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空间实体与薄板组合结构的数学模型

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摘 要

本文利用变换将空间实体与薄板的组合结构转化为较易研究的等价问题，进而用摄动的思想研究了在体力作用下，依赖于板厚度的位移解簇的收敛性及其极限满足的变分方程。

关键词 组合结构 变换 连接条件 收敛

一、引 言

组合结构是指由梁、杆、薄板、薄壳之类元件组合而成的结构。在实际工程设计特别是航空航天等高科技领域中，组合结构是经常用到的。对组合结构进行分析研究的难点在于，对不同元件交接区域相互藕合的合理描述。八十年代初，我国冯康教授等开始研究并提出组合结构分析的数学理论；近些年来，法国学者 P. G. Ciarlet 领导的课题组开始较全面地研究这一类问题，他们在文章[1]研究了一类三维与二维组合弹性结构问题，其边界条件是板的一侧面固定。本文研究的问题的边界条件是三维体的一侧面固定，最后得到了与前者完全不同且应用更为广泛的数学模型。

二、空间原问题

设a_1，a_3，b_1，b_2和b_3为正实数，板厚为2ε，嵌入空间实体的纵向长度为β，且b_3和a_3远大于ε。

对于图1所示的构形有如下描述：

$\omega=\{(x_1,x_2)\in R^2\,|\,0<x_1<b_1,\ |x_2|<b_2\}$；

$\omega_\beta=\omega|_{0<x_1<\beta}$

$\Omega^\varepsilon=\omega\times(-\varepsilon,\varepsilon)$；$\Omega^\varepsilon_\beta=\omega_\beta\times(-\varepsilon,\varepsilon)$；

$O=\{(x_1,x_2,x_3)\in R^3\,|\,-a_1<x_1<\beta,$
$\quad |x_2|<b_2,\ -a_3<x_3<b_3\}$；

$O^\varepsilon_\beta=O-\Omega^\varepsilon_\beta$；$S^\varepsilon=O\cup\Omega^\varepsilon$；$\Gamma=O|_{x_1=-a_1}$

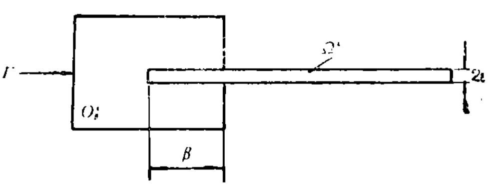

图1 组合结构S^ε的横断面（x_2=const.）

设空间实体与板的Lamé常数分别为$\tilde{\lambda}^\varepsilon$，$\tilde{\mu}^\varepsilon$和$\lambda^\varepsilon$，$\mu^\varepsilon$，作用于$S^\varepsilon$的体力密度为$f^\varepsilon$，未知位移场$u^\varepsilon$满足边界条件$u^\varepsilon(x^\varepsilon)=0$，$x^\varepsilon\in\Gamma$。

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由最小位能原理得问题$P(I)$：

求$u^\varepsilon \in V^\varepsilon$，使得

$$J^\varepsilon(u^\varepsilon)=\inf_{v^\varepsilon \in V^\varepsilon} J^\varepsilon(v^\varepsilon)$$

其中
$$J^\varepsilon(v^\varepsilon) \triangleq \frac{1}{2}\int_{O_\beta^\varepsilon} \tilde{A}^\varepsilon e(v^\varepsilon):e(v^\varepsilon)\,dx^\varepsilon + \frac{1}{2}\int_{\Omega^\varepsilon} A^\varepsilon e(v^\varepsilon):e(v^\varepsilon)\,dx^\varepsilon - \int_{S^\varepsilon} f^\varepsilon \cdot v^\varepsilon dx^\varepsilon$$

$$A^\varepsilon \triangleq \lambda^\varepsilon \delta_{ij}\delta_{kl}+\mu^\varepsilon(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$$

$$\tilde{A}^\varepsilon \triangleq \tilde{\lambda}^\varepsilon \delta_{ij}\delta_{kl}+\tilde{\mu}^\varepsilon(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$$

$$e_{ij}(v^\varepsilon) \triangleq \frac{1}{2}(\partial_i^\varepsilon v_j^\varepsilon+\partial_j^\varepsilon v_i^\varepsilon); \quad \partial_i^\varepsilon v_j^\varepsilon \triangleq \partial v_j^\varepsilon/\partial x_i^\varepsilon$$

$$V^\varepsilon \triangleq \{v^\varepsilon \in (H^1(S^\varepsilon))^3 \mid v^\varepsilon(x^\varepsilon)=0,\ x^\varepsilon \in \Gamma\}$$

用应力和位移表示的连接条件分别为：

$$\tilde{A}^\varepsilon e(u^\varepsilon)\tilde{n}^\varepsilon + A^\varepsilon e(u^\varepsilon) n^\varepsilon = 0, \quad x^\varepsilon \in \partial O_\beta^\varepsilon \cap \partial\Omega^\varepsilon$$

$$(u^\varepsilon \mid O_\beta^\varepsilon)\mid_{\partial O_\beta^\varepsilon \cap \partial\Omega^\varepsilon} = (u^\varepsilon \mid \Omega^\varepsilon)\mid_{\partial O_\beta^\varepsilon \cap \partial\Omega^\varepsilon}$$

上式反映了薄板与三维体是粘接的，不能有相对滑动。

三、变 换 问 题

为得到较易研究而又和问题$P(I)$等价的新问题，首先利用交换将O和Ω^ε的公共部分Ω分离，即做变换

$$\Pi:\ \Omega^\varepsilon \to \Omega \triangleq \omega \times (-1,1)$$
$$[x_\alpha = x_\alpha^\varepsilon,\ x_3 = x_3^\varepsilon/\varepsilon]$$
$$O \to \tilde{\Omega} \triangleq O+T$$

其中$O+T$表示将O中的点经过T平移之后所得的集合，即

$$\{\tilde{x}\in R^3 \mid \tilde{x}_1 = x_1^\varepsilon + t_1,\ \tilde{x}_2 = x_2^\varepsilon,\ \tilde{x}_3 = x_3^\varepsilon,\ x^\varepsilon \in O\}$$

t_1的选取应使得$\{\tilde{\Omega}\}^- \cap \bar{\Omega} = \phi$.

为了叙述方便，约定希腊字母和拉丁字母做角标时，分别取集自合$\{1,2\}$和$\{1,2,3\}$，点或集合记号上方加符号～表示是经过T平移之后所得的。

其次，设空间实体和板的Lamé常数满足

$$\tilde{\lambda}^\varepsilon = \varepsilon^{-2}\tilde{\lambda},\quad \tilde{\mu}^\varepsilon = \varepsilon^{-2}\tilde{\mu},\quad \lambda^\varepsilon = \varepsilon^{-3}\lambda,\quad \mu^\varepsilon = \varepsilon^{-3}\mu$$

$\tilde{\lambda}$，$\tilde{\mu}$和λ，μ是独立于常数ε的。

最后设体力及与其相应的位移场为下列形式

$$f_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^{-1} f_\alpha(x),\quad f_3^\varepsilon(x^\varepsilon) = f_3(x) \qquad (x^\varepsilon \in \Omega^\varepsilon)$$

$$f^\varepsilon(x^\varepsilon) = \tilde{f}(\tilde{x}) \qquad (x^\varepsilon \in O_\beta^\varepsilon)$$

$$u_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 u_\alpha(\varepsilon)(x),\quad u_3^\varepsilon(x^\varepsilon) = \varepsilon u_3(\varepsilon)(x) \qquad (x^\varepsilon \in \bar{\Omega}^\varepsilon)$$

$$u^\varepsilon(x^\varepsilon) = \varepsilon^2 \tilde{u}(\varepsilon)(\tilde{x}) \qquad (x^\varepsilon \in \bar{O})$$

进而得到变换以后在上述条件下的边界条件

$$\tilde{u}(\varepsilon)(\tilde{x}) = 0 \qquad (\tilde{x} \in \tilde{\Gamma} \triangleq \Gamma + T)$$

和位移连接条件

$$u_\alpha(\varepsilon)(x)=\tilde{u}_\alpha(\varepsilon)(\tilde{x}),\quad u_3(\varepsilon)(x)=\varepsilon\tilde{u}_3(\varepsilon)(\tilde{x})$$

$\tilde{x}\in\tilde{\Omega}_\beta^\varepsilon \triangleq \Omega_\beta^\varepsilon+T$，$x\in\Omega_\beta\triangleq\omega_\beta\times(-1,1)$ 且 $\Pi^{-1}(\tilde{x})=\Pi^{-1}(x)$

经过变换以后的问题P(Ⅱ)为

求$(\tilde{u}(\varepsilon),u(\varepsilon))\in V(\varepsilon)$，使得

$$J(\tilde{u}(\varepsilon),u(\varepsilon))=\inf_{(\tilde{v},v)\in V(\varepsilon)} J(\tilde{v},v)$$

其中 $$J(\tilde{v},v)=\frac{1}{2}\int_{\tilde{\Omega}_\beta^\varepsilon}\tilde{A}e(\tilde{v}):e(\tilde{v})d\tilde{x}-\int_{\tilde{\Omega}_\beta^\varepsilon}\tilde{f}\cdot\tilde{v}d\tilde{x}$$

$$+\frac{1}{2}\int_\Omega\{\lambda e_{\alpha\alpha}(v)e_{\beta\beta}(v)+2\mu e_{\alpha\beta}(v)e_{\alpha\beta}(v)\}dx$$

$$+\frac{1}{\varepsilon^2}\int_\Omega\{\lambda e_{\alpha\alpha}(v)e_{33}(v)+2\mu e_{\alpha3}(v)e_{\alpha3}(v)\}dx$$

$$+\frac{1}{\varepsilon^4}\int_\Omega\frac{\lambda+2\mu}{2}e_{33}(v)e_{33}(v)dx-\int_\Omega f\cdot vdx$$

$$V(\varepsilon)\triangleq\{(\tilde{v},v)\in(H^1(\tilde{\Omega}))^3\times(H^1(\Omega))^3\,|\,\tilde{v}=0,\ \tilde{x}\in\tilde{\Gamma};$$
$$v_\alpha=\tilde{v}_\alpha,\ v_3=\varepsilon\tilde{v}_3,\ \tilde{x}\in\tilde{\Omega}_\beta^\varepsilon,\ x\in\Omega_\beta \text{且} \Pi^{-1}(\tilde{x})=\Pi^{-1}(x)\}$$

由问题P(Ⅰ)的解存在唯一可推得问题P(Ⅱ)的解也存在且唯一。

四、函数偶簇 $(\tilde{u}(\varepsilon),u(\varepsilon))_{\varepsilon>0}$的收敛性

1. $(\tilde{u}(\varepsilon),u(\varepsilon))_{\varepsilon>0}$的弱收敛性

利用空间$V(\varepsilon)$上的范数

$$|(\tilde{v},v)|\triangleq\{|e(\tilde{v})|_{0,\tilde{\Omega}}^2+|e(v)|_{0,\Omega}^2\}^{\frac{1}{2}}$$

和 $$\|(\tilde{v},v)\|\triangleq\{\|\tilde{v}\|_{1,\tilde{\Omega}}^2+\|v\|_{1,\Omega}^2\}^{\frac{1}{2}}$$

之间的关系：存在不依赖于ε的常数C，使得

$$\|(\tilde{v},v)\|\leqslant C|(\tilde{v},v)|$$

可得到以下结论。

定理1 （有界性）函数偶簇$(\tilde{u}(\varepsilon),u(\varepsilon))_{\varepsilon>0}$在空间$(H^1(\tilde{\Omega}))^3\times(H^1(\Omega))^3$中关于$\varepsilon$是一致有界的。

推论1 （弱收敛性）存在着函数偶簇 $(\tilde{u}(\varepsilon),u(\varepsilon))_{\varepsilon>0}$ 的一个子序列，仍记为 $(\tilde{u}(\varepsilon),u(\varepsilon))_{\varepsilon>0}$，和一个函数偶$(\tilde{u},u)\in(H^1(\tilde{\Omega}))^3\times(H^1(\Omega))^3$，使得

$$\tilde{u}(\varepsilon)\to\tilde{u}(\varepsilon\to0)\text{于}(H^1(\tilde{\Omega}))^3$$
$$\tilde{u}=0\qquad(\tilde{x}\in\tilde{\Gamma})$$
$$u(\varepsilon)\to u(\varepsilon\to0)\text{于}(H^1(\Omega))^3$$

推论2 函数u在下列空间中

$$V_{K,L}(\Omega)\triangleq\{v\in(H^1(\Omega))^3\,|\,e_{i3}=0,\ x\in\Omega\}$$
$$=\{v\in(H^1(\Omega))^3\,|\,v_\alpha=\eta_\alpha-x_3\partial_\alpha\eta_3,$$
$$v_3=\eta_3,\eta_\alpha\in H^1(\omega),\ \eta_3\in H^2(\omega)\}$$

推论3 弱极限$(\tilde{u},u)$满足连接条件

$$u_3|_{\omega_\beta}=0 \text{ 和 } u_\alpha|_{\omega_\beta}=\tilde{u}_\alpha|_{\tilde{\omega}_\beta}$$

综合上述推论，弱极限$(\tilde{u},u)$是下列空间的元素：

$$[H^1(\tilde{\Omega})\times V_{K,L}(\Omega)]_\beta \triangleq \{(\tilde{v},v)\in(H^1(\tilde{\Omega}))^3\times V_{K,L}(\Omega)|$$
$$\tilde{v}=0,\ \tilde{x}\in\tilde{\Gamma};\ v_\alpha|_{\omega_\beta}=\tilde{v}_\alpha|_{\tilde{\omega}_\beta},\ v_3|_{\omega_\beta}=0\}$$

2. 弱极限$(\tilde{u},u)$满足的变分方程

定理2 设$(\tilde{v},v)\in[H^1(\tilde{\Omega})\times V_{K,L}(\Omega)]_\beta$，它满足$\mathrm{Supp}(\tilde{v})\subset\{\tilde{x}\in\tilde{\Omega}|\tilde{x}_1\leqslant\tilde{O}\}$且$v=0$，或者$\tilde{v}|_{\tilde{\omega}}\in H^1(\tilde{\omega})$，那么存在着函数偶簇$(\tilde{v}(\varepsilon),v(\varepsilon))_{\varepsilon>0}$使得对任意的$\varepsilon>0$，有

$$(\tilde{v}(\varepsilon),v(\varepsilon))\in V(\varepsilon);\ v(\varepsilon)\in V_{K,L}(\Omega)$$

和

$$\|v(\varepsilon)-v\|_{1,\Omega}\to 0\quad(\varepsilon\to 0)$$
$$\|\tilde{v}(\varepsilon)-\tilde{v}\|_{1,\tilde{\Omega}}\to 0\quad(\varepsilon\to 0)$$

推论4 设 $(\tilde{v},v)\in[H^1(\tilde{\Omega})\times V_{K,L}(\Omega)]_\beta$，满足定理2的条件，那么弱极限$(\tilde{u},u)\in[H^1(\tilde{\Omega})\times V_{K,L}(\Omega)]_\beta$满足变分方程：

$$\int_{\tilde{\Omega}}\tilde{A}e(\tilde{u}):e(\tilde{v})d\tilde{x}-\int_{\tilde{\Omega}}\tilde{f}\cdot\tilde{v}d\tilde{x}$$
$$+\int_\omega\frac{4}{3}\left\{\mu\partial_{\alpha\beta}\xi_3+\frac{\lambda\mu}{\lambda+2\mu}\Delta\xi_3\delta_{\alpha\beta}\right\}\partial_{\alpha\beta}\eta_3 d\omega$$
$$-\int_\omega f_3^0\eta_3 d\omega+\int_\omega f_\alpha^1\partial_\alpha\eta_3 d_\omega$$
$$+\int_\omega 4\left\{\mu\frac{\partial_\alpha\xi_\beta+\partial_\beta\xi_\alpha}{2}+\frac{\lambda\mu}{\lambda+2\mu}\partial_\tau\xi_\tau\delta_{\alpha\beta}\right\}\partial_\alpha\eta_\beta d\omega$$
$$-\int_\omega f_\alpha^0\eta_\alpha d\omega=0$$

其中

$$v_\alpha=\eta_\alpha-x_3\partial_\alpha\eta_3,\ v_3=\eta_3$$
$$u_\alpha=\xi_\alpha-x_3\partial_\alpha\xi_3,\ u_3=\xi_3$$
$$\eta_\alpha,\xi_\alpha\in H^1(\omega),\ \eta_3,\xi_3\in H_0^2(\omega)$$
$$f^0\triangleq\int_{-1}^1 f dx_3,\ f^1\triangleq\int_{-1}^1 x_3 f dx_3$$
$$H_0^2(\omega)\triangleq\{v\in H^2(\omega)|v=0,\ x\in\omega_\beta\}$$

为分离上述变分方程，我们引用文献[1]中的结论为定理3。

定理3 对任意一个向量函数$\tilde{v}\in(H^1(\tilde{\Omega}))^3$，它满足$\tilde{v}|_{\tilde{\omega}_\beta}=0$，那么存在着向量函数序列$\tilde{\nu}^n$和$\tilde{S}^n(n\geqslant1)$，满足

$$\tilde{\nu}^n\in(H^1(\tilde{\Omega}))^3,\ \tilde{\nu}^n|_{\tilde{\omega}}\in[H^1(\tilde{\omega})]^3$$
$$\tilde{S}^n\in(H^1(\tilde{\Omega}))^3,\ \mathrm{supp}\tilde{S}^n\subset\{\tilde{x}\in\tilde{\Omega}|\tilde{x}_1\leqslant 0\}$$
$$\tilde{\nu}^n+\tilde{S}^n\to\tilde{v},\ (n\to\infty)\text{于}(H^1(\tilde{\Omega}))^3$$

证明参阅[1]和[3]。将推论4中的测试函数$(\tilde{v},v)$取为特殊的函数偶，便可得出下面两个结论。

推论5 函数偶$(\tilde{u},(\xi_1,\xi_2))$是下列空间的元素

$$[H^1(\tilde{\Omega})\times(H^1(\omega))^2]_\beta\triangleq\{(\tilde{v},(\eta_1,\eta_2))\in(H^1(\tilde{\Omega}))^3\times(H^1(\omega))^2|$$
$$\tilde{v}=0,\ \tilde{x}\in\tilde{\Gamma};\ \tilde{v}_\alpha|_{\tilde{\omega}_\beta}=\eta_\alpha|_{\omega_\beta}\}$$

并满足方程

$$\int_{\tilde{\Omega}} \tilde{A} e(\tilde{u}) : e(\tilde{v}) d\tilde{x} - \int_{\tilde{\Omega}} \tilde{f} \cdot \tilde{v} d\tilde{x}$$

$$+ \int_{\omega} 4\mu \left\{ \frac{\partial_\alpha \zeta_\beta + \partial_\beta \zeta_\alpha}{2} + \frac{\lambda}{\lambda + 2\mu} \partial_\tau \xi_\tau \delta_{\alpha\beta} \right\} \partial_\alpha \eta_\beta d\omega$$

$$- \int_{\omega} f_\alpha^0 \eta_\alpha d\omega = 0 \quad (\forall (\tilde{v}, (\eta_1, \eta_2)) \in [H^1(\tilde{\Omega}) \times (H^1(\omega))^2]_\beta)$$

此变分方程是强制的，故有唯一解。

推论6 函数$\xi_3 \in H_0^2(\omega)$是变分方程

$$\int_{\omega} \frac{4}{3} \left\{ \mu \partial_{\alpha\beta} \xi_3 + \frac{\lambda\mu}{\lambda + 2\mu} \Delta \xi_3 \delta_{\alpha\beta} \right\} \partial_{\alpha\beta} \eta_3 d\omega$$

$$- \int_{\omega} f_3^0 \eta_3 d\omega + \int_{\omega} f_\alpha^1 \partial_\alpha \eta_3 d\omega = 0 \quad (\forall \eta_3 \in H_0^2(\omega))$$

的唯一解。

推论6说明ξ_3是一弯曲板其Lamé常数分别为$\tilde{\lambda} \triangleq \frac{4}{3} \frac{\lambda\mu}{\lambda + 2\mu}$和$\tilde{\mu} \triangleq \frac{2}{3}\mu$时，在横向载荷$f_3^0$和力矩$f_\alpha^1$作用下的横向位移。

利用弱极限$(\tilde{u}, u)$满足的变分方程，我们可以进一步得到以下定理。

定理4 函数偶簇$(\tilde{u}(\varepsilon), u(\varepsilon))_{\varepsilon > 0}$在空间$(H^1(\tilde{\Omega}))^3 \times (H^1(\Omega))^3$中强收敛到$(\tilde{u}, u)$。

五、结　　论

至此，我们研究了变换后变分方程解簇的收敛性及其极限满足的变分方程。这一模型适用于空间实体与板的杨氏模量之比是$O(\varepsilon)$的情形。能否将上述条件放宽以及有关此模型的数值方面的应用还需进一步研究。

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Mathematical Model of a Junction between Linear Elastomer and Thin Plate

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Abstract

In this paper, a kind of junction problem between linear elastomer and thin plate is changed, through a well scaling, into an equivalent problem which is studied easily. Then, using an idea of disturbance, we have studied the convergence of the displacement vector field under the action of body force, which is depending on the thickness of the plate. At last, the variational equations of the limit are obtained.

Key words junction, scaling, junction condition, convergence