f(t,y,y',y'',y''')$$

$$y''(a)=s,\quad y(b)=b_0,\quad y'(b)=b_1,\quad y''(c)=c_2$$

有解$y_s(t)$，满足

$$\psi(t)\leqslant y_s(t)\leqslant\phi(t),\quad t\in[a,c];\quad \phi'(t)\leqslant y_s'(t)\leqslant\psi'(t),\quad t\in[a,b]$$

$$\psi'(t)\leqslant y_s'(t)\leqslant\phi'(t),\quad t\in[b,c];\quad \psi''(t)\leqslant y_s''(t)\leqslant\phi''(t),\quad t\in[a,c]$$

记$\pi(s)=\{y_s(t):\psi''(a)\leqslant s\leqslant\phi''(a)\}$，显然$\pi(s)\neq\phi$。

下面分两种情况讨论：

(i) $\psi''(a)=\phi''(a)$

这时，由$y_s''(a)=\psi''(a)$及 $\psi''(t)\leqslant y_s''(t)$知，$\psi'''(a)\leqslant y_s'''(a)$。再由$(A_3)$知

$$g(y_s(a),y_s'(a),y_s''(a),y_s'''(a))\leqslant g(\psi(a),\psi'(a),\psi''(a),\psi'''(a))\leqslant 0$$

另一方面，由$y_s''(a)=\phi''(a)$及$y_s''(t)\leqslant\phi''(t)$知，$y_s'''(a)\leqslant\phi'''(a)$。再由$(A_3)$知

$$g(y_s(a),y_s'(a),y_s''(a),y_s'''(a))\geqslant g(\phi(a),\phi'(a),\phi''(a),\phi'''(a))\geqslant 0$$

从而

$$g(y_s(a),y_s'(a),y_s''(a),y_s'''(a))=0$$

这说明，当$\psi''(a)=\phi''(a)$时，边值问题(*)，(2.9)有解$y(t)$，并满足(2.5)式。

(ii) $\psi''(a)<\phi''(a)$

用反证法。假设对于$\forall y_s(t)\in\pi(s)$，$y_s(t)$均不是边值问题(*)，(2.9)的解，这时，

$$g(y_s(a),y_s'(a),y_s''(a),y_s'''(a))\neq 0 \tag{2.10}$$

由情形(i)的讨论及(2.10)知：

(i) 若$y_s''(t)\in\pi(\psi''(a))$，则$g(y_s(a),y_s'(a),y_s''(a),y_s'''(a))<0$。

(ii) 若$y_s''(t)\in\pi(\phi''(a))$，则$g(y_s(a),y_s'(a),y_s''(a),y_s'''(a))>0$。

记$E=\{y_s(t):y_s(t)\in\pi(s)$ 且 $g(y_s(a),y_s'(a),y_s''(a),y_s'''(a))<0\}$。显然$E\neq\phi$。

记$s_0=\sup\{y_s''(a):y_s(t)\in E\}$。

由s_0的定义知，存在 $y_n(t)\in E$ $(n=1,2,\cdots)$，满足 $y_n''(a)=s_n\to s_0$ $(n\to\infty)$。根据引理1

知，边值问题

$$y^{(4)}=f(t,y,y',y'',y''')$$

$$y''(a)=s_0,\ y(b)=b_0,\ y'(b)=b_1,\ y''(c)=c_2$$

有解$y_0(t)$，并满足

$$\psi(t)\leqslant y_0(t)\leqslant\phi(t),\quad t\in[a,c];\ \phi'(t)\leqslant y_0'(t)\leqslant\psi'(t),\quad t\in[a,b]$$

$$\psi'(t)\leqslant y_0'(t)\leqslant\phi'(t),\quad t\in[b,c];\ \psi''(t)\leqslant y_0''(t)\leqslant\phi''(t),\quad t\in[a,c]$$

由 $g(y_n(a),y_n'(a),y_n''(a),y_n'''(a))<0$及(2.10)知，$g(y_0(a),y_0'(a),y_0''(a),y_0'''(a))<0$，即$y_0(t)\in E$，$s_0<\phi''(a)$。

如果用$y_0(t)$代替引理3中的下解$\psi(t)$，上解仍用$\phi(t)$表示，由引理3知，若 $y_0''(a)=s_0\leqslant s\leqslant\phi''(a)$，$y_0(b)\leqslant b_0\leqslant\phi(b)$，$b_1=y_0'(b)=\phi'(b)$，$y_0''(c)\leqslant c_2\leqslant\phi''(c)$，则边值问题

$$y^{(4)}=f(t,y,y',y'',y''')$$

$$y''(a)=s,\ y(b)=b_0,\ y'(b)=b_1,\ y''(c)=c_2$$

有解$\bar{y}_s(t)$，且满足

$$y_0(t)\leqslant\bar{y}_s(t)\leqslant\phi(t),\quad t\in[a,c];\ \phi'(t)\leqslant\bar{y}_s'(t)\leqslant y_0'(t),\quad t\in[a,b]$$

$$y_0'(t)\leqslant\bar{y}_s'(t)\leqslant\phi'(t),\quad t\in[b,c];\ y_0''(t)\leqslant\bar{y}_s''(t)\leqslant\phi''(t),\quad t\in[a,c]$$

记$\bar{\pi}(s)=\{\bar{y}_s(t):y_0''(a)\leqslant s\leqslant\phi''(a)\}$。显然$\bar{\pi}(s)\neq\phi$。

因为 $s_0<\phi''(a)$，则存在 $n\in N$，使 $s_0+1/n<\phi''(a)$。根据引理1知，函数列 $\{\bar{y}_n(t)\}\subset\bar{\pi}(s+1/n)$存在子序列$\{\bar{y}_{n_k}(t)\}$在$[a,c]$上一致收敛于边值问题

$$y^{(4)}=f(t,y,y',y'',y''')$$

$$y''(a)=s_0,\ y(b)=b_0,\ y'(b)=b_1,\ y''(c)=c_2$$

的解$\bar{y}_0(t)$，且满足

$$y_0(t)\leqslant\bar{y}_0(t)\leqslant\phi(t),\quad t\in[a,c];\ \phi'(t)\leqslant\bar{y}_0'(t)\leqslant y_0'(t),\quad t\in[a,b]$$

$$y_0'(t)\leqslant\bar{y}_0'(t)\leqslant\phi'(t),\quad t\in[b,c];\ y_0''(t)\leqslant\bar{y}_0''(t)\leqslant\phi''(t),\quad t\in[a,c]$$

由s_0的定义及(2.10)知，

$$g(\bar{y}_0(a),\bar{y}_0'(a),\bar{y}_0''(a),\bar{y}_0'''(a))>0 \tag{2.11}$$

另一方面，由$y_0''(a)=\bar{y}_0''(a)$及$y_0''(t)\leqslant\bar{y}_0''(t)$ 知，$y_0'''(a)\leqslant\bar{y}_0'''(a)$，再由$(A_3)$知，

$$g(\bar{y}_0(a),\bar{y}_0'(a),\bar{y}_0''(a),\bar{y}_0'''(a))\leqslant g(y_0(a),y_0'(a),y_0''(a),y_0'''(a))<0$$

这与(2.11)式相矛盾。因此，当$\psi''(a)<\phi''(a)$ 时，边值问题(*)，(2.9)有解 $y(t)$，且满足(2.5)式。

定理2　假设条件(H_1),(H_2),(A_1),(A_2),(A_3)，(A_4)成立，如果 $\phi'(b)=b_1=\psi'(b)$，$\psi''(c)\leqslant c_2\leqslant\phi''(c)$，则边值问题

$$y^{(4)}=f(t,y,y',y'',y''')$$

$$g(y(a),y'(a),y''(a),y'''(a))=0,\ h(y(b),y''(b))=0,\ y'(b)=b_1,\ y''(c)=c_2$$

有解$y(t)$，且满足(2.5)式。

证明　对任意的$\psi(b)\leqslant s\leqslant\phi(b)$，根据定理1知，边值问题

$$y^{(4)}=f(t,y,y',y'',y''')$$

$$g(y(a),y'(a),y''(a),y'''(a))=0,\ y(b)=s,\ y'(b)=b_1,\ y''(c)=c_2$$

有解$y_s(t)$，且满足

$$\psi(t)\leqslant y_s(t)\leqslant\phi(t),\quad t\in[a,c];\ \phi'(t)\leqslant y_s'(t)\leqslant\psi'(t),\quad t\in[a,b]$$

$$\psi'(t)\leqslant y_*'(t)\leqslant\phi'(t),\quad t\in[b,c];\ \psi''(t)\leqslant y_*''(t)\leqslant\phi''(t),\quad t\in[a,c]$$

于是分$\psi(b)=\phi(b)$及$\psi(b)<\phi(b)$这两种情况加以讨论，类似于定理1的证明，便不难完成定理的证明。

仿定理1的证明，不难得到如下的

定理3 假设条件(H_1)，(H_2)，(A_1)，(A_2)，(A_3)，(A_4)，(A_5)成立，如果$\phi'(b)=b_1=\psi'(b)$，则边值问题

$$y^{(4)}=f(t,y,y',y'',y''')$$
$$g(y(a),y'(a),y''(a),y'''(a))=0,\ h(y(b),y''(b))=0,\ y'(b)=b_1,$$
$$K(y(c),y'(c),y''(c),y'''(c))=0$$

有解$y(t)$，且满足(2.5)式。

推论1 假设条件(H_1),(H_2),(A_1),(A_2)成立，且

$$a_0\psi(a)+a_1\psi'(a)+a_2\psi''(a)+a_3\psi'''(a)\leqslant y_0\leqslant a_0\phi(a)+a_1\phi'(a)+a_2\phi''(a)+a_3\phi'''(a)$$
$$b_0\phi(b)+b_2\phi''(b)\leqslant y_1\leqslant b_0\psi(b)+b_2\psi''(b)$$
$$c_0\psi(c)+c_1\psi'(c)+c_2\psi''(c)+c_3\psi'''(c)\leqslant y_2\leqslant c_0\phi(c)+c_1\phi'(c)+c_2\phi''(c)+c_3\phi'''(c)$$

其中，$a_0,a_3,c_0,c_1\leqslant0$；$a_1,b_2,c_3\geqslant0$；$\sum_{i=0}^{3}|a_i|\neq0$，$|b_0|+|b_2|\neq0$，$\sum_{i=0}^{3}|c_i|\neq0$，则若$b_1=\phi'(b)=\psi'(b)$，方程(*)在边界条件

$$a_0y(a)+a_1y'(a)+a_2y''(a)+a_3y'''(a)=y_0,\quad b_0y(b)+b_2y''(b)=y_1$$
$$y'(b)=b_1,\quad c_0y(c)+c_1y'(c)+c_2y''(c)+c_3y'''(c)=y_2$$

下有解，且满足(2.5)式。

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Existence of Solutions of Three-Point Boundary Value Problems for Nonlinear Fourth Order Differential Equation

Gao Yongxin

(Department of Basic Science, Northeast Institute of Electric Power, Jilin 132012, *P. R. China)*

Abstract

In this paper, the author uses the methods in [1,2] to study the existence of solutions of three point boundary value problems for nonlinear fourth order differential equation

$$y^{(4)}=f(t,y,y',y'',y''') \tag{*}$$

with the boundary conditions

$$\left.\begin{array}{l} g(y(a),y'(a),y''(a),y'''(a))=0,\ h(y(b),y''(b))=0,\ y'(b)=b_1 \\ k(y(c),y'(c),y''(c),y'''(c))=0 \end{array}\right\} \tag{**}$$

For the boundary value problems of nonlinear fourth order differential equation

$$y^{(4)}=f(t,y,y',y',y''')$$

many results have been given at the present time. But the existence of solutions of boundary value problem (*), (**) studied in this paper has not been covered by the above researches. Moreover, the corollary of the important theorem in this paper, i. e. existence of solutions of the boundary value problem of equation(*) with the following boundary conditions

$$a_0y(a)+a_1y'(a)+a_2y''(a)+a_3y'''(a)=y_0,\ b_0y(b)+b_2y''(b)=y_1$$

$$y'(b)=y_2,\ c_0y(c)+c_1y'(c)+c_2y''(c)+c_3y'''(c)=y_3$$

has not been dealt with in previous works.

Key words nonlinear fourth order differential equation, three-point boundary value problems, existence of solutions