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薄壳稳定的变分原理

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摘　要

本文按照易曲物体的形变理论来确定薄壳的内力和内矩，应变能以及外力的功。从而根据虚位移原理求得临载荷的能量准则，并导出稳定问题的平衡方程和边界条件。

关键词　壳体　稳定　临界载荷

一、薄壳的变形和应变能

按照易曲物体的形变理论[1]，薄壳任一点在坐标线α_1，α_2和中面法线方向的位移$\hat{u}$，$\hat{v}$，$\hat{w}$可表为

$$\hat{u}=u+z\vartheta,\quad \hat{v}=v+z\psi,\quad \hat{w}=w+z\chi \tag{1.1}$$

式中u，v，w为壳体中面的位移。由直法线假设可得

$$\left.\begin{aligned}\vartheta&=-e_{13}(1+e_{22})+e_{12}e_{23}\\ \psi&=-e_{23}(1+e_{11})+e_{21}e_{13}\\ \chi&=-(e_{13}^2+e_{23}^2)/2\end{aligned}\right\} \tag{1.2}$$

式中

$$\left.\begin{aligned}e_{11}&=\frac{1}{A_1}\frac{\partial u}{\partial\alpha_1}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}v+\frac{w}{R_1},\quad & e_{22}&=\frac{1}{A_2}\frac{\partial v}{\partial\alpha_2}+\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}u+\frac{w}{R_2}\\ e_{12}&=\frac{1}{A_1}\frac{\partial v}{\partial\alpha_1}-\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}u,\quad & e_{21}&=\frac{1}{A_2}\frac{\partial u}{\partial\alpha_2}-\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}v\\ e_{13}&=\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1}-\frac{u}{R_1},\quad & e_{23}&=\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2}-\frac{v}{R_2}\end{aligned}\right\} \tag{1.3}$$

壳体任一点的单位伸长和切应变为

$$\left.\begin{aligned}\varepsilon_{11}&=\frac{e_{11}+zk_{11}}{1+z/R_1}+\frac{(e_{11}+zk_{11})^2+(e_{12}+zk_{12})^2+(e_{13}+zk_{13})^2}{2(1+z/R_1)^2}\\ \varepsilon_{22}&=\frac{e_{22}+zk_{22}}{1+z/R_2}+\frac{(e_{21}+zk_{21})^2+(e_{22}+zk_{22})^2+(e_{23}+zk_{23})^2}{2(1+z/R_2)^2}\\ \varepsilon_{12}&=\frac{e_{12}+zk_{12}}{1+z/R_1}+\frac{e_{21}+zk_{21}}{1+z/R_2}+\frac{e_{11}+zk_{11}}{1+z/R_1}\frac{e_{21}+zk_{21}}{1+z/R_2}\\ &\quad+\frac{e_{12}+zk_{12}}{1+z/R_1}\frac{e_{22}+zk_{22}}{1+z/R_2}+\frac{e_{13}+zk_{13}}{1+z/R_1}\frac{e_{23}+zk_{23}}{1+z/R_2}\end{aligned}\right\} \tag{1.4}$$

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式中

$$\left.\begin{aligned}&k_{11}=\frac{1}{A_1}\frac{\partial\vartheta}{\partial\alpha_1}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\psi+\frac{\chi}{R_1},\quad k_{22}=\frac{1}{A_2}\frac{\partial\psi}{\partial\alpha_2}+\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\vartheta+\frac{\chi}{R_2}\\&k_{12}=\frac{1}{A_1}\frac{\partial\psi}{\partial\alpha_1}-\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\vartheta,\qquad k_{21}=\frac{1}{A_2}\frac{\partial\vartheta}{\partial\alpha_2}-\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\psi\\&k_{13}=\frac{1}{A_1}\frac{\partial\chi}{\partial\alpha_1}-\frac{\vartheta}{R_1},\qquad k_{23}=\frac{1}{A_2}\frac{\partial\chi}{\partial\alpha_2}-\frac{\psi}{R_2}\end{aligned}\right\}\tag{1.5}$$

按照均匀和各向同性体的虎克定律，当忽略壳体中面法应力σ_{33}时，壳体的内力T_1,T_2,T_{12}, T_{21}和内矩M_1, M_2, M_{12}, M_{21}以及应变能A为[2]

$$\left.\begin{aligned}&T_1=\int\sigma_{11}\left(1+\frac{z}{R_2}\right)dz,\quad M_1=\int\sigma_{11}\left(1+\frac{z}{R_2}\right)zdz\\&T_2=\int\sigma_{22}\left(1+\frac{z}{R_1}\right)dz,\quad M_2=\int\sigma_{22}\left(1+\frac{z}{R_1}\right)zdz\\&T_{12}=\int\sigma_{12}\left(1+\frac{z}{R_2}\right)dz,\quad M_{12}=\int\sigma_{12}\left(1+\frac{z}{R_2}\right)zdz\\&T_{21}=\int\sigma_{12}\left(1+\frac{z}{R_1}\right)dz,\quad M_{21}=\int\sigma_{12}\left(1+\frac{z}{R_1}\right)zdz\end{aligned}\right\}\tag{1.6}$$

$$A=\frac{1}{2}\iiint(\sigma_{11}\varepsilon_{11}+\sigma_{22}\varepsilon_{22}+\sigma_{12}\varepsilon_{12})A_1A_2\left(1+\frac{z}{R_1}\right)\left(1+\frac{z}{R_2}\right)d\alpha_1d\alpha_2dz\tag{1.7}$$

式中

$$\sigma_{11}=\frac{E}{1-\mu^2}(\varepsilon_{11}+\mu\varepsilon_{22}),\quad\sigma_{22}=\frac{E}{1-\mu^2}(\varepsilon_{22}+\mu\varepsilon_{11}),\quad\sigma_{12}=\frac{E}{2(1+\mu)}\varepsilon_{12}\tag{1.8}$$

E为弹性模数，μ为泊桑比.

二、虚位移原理

将虚位移原理应用到变形物体，设u, v, w为载荷引起的真实位移，δu, δv, δw为由承受载荷的平衡位置算起的虚位移. 应变能的增量δA等于外力在虚位移上所作的功.

$$\delta A=\delta R_1+\delta R_2\tag{2.1}$$

式中δR_1为载荷的功，δR_2为边界力所作的功. 作用在壳体上的载荷有体积力和表面力，可近似地认为作用在壳体中面上的分布载荷. 设q_1, q_2, q_n为这些载荷沿α_1, α_2和法线n方向的分量，载荷的虚功为

$$\delta R_1=\iint(q_1^*\delta u+q_2^*\delta v+q_n^*\delta w)A_1'A_2'd\alpha_1d\alpha_2\tag{2.2}$$

式中

$$\left.\begin{aligned}&q_1^*=q_1(1+e_{11})+q_2e_{21}-q_ne_{13},\quad q_2^*=q_2(1+e_{22})+q_1e_{12}-q_ne_{23}\\&q_n^*=q_n+q_1e_{13}+q_2e_{23},\quad A_1'A_2'=A_1A_2(1+e_{11})(1+e_{22})\end{aligned}\right\}\tag{2.3}$$

q_1^*, q_2^*, q_n^*为壳体上分布载荷的分量. 通常体积力，如重力和惯性力，其大小和方向是不变的. 表面力，如液体或气体的压力，其方向是变的. 欲使积分不是对已变形位置进行，可将(2.3)式代入(2.2)式后再积分.

如图1所示，设 $\boldsymbol{T}_1$，$\boldsymbol{T}_{12}$，$\bar{\boldsymbol{V}}_1$，$\bar{\boldsymbol{M}}_1$ 为边界力沿壳体中面轴线 α_2 的分量，$\boldsymbol{T}_2$，$\boldsymbol{T}_{21}$，$\bar{\boldsymbol{V}}_2$，$\bar{\boldsymbol{M}}_2$为边界力沿壳体中面轴线α_1的分量．$\bar{P}$为角点力．通常这些力是不变的．这些力的虚功是

$$\delta R_2=\int(\boldsymbol{T}_1\delta u+\boldsymbol{T}_{12}\delta v+\bar{V}_1\delta w+\bar{M}_1\delta\vartheta)A_2d\alpha_2+\int(\boldsymbol{T}_{21}\delta u+\boldsymbol{T}_2\delta v+\bar{V}_2\delta w+\bar{M}_2\delta\psi)A_1d\alpha_1+\bar{P}\delta w \tag{2.4}$$

若边界位移是给定的，则边界位移的变分等于零．

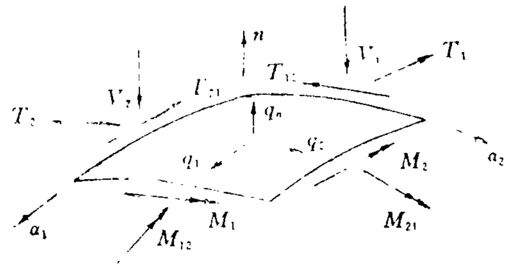

图 1

三、确定临界载荷的能量准则

当壳体的载荷等于临界载荷，则壳体将同时有两个无限接近的平衡位置．取第一个平衡位置处于线弹性应力状态．此时壳体中面任一点的位移分量为 u_0，v_0，w_0，而另一平衡位置的为

$$u=u_0+\alpha u_1,\quad v=v_0+\alpha v_1,\quad w=w_0+\alpha w_1 \tag{3.1}$$

式中αu_1，αv_1，αw_1为从初始平衡位置移到新平衡位置的附加位移，而且假设u_1，v_1，w_1为有限量，α是一个不依赖于α_1和α_2的无限小量．

对于薄壳的非线性变形，通常可略去中面曲率变化和扭率的非线性项，而仅保留中面单位伸长和切应变的非线性项．相应地将(1.2)式和(1.4)式改变为[3]

$$\vartheta=-e_{13},\quad \psi=-e_{23},\quad \chi=0 \tag{3.2}$$

$$\left.\begin{aligned}
\varepsilon_{11}&=\frac{e_{11}+zk_{11}}{1+z/R_1}+\frac{1}{2}(e_{11}^2+e_{12}^2+e_{13}^2)\\
\varepsilon_{22}&=\frac{e_{22}+zk_{22}}{1+z/R_2}+\frac{1}{2}(e_{21}^2+e_{22}^2+e_{23}^2)\\
\varepsilon_{12}&=\frac{e_{12}+zk_{12}}{1+z/R_1}+\frac{e_{21}+zk_{21}}{1+z/R_2}+e_{11}e_{21}+e_{12}e_{22}+e_{13}e_{23}
\end{aligned}\right\} \tag{3.3}$$

将(3.1)式代入(1.3)式可得

$$\left.\begin{aligned}
&e_{11}=e_{11}^{\circ}+\alpha e_{11}',\quad e_{12}=e_{12}^{\circ}+\alpha e_{12}',\quad e_{13}=e_{13}^{\circ}+\alpha e_{13}'\\
&e_{21}=e_{21}^{\circ}+\alpha e_{21}',\quad e_{22}=e_{22}^{\circ}+\alpha e_{22}',\quad e_{23}=e_{23}^{\circ}+\alpha e_{23}'
\end{aligned}\right\} \tag{3.4}$$

式中

$$\left.\begin{aligned}
&e_{11}'=\frac{1}{A_1}\frac{\partial u_1}{\partial\alpha_1}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}v_1+\frac{w_1}{R_1},\quad e_{22}'=\frac{1}{A_2}\frac{\partial v_1}{\partial\alpha_2}+\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}u_1+\frac{w_1}{R_2}\\
&e_{12}'=\frac{1}{A_1}\frac{\partial v_1}{\partial\alpha_1}-\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}u_1,\qquad e_{21}'=\frac{1}{A_2}\frac{\partial u_1}{\partial\alpha_2}-\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}v_1\\
&e_{13}'=\frac{1}{A_1}\frac{\partial w_1}{\partial\alpha_1}-\frac{u_1}{R_1},\qquad e_{23}'=\frac{1}{A_2}\frac{\partial w_1}{\partial\alpha_2}-\frac{v_1}{R_2}
\end{aligned}\right\} \tag{3.5}$$

e_{11}°，…分别与上式e_{11}'，…相似，但上式右边各项的下标“1”改为“0”．将(3.4)式代入(3.2)

式然后代入(1.5)式可得

$$k_{11}=k^{\circ}_{11}+\alpha k'_{11},\quad k_{22}=k^{\circ}_{22}+\alpha k'_{22},\quad k_{12}=k^{\circ}_{12}+\alpha k'_{12},\quad k_{21}=k^{\circ}_{21}+\alpha k'_{21} \tag{3.6}$$

式中

$$\left.\begin{aligned} k'_{11}&=-\frac{1}{A_1}\frac{\partial e'_{13}}{\partial\alpha_1}-\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}e'_{23},\quad k'_{22}=-\frac{1}{A_2}\frac{\partial e'_{23}}{\partial\alpha_2}-\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}e'_{13}\\ k'_{12}&=-\frac{1}{A_1}\frac{\partial e'_{23}}{\partial\alpha_1}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}e'_{13},\quad k'_{21}=-\frac{1}{A_2}\frac{\partial e'_{13}}{\partial\alpha_2}+\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}e'_{23}\end{aligned}\right\} \tag{3.7}$$

k°_{11}，…和上式相似，仅将上标“′”改为“°”。

将(3.4)式和(3.6)式代入(3.3)式并略去第一平衡位置线弹性应力状态的非线性项可得

$$\left.\begin{aligned}\varepsilon_{11}&=\frac{e^{\circ}_{11}+zk^{\circ}_{11}}{1+z/R_1}+\alpha\frac{e'_{11}+zk'_{11}}{1+z/R_1}+\frac{\alpha^2}{2}(e'^2_{11}+e'^2_{12}+e'^2_{13})\\ \varepsilon_{22}&=\frac{e^{\circ}_{22}+zk^{\circ}_{22}}{1+z/R_2}+\alpha\frac{e'_{22}+zk'_{22}}{1+z/R_2}+\frac{\alpha^2}{2}(e'^2_{21}+e'^2_{22}+e'^2_{23})\\ \varepsilon_{12}&=\frac{e^{\circ}_{12}+zk^{\circ}_{12}}{1+z/R_1}+\frac{e^{\circ}_{21}+zk^{\circ}_{21}}{1+z/R_2}+\alpha\left(\frac{e'_{12}+zk'_{12}}{1+z/R_1}+\frac{e'_{21}+zk'_{21}}{1+z/R_2}\right)\\ &\quad+\alpha^2(e'_{11}e'_{21}+e'_{12}e'_{22}+e'_{13}e'_{23})\end{aligned}\right\} \tag{3.8}$$

将上式展成z的幂级数代入(1.8)式然后代入(1.6)式和(1.7)式．积分后仅保留z和α的二次幂可得

$$\left.\begin{aligned} &T_1=T^{\circ}_1+\alpha T'_1+\alpha^2T''_1,\quad M_1=M^{\circ}_1+\alpha M'_1\\ &T_2=T^{\circ}_2+\alpha T'_2+\alpha^2T''_2,\quad M_2=M^{\circ}_2+\alpha M'_2\\ &T_{12}=T^{\circ}_{12}+\alpha T'_{12}+\alpha^2T''_{12},\quad M_{12}=M^{\circ}_{12}+\alpha M'_{12}\\ &T_{21}=T^{\circ}_{21}+\alpha T'_{21}+\alpha^2T''_{21},\quad M_{21}=M^{\circ}_{21}+\alpha M'_{21}\end{aligned}\right\} \tag{3.9}$$

$$A=A^{\circ}+\alpha A^{\circ\prime}+\alpha^2(A'+A'') \tag{3.10}$$

式中

$$\left.\begin{aligned} T'_1&=\frac{E\delta}{1-\mu^2}\left[e'_{11}+\mu e'_{22}-\frac{\delta^2}{12}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\left(k'_{11}-\frac{e'_{11}}{R_1}\right)\right]\\ T'_2&=\frac{E\delta}{1-\mu^2}\left[e'_{22}+\mu e'_{11}+\frac{\delta^2}{12}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\left(k'_{22}-\frac{e'_{22}}{R_2}\right)\right]\\ T'_{12}&=\frac{E\delta}{2(1+\mu)}\left[e'_{12}+e'_{21}-\frac{\delta^2}{12}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\left(k'_{12}-\frac{e'_{12}}{R_1}\right)\right]\\ T'_{21}&=\frac{E\delta}{2(1+\mu)}\left[e'_{12}+e'_{21}+\frac{\delta^2}{12}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\left(k'_{21}-\frac{e'_{21}}{R_2}\right)\right]\\ M'_1&=\frac{E\delta^3}{12(1-\mu^2)}\left[k'_{11}+\mu k'_{22}-\left(\frac{1}{R_1}-\frac{1}{R_2}\right)e'_{11}\right]\\ M'_2&=\frac{E\delta^3}{12(1-\mu^2)}\left[k'_{22}+\mu k'_{11}+\left(\frac{1}{R_1}-\frac{1}{R_2}\right)e'_{22}\right]\\ M'_{12}&=\frac{E\delta^3}{24(1+\mu)}\left[k'_{12}+k'_{21}-\left(\frac{1}{R_1}-\frac{1}{R_2}\right)e'_{12}\right]\\ M'_{21}&=\frac{E\delta^3}{24(1+\mu)}\left[k'_{12}+k'_{21}+\left(\frac{1}{R_1}-\frac{1}{R_2}\right)e'_{21}\right]\end{aligned}\right\} \tag{3.11}$$

$$A'=\frac{E\delta}{2(1-\mu^2)}\iint\Big\{e_{11}^{\prime 2}+e_{22}^{\prime 2}+2\mu e_{11}^{\prime}e_{22}^{\prime}+\frac{1-\mu}{2}(e_{12}^{\prime}+e_{21}^{\prime})^2$$
$$+\frac{\delta^2}{12}\Big[k_{11}^{\prime 2}+k_{22}^{\prime 2}+2\mu k_{11}^{\prime}k_{22}^{\prime}+\frac{1-\mu}{2}(k_{12}^{\prime}+k_{21}^{\prime})^2\Big]$$
$$-\Big(\frac{1}{R_1}-\frac{1}{R_2}\Big)\Big[2e_{11}^{\prime}k_{11}^{\prime}-2e_{22}^{\prime}k_{22}^{\prime}-\frac{e_{11}^{\prime 2}}{R_1}+\frac{e_{22}^{\prime 2}}{R_2}+\frac{1-\mu}{2}\Big(2e_{12}^{\prime}k_{12}^{\prime}$$
$$-2e_{21}^{\prime}k_{21}^{\prime}-\frac{e_{12}^{\prime 2}}{R_1}+\frac{e_{21}^{\prime 2}}{R_2}\Big)\Big]\Big\}A_1A_2d\alpha_1d\alpha_2 \tag{3.12}$$

式中δ为壳体的厚度．$T_1^{\circ},\cdots$，A°与以上各式相似，仅各项上标“$'$”改为“$\circ$”，且通常略去含因子$1/R_1-1/R_2$的项．

$$T_1''=\frac{E\delta}{2(1-\mu^2)}[e_{11}^{\prime 2}+e_{12}^{\prime 2}+e_{13}^{\prime 2}+\mu(e_{21}^{\prime 2}+e_{22}^{\prime 2}+e_{23}^{\prime 2})]$$

$$T_2''=\frac{E\delta}{2(1-\mu^2)}[e_{21}^{\prime 2}+e_{22}^{\prime 2}+e_{23}^{\prime 2}+\mu(e_{11}^{\prime 2}+e_{12}^{\prime 2}+e_{13}^{\prime 2})]$$

$$T_{12}''=T_{21}''=\frac{E\delta}{2(1+\mu)}(e_{11}^{\prime}e_{21}^{\prime}+e_{21}^{\prime}e_{22}^{\prime}+e_{13}^{\prime}e_{23}^{\prime})$$

应用到(3.11)式还可得

$$A^{\circ\prime}=\iint(T_1^{\circ}e_{11}^{\prime}+M_1^{\circ}k_{11}^{\prime}+T_2^{\circ}e_{22}^{\prime}+M_2^{\circ}k_{22}^{\prime}+T_{12}^{\circ}e_{12}^{\prime}+M_{12}^{\circ}k_{12}^{\prime}+T_{21}^{\circ}e_{21}^{\prime}$$
$$+M_{21}^{\circ}k_{21}^{\prime})A_1A_2d\alpha_1d\alpha_2 \tag{3.13}$$

$$A'=\frac{1}{2}\iint(T_1^{\prime}e_{11}^{\prime}+M_1^{\prime}k_{11}^{\prime}+T_2^{\prime}e_{22}^{\prime}+M_2^{\prime}k_{22}^{\prime}+T_{12}^{\prime}e_{12}^{\prime}+M_{12}^{\prime}k_{12}^{\prime}$$
$$+T_{21}^{\prime}e_{21}^{\prime}+M_{21}^{\prime}k_{21}^{\prime})A_1A_2d\alpha_1d\alpha_2 \tag{3.14}$$

$$A''=\frac{1}{2}\iint\Big[\Big(T_1^{\circ}+\frac{M_1^{\circ}}{R_1}\Big)(e_{11}^{\prime 2}+e_{12}^{\prime 2}+e_{13}^{\prime 2})+\Big(T_2^{\circ}+\frac{M_2^{\circ}}{R_2}\Big)(e_{21}^{\prime 2}+e_{22}^{\prime 2}+e_{23}^{\prime 2})$$
$$+\Big(T_{12}^{\circ}+\frac{M_{12}^{\circ}}{R_1}+T_{21}^{\circ}+\frac{M_{21}^{\circ}}{R_2}\Big)(e_{11}^{\prime}e_{21}^{\prime}+e_{12}^{\prime}e_{22}^{\prime}+e_{13}^{\prime}e_{23}^{\prime})\Big]A_1A_2d\alpha_1d\alpha_2 \tag{3.15}$$

A°与(3.14)式相似，但各项的上标“$'$”改为“$\circ$”．

当应用虚位移原理时，仅变化附加位移αu_1，αv_1，αw_1，则

$$\delta u=\alpha\delta u_1,\quad \delta v=\alpha\delta v_1,\quad \delta w=\alpha\delta w_1 \tag{3.16}$$

对于稳定问题，通常载荷是指与壳体变形有关的分布力．将(3.4)式代入(2.3)式然后和(3.14)式一齐代入(2.2)式，略去α^3的项可得

$$\delta R_1=\alpha\delta R_1^{\prime}+\alpha^2\delta R_1''$$

式中

$$\delta R_1^{\prime}=\iint(q_1\delta u_1+q_2\delta v_1+q_n\delta w_1)A_1A_2d\alpha_1d\alpha_2 \tag{3.17}$$

$$\delta R_1''=\iint\{[q_1(e_{11}^{\prime}+e_{22}^{\prime})+q_1e_{12}^{\prime}+q_2e_{21}^{\prime}-q_ne_{13}^{\prime}]\delta u_1$$
$$+[q_2(e_{11}^{\prime}+e_{22}^{\prime})+q_1e_{12}^{\prime}+q_2e_{22}^{\prime}-q_ne_{23}^{\prime}]\delta v_1$$
$$+[q_n(e_{11}^{\prime}+e_{22}^{\prime})+q_1e_{13}^{\prime}+q_2e_{23}^{\prime}]\delta w_1\}A_1A_2d\alpha_1d\alpha_2 \tag{3.18}$$

将(3.14)式代入(2.4)式并应用到(3.2)式和(3.4)式可得

$$\delta R_2=\alpha\delta R_2^{\prime}=\alpha\int(\boldsymbol{T}_1\delta u_1+\boldsymbol{T}_{12}\delta v_1+\boldsymbol{V}_1\delta w_1-\boldsymbol{M}_1\delta e_{13}^{\prime})A_2d\alpha_2$$
$$+\alpha\int(\boldsymbol{T}_{21}\delta u_1+\boldsymbol{T}_2\delta v_1+\boldsymbol{V}_2\delta w_1-\boldsymbol{M}_2\delta e_{23}^{\prime})A_1d\alpha_1+\alpha\bar{\boldsymbol{P}}\delta w_1 \tag{3.19}$$

因此第二个平衡位置的虚位移原理为

$$\delta A=\alpha\delta A^{\circ\prime}+\alpha^2\delta(A'+A'')=\alpha(\delta R_1^{\prime}+\delta R_2^{\prime})+\alpha^2\delta R_1''$$

$\delta A^{\circ}=0$是由于A°不依赖于u_1，v_1，w_1，而$\delta A^{\circ\prime}$，$\delta R_1^{\prime}$和$\delta R_2^{\prime}$三项洽好组成第一平衡位置的虚位移原理的数学公式

$$\delta A^{\circ\prime}=\delta R_1^{\prime}+\delta R_2^{\prime} \tag{3.20}$$

比较以上二式可得弹性稳定的变分公式为

$$\delta A'+\delta A''=\delta R_1'' \tag{3.21}$$

四、确定临界载荷的微分方程

临界载荷通常与边界约束无关．因此取第一平衡位置处于薄膜应力状态．将(3.5)式代入(3.13)式然后和(3.17)式以及(3.19)式一起代入(3.20)式．令$M_1^{\circ}=M_2^{\circ}=M_{12}^{\circ}=M_{21}^{\circ}=\boldsymbol{V}_1=\boldsymbol{V}_2=\boldsymbol{M}_1=\boldsymbol{M}_2=\bar{\boldsymbol{P}}=0$，$T_{12}^{\circ}=T_{21}^{\circ}$，积分后可得

$$\delta(A^{\circ\prime}-R_1^{\prime}-R_2^{\prime})=-\iint\Big[\Big(\frac{\partial A_2T_1^{\circ}}{\partial\alpha_1}+\frac{\partial A_1T_{12}^{\circ}}{\partial\alpha_2}+\frac{\partial A_1}{\partial\alpha_2}T_{12}^{\circ}\cdot$$
$$-\frac{\partial A_2}{\partial\alpha_2}T_2^{\circ}+A_1A_2q_1\Big)\delta u_1+\Big(\frac{\partial A_2T_{12}^{\circ}}{\partial\alpha_1}+\frac{\partial A_1T_2^{\circ}}{\partial\alpha_2}+\frac{\partial A_2}{\partial\alpha_1}T_{12}^{\circ}$$
$$-\frac{\partial A_1}{\partial\alpha_1}T_1^{\circ}+A_1A_2q_2\Big)\delta v_1+A_1A_2\Big(q_n-\frac{T_1^{\circ}}{R_1}-\frac{T_2^{\circ}}{R_2}\Big)\delta w_1\Big]d\alpha_1d\alpha_2$$
$$+\int[(T_1^{\circ}-\boldsymbol{T}_1)\delta u_1+(T_{12}^{\circ}-\boldsymbol{T}_{12})\delta v_1]A_2d\alpha_2+\int[(T_{12}^{\circ}$$
$$-\boldsymbol{T}_{21})\delta u_1+(T_2^{\circ}-\boldsymbol{T}_2)\delta v_1]A_1d\alpha_1=0 \tag{4.1}$$

由于积分的任意性，上式双重积分号下各项等于零组成第一平衡位置薄膜应力的微分方程．线积分号下各项等于零组成外力的边界条件．(3.14)式的变分为[2]

$$\delta A'=\iint(T_1^{\prime}\delta e_{11}^{\prime}+M_1^{\prime}\delta k_{11}^{\prime}+T_2^{\prime}\delta e_{22}^{\prime}+M_2^{\prime}\delta k_{22}^{\prime}+T_{12}^{\prime}\delta e_{12}^{\prime}+M_{12}^{\prime}\delta k_{12}^{\prime}$$
$$+T_{21}^{\prime}\delta e_{21}^{\prime}+M_{21}^{\prime}\delta k_{21}^{\prime})A_1A_2d\alpha_1d\alpha_2 \tag{4.2}$$

将(3.5)式和(3.7)式代入上式，然后和(3.15)式，(3.18)式一起代入(3.21)式，积分后可得

$$\delta(A'+A''-R_1'')=-\iint\Big\{\Big[\frac{\partial A_2T_1^{\prime}}{\partial\alpha_1}+\frac{\partial A_1T_{21}^{\prime}}{\partial\alpha_2}+\frac{\partial A_1}{\partial\alpha_2}T_{12}^{\prime}-\frac{\partial A_2}{\partial\alpha_1}T_2^{\prime}$$
$$+\frac{1}{R_1}\Big(\frac{\partial A_2M_1^{\prime}}{\partial\alpha_1}+\frac{\partial A_1M_{21}^{\prime}}{\partial\alpha_2}+\frac{\partial A_1}{\partial\alpha_2}M_{12}^{\prime}-\frac{\partial A_2}{\partial\alpha_1}M_2^{\prime}\Big)$$
$$+\frac{\partial A_2T_1^{\circ}e_{11}^{\prime}}{\partial\alpha_1}+\frac{\partial A_1T_{21}^{\circ}e_{11}^{\prime}}{\partial\alpha_2}+\frac{\partial A_1}{\partial\alpha_2}T_{12}^{\circ}e_{22}^{\prime}-\frac{\partial A_2}{\partial\alpha_1}T_2^{\circ}e_{22}$$
$$+\frac{\partial A_2T_{12}^{\circ}e_{21}^{\prime}}{\partial\alpha_1}+\frac{\partial A_1T_2^{\circ}e_{21}^{\prime}}{\partial\alpha_2}+\frac{\partial A_1}{\partial\alpha_2}T_1^{\circ}e_{12}^{\prime}-\frac{\partial A_2}{\partial\alpha_1}T_{21}^{\circ}e_{12}$$

$$+\frac{A_1A_2}{R_1}(T^{\circ}_{1}e'_{13}+T^{\circ}_{12}e'_{23})+A_1A_2q_1(e'_{11}+e'_{22})$$

$$+A_1A_2(q_1e'_{11}+q_2e'_{21}-q_ne'_{13})\Big]\delta u_1+\Big[\frac{\partial A_1T'_{2}}{\partial\alpha_2}+\frac{\partial A_2T'_{12}}{\partial\alpha_1}$$

$$+\frac{\partial A_2}{\partial\alpha_1}T'_{21}-\frac{\partial A_1}{\partial\alpha_2}T'_{1}+\frac{1}{R_2}\Big(\frac{\partial A_1M'_{2}}{\partial\alpha_2}+\frac{\partial A_2M'_{12}}{\partial\alpha_1}+\frac{\partial A_2}{\partial\alpha_1}M'_{21}$$

$$-\frac{\partial A_1}{\partial\alpha_2}M'_{1}\Big)+\frac{\partial A_1T^{\circ}_{2}e'_{22}}{\partial\alpha_2}+\frac{\partial A_2T^{\circ}_{12}e'_{22}}{\partial\alpha_1}+\frac{\partial A_2}{\partial\alpha_1}T'_{21}e'_{11}-\frac{\partial A_1}{\partial\alpha_2}T'_{1}e'_{11}$$

$$+\frac{\partial A_1T^{\circ}_{21}e'_{12}}{\partial\alpha_2}+\frac{\partial A_2T^{\circ}_{1}e'_{12}}{\partial\alpha_1}+\frac{\partial A_2}{\partial\alpha_1}T^{\circ}_{2}e'_{21}-\frac{\partial A_1}{\partial\alpha_2}T^{\circ}_{12}e'_{21}$$

$$+\frac{A_1A_2}{R_2}(T^{\circ}_{2}e'_{23}+T^{\circ}_{21}e'_{13})+A_1A_2q_2(e'_{11}+e'_{22})$$

$$+A_1A_2(q_1e'_{12}+q_2e'_{22}-q_ne'_{23})\Big]\delta v_1+\Big\{\frac{\partial}{\partial\alpha_1}\Big[\frac{1}{A_1}\Big(\frac{\partial A_2M'_{1}}{\partial\alpha_1}$$

$$+\frac{\partial A_1M'_{21}}{\partial\alpha_2}+\frac{\partial A_1}{\partial\alpha_2}M'_{12}-\frac{\partial A_2}{\partial\alpha_1}M'_{2}\Big)\Big]+\frac{\partial}{\partial\alpha_2}\Big[\frac{1}{A_2}\Big(\frac{\partial A_1M'_{2}}{\partial\alpha_2}$$

$$+\frac{\partial A_2M'_{12}}{\partial\alpha_1}+\frac{\partial A_2}{\partial\alpha_1}M'_{21}-\frac{\partial A_1}{\partial\alpha_2}M'_{1}\Big)\Big]-A_1A_2\Big(\frac{T'_{1}}{R_1}+\frac{T'_{2}}{R_2}\Big)$$

$$+\frac{\partial A_2T^{\circ}_{1}e'_{13}}{\partial\alpha_1}+\frac{\partial A_1T^{\circ}_{21}e'_{13}}{\partial\alpha_2}+\frac{\partial A_2T^{\circ}_{12}e'_{23}}{\partial\alpha_1}+\frac{\partial A_1T^{\circ}_{2}e'_{13}}{\partial\alpha_2}$$

$$-\frac{A_1A_2}{R_1}(T^{\circ}_{1}e'_{11}+T^{\circ}_{12}e'_{21})-\frac{A_1A_2}{R_2}(T^{\circ}_{2}e'_{22}+T^{\circ}_{21}e'_{12})$$

$$+A_1A_2q_n(e'_{11}+e'_{22})+A_1A_2(q_1e'_{13}+q_2e'_{23})\Big\}\delta w_1\Big\}d\alpha_1d\alpha_2$$

$$+\int\Big\{(T'_{1}+T^{\circ}_{1}e'_{11}+T^{\circ}_{12}e'_{21})\,\delta u_1+\Big(T'_{12}+\frac{M'_{12}}{R_2}+T^{\circ}_{1}e'_{12}+T^{\circ}_{12}e'_{22}\Big)\delta v_1$$

$$+\Big[\frac{1}{A_1A_2}\Big(\frac{\partial A_2M'_{1}}{\partial\alpha_1}+\frac{\partial A_1M'_{21}}{\partial\alpha_2}+\frac{\partial A_1}{\partial\alpha_2}M'_{12}-\frac{\partial A_2}{\partial\alpha_1}M'_{2}\Big)$$

$$+\frac{1}{A_2}\frac{\partial M'_{12}}{\partial\alpha_2}+T^{\circ}_{1}e'_{13}+T^{\circ}_{12}e'_{23}\Big]\delta w_1-M'_{1}\delta e'_{13}\Big\}A_2d\alpha_2$$

$$+\int\Big\{\Big(T'_{21}+\frac{M'_{21}}{R_1}+T^{\circ}_{2}e'_{21}+T^{\circ}_{12}e'_{11}\Big)\delta u_1+\Big(T'_{2}+T^{\circ}_{2}e'_{22}+T^{\circ}_{21}e'_{12}\Big)\delta v_1$$

$$+\Big[\frac{1}{A_1A_2}\Big(\frac{\partial A_1M'_{2}}{\partial\alpha_2}+\frac{\partial A_2M'_{12}}{\partial\alpha_1}+\frac{\partial A_2}{\partial\alpha_1}M'_{21}-\frac{\partial A_1}{\partial\alpha_2}M'_{1}\Big)$$

$$+\frac{1}{A_1}\frac{\partial M'_{21}}{\partial\alpha_1}+T^{\circ}_{2}e'_{23}+T^{\circ}_{21}e'_{13}\Big]\delta w_1-M'_{2}\delta e'_{23}\Big\}A_1d\alpha_1$$

$$+(M'_{12}+M'_{21})\delta w_1=0 \tag{4.3}$$

由于变分的任意性，上式双重积分号下各项等于零组成临界载荷的第二个平衡位置的微分方程[4]．线积分号下各项等于零以及最后一项等于零组成外力的边界条件和角点条件．

五、用位移变分法求临界载荷

将(3.12)式，(3.15)式和(3.18)式代入(3.21)式可得求解临界载荷的变分公式．为简化计算，首先可略去(3.12)式中含因子$(1/R_1-1/R_2)$的项．其次考虑到转动角e'_{13}和e'_{23}大大超过应变e'_{11}，e'_{12}，e'_{21}和e'_{22}[1]．相应地在(3.15)式和(3.18)式中可略去后者而保留前者．因此我们得到一实用的一般性的薄壳稳定的变分公式

$$\delta\iint\frac{E\delta}{2(1-\mu^2)}\Big\{e'^{2}_{11}+e'^{2}_{22}+2\mu e'_{11}e'_{22}+\frac{1-\mu}{2}(e'_{12}+e'_{21})^2$$

$$+\frac{\delta^2}{12}\Big[k'^{2}_{11}+k'^{2}_{22}+2\mu k'_{11}k'_{22}+\frac{1-\mu}{2}(k'_{12}+k'_{21})^2\Big]$$

$$+\frac{1}{2}T^{\circ}_{1}e'^{2}_{13}+\frac{1}{2}T^{\circ}_{2}e'^{2}_{23}+T^{\circ}_{12}e'_{13}e'_{23}\Big\}A_1A_2d\alpha_1d\alpha_2$$

$$+\iint[q_n(e'_{13}\delta u_1+e'_{23}\delta v_1)-(q_1e'_{13}+q_2e'_{23})\,\delta w_1]A_1A_2d\alpha_1d\alpha_2=0 \qquad (5.1)$$

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The Variational Principle of Thin Shell in Stability

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Abstract

This paper gives the resultant forces and moments, strain energy and work of external forces on the basis of the deformation theory of flexible body. Therefore, in accordance with the principle of virtual displacement, the energy criterion of critical load is obtained and the equailibrium equations and boundary conditions of stability problem are derived.

Key words shell, stability, critical load