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高速扩展平面应力裂纹尖端的各向异性塑性场的补充研究（Ⅰ）*

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摘　要

文献[1]的结果对于$\beta\geqslant2$的情形不适用．为此，我们用文献[1]和[2]的方法导出了$\beta=2$和$\beta>2$两者的高速扩展平面应力裂纹尖端的各向异性塑性场的一般表达式．

关键词　高速扩展　平面应力　裂纹尖端　各向异性塑性场　一般表达式　塑性区

一、引　言

我们知道，文献[1]的结果不适用于$\beta\geqslant2$的情形．为此，我们用文献[1]和[2]的方法来研究$\beta\geqslant2$的情形的高速扩展平面应用裂纹尖端的各向异性塑性场．

在理想弹塑性材料中，高速扩展裂纹尖端的应力分量都只是θ的函数．利用这个条件以及定常运动方程，$\beta=2$和$\beta>2$两者的Hill屈服条件与弹塑性本构方程，我们导出了$\beta=2$和$\beta>2$两者的高速扩展平面应力裂纹尖端的各向异性塑性场的一般表达式．将这些一般表达式用于两种各向异性塑性特殊情形，我们就得到$\beta=2$和$\beta>2$两者的两种特殊情形的高速扩展平面应力裂纹尖端的各向异性塑性场的一般表达式．

图1表示沿裂纹线高速扩展平面应力裂纹的尖端几何．(x_1,y_1,z_1)和(x,y,z)分别是静止坐标系和运动坐标系．这些坐标轴亦是各向异性塑性主轴．运动坐标系的原点放在高速扩展平面应力裂纹的尖点O上．裂纹尖端的速度为$c=\text{const}$．设裂纹作定常运动，则有如下关系：

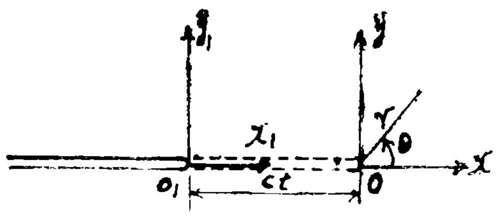

图　1

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$$\frac{\partial}{\partial t}=-c\frac{\partial}{\partial x},\quad \frac{\partial^2}{\partial t^2}=c^2\frac{\partial^2}{\partial x^2} \tag{1.1}$$

今后取

$$\alpha=c/\sqrt{\mu/\rho}\leqslant 1 \tag{1.2}$$

其中，$c_s=\sqrt{\mu/\rho}$ 为剪切波波速，μ为剪切弹性模量，ρ是材料的密度。

二、β＝2的一般表达式

对于$\beta=2$的情形，在运动坐标系$Oxyz$中，我们有如下的偏微分方程组[1]：

$$\left.\begin{aligned}
&X\frac{\partial\sigma_+}{\partial x}+X\frac{\partial\sigma_-}{\partial x}+T\frac{\partial\sigma_{zy}}{\partial y}-\rho c^2\frac{\partial u_z}{\partial x}=0\\
&Y\frac{\partial\sigma_+}{\partial y}-Y\frac{\partial\sigma_-}{\partial y}+T\frac{\partial\sigma_{zy}}{\partial x}-\rho c^2\frac{\partial v_z}{\partial x}=0\\
&X\frac{\partial u_z}{\partial x}+Y\frac{\partial v_z}{\partial y}-\left(D_1\frac{\partial\sigma_+}{\partial x}+D_2\frac{\partial\sigma_-}{\partial x}\right)=0\\
&X\sigma_{zy}\frac{\partial u_z}{\partial x}-4T\sigma_-\frac{\partial u_z}{\partial y}-Y\sigma_{zy}\frac{\partial v_z}{\partial y}-4T\sigma_-\frac{\partial v_z}{\partial x}\\
&\quad-\sigma_{zy}\left(D_2\frac{\partial\sigma_+}{\partial x}+D_3\frac{\partial\sigma_-}{\partial x}\right)+4D_4\sigma_-\frac{\partial\sigma_{zy}}{\partial x}=0
\end{aligned}\right\} \tag{2.1}$$

其中[1]

$$\sigma_+=\frac{1}{2}\left(\frac{\sigma_x}{X}+\frac{\sigma_y}{Y}\right),\quad \sigma_-=\frac{1}{2}\left(\frac{\sigma_x}{X}-\frac{\sigma_y}{Y}\right),\quad \sigma_{zy}=\frac{\tau_{zy}}{T} \tag{2.2}$$

$$\left.\begin{aligned}
&D_1=\frac{X^2+Y^2-2\nu XY}{E},\quad D_2=\frac{X^2-Y^2}{E}\\
&D_3=\frac{X^2+Y^2+2\nu XY}{E},\quad D_4=\frac{T^2}{\mu}
\end{aligned}\right\} \tag{2.3}$$

$\beta=2$的Hill屈服条件为[1]：

$$4\sigma_-^2+\sigma_{zy}^2=1 \tag{2.4}$$

若取[2]

$$\sigma_-=-\frac{1}{2}\cos\varphi,\quad \sigma_{zy}=\sin\varphi \tag{2.5}$$

则式(2.4)恒被满足。φ只是θ的函数。

我们知道，σ_+，σ_-，σ_{zy}，u_z和v_z都只是θ的函数[1]。将(2.5)代入(2.1)，并采用如下变换：

$$\frac{\partial}{\partial x}=-\frac{\sin\theta}{r}\frac{d}{d\theta},\quad \frac{\partial}{\partial y}=\frac{\cos\theta}{r}\frac{d}{d\theta} \tag{2.6}$$

我们就得到关于新变量$d\sigma_+/d\theta$，$d\varphi/d\theta$，$du_z/d\theta$和$dv_z/d\theta$的方程组：

$$
\left.\begin{aligned}
&X\sin\theta\frac{d\sigma_+}{d\theta}+\frac{1}{2}(X\sin\theta\sin\varphi-2T\cos\theta\cos\varphi)\frac{d\varphi}{d\theta}-\alpha^2\mu\sin\theta\frac{du_z}{d\theta}=0\\
&Y\cos\theta\frac{d\sigma_+}{d\theta}-\frac{1}{2}(Y\cos\theta\sin\varphi+2T\sin\theta\cos\varphi)\frac{d\varphi}{d\theta}+\alpha^2\mu\sin\theta\frac{dv_z}{d\theta}=0\\
&D_1\sin\theta\frac{d\sigma_+}{d\theta}+\frac{D_2}{2}\sin\theta\sin\varphi\frac{d\varphi}{d\theta}-X\sin\theta\frac{du_z}{d\theta}+Y\cos\theta\frac{dv_z}{d\theta}=0\\
&D_1\sin\theta\frac{d\sigma_+}{d\theta}+\frac{\sin\theta}{2}(D_3\sin^2\varphi+2D_4\cos^2\varphi)\frac{d\varphi}{d\theta}\\
&\quad-(X\sin\theta\sin\varphi-2T\cos\theta\cos\varphi)\frac{du_z}{d\theta}-(Y\cos\theta\sin\varphi+2T\sin\theta\cos\varphi)\frac{dv_z}{d\theta}=0
\end{aligned}\right\}\tag{2.7}
$$

根据(2.7)，我们得到下面两种塑性区：

1. **均匀塑性区**($d\sigma_+/d\theta=0$，$d\varphi/d\theta=0$，$du_z/d\theta=0$，$dv_z/d\theta=0$)

在均匀塑性区内有

$$\sigma_+=\text{const},\ \varphi=\text{const},\ u_z=\text{const},\ v_z=\text{const}\tag{2.8}$$

所以均匀塑性区是均匀应力区．

2. **非均匀塑性区**($d\sigma_+/d\theta\neq0$，$d\varphi/d\theta\neq0$，$du_z/d\theta\neq0$，$dv_z/d\theta\neq0$)

非均匀塑性区的存在条件是(2.7)的系数行列式为零，即

$$A\sin^2\varphi+B\cos^2\varphi+C\sin\varphi\cos\varphi=0\tag{2.9}$$

其中

$$
\left.\begin{aligned}
&A=Z^2Y^2\sin^2\theta\cdot\left\{\alpha^2\left[1-\frac{\alpha^2}{2}(1-\nu)\sin^2\theta\right]-2(1+\nu)\cos^2\theta\right\}\\
&B=T^2\left\{(X^2+Y^2-2\nu XY)\alpha^2\sin^2\theta\cdot\left(1-\frac{\alpha^2\sin^2\theta}{2}\right)\right.\\
&\qquad\left.+(1+\nu)[(X^2\sin^2\theta+Y^2\cos^2\theta)\alpha^2\sin^2\theta-2(X\sin^2\theta-Y\cos^2\theta)^2]\right\}\\
&C=TXY[(X-Y)\alpha^2\sin^2\theta-2(1+\nu)(X\sin^2\theta-Y\cos^2\theta)]
\end{aligned}\right\}\tag{2.10}
$$

当$\alpha=0$时，式(2.9)变为：

$$[T(X\sin^2\theta-Y\cos^2\theta)\cos\varphi+XY\sin\theta\cos\theta\sin\varphi]^2=0\tag{2.11}$$

由此得到：

$$\mathrm{tg}\varphi=\frac{2T(Y\cos^2\theta-X\sin^2\theta)}{XY\sin2\theta}\tag{2.12}$$

这就是文献[2]中的式(2.8)．所以，本文所得的结果是正确的．

利用(2.7)的前三式解出$d\sigma_+/d\theta$，$du_z/d\theta$和$dv_z/d\theta$，然后积分得到：

$$
\begin{aligned}
&\sigma_+=\sigma_{+0}+\int_{\varphi_0}^{\varphi}\left\{\frac{(\alpha^2\mu D_2-X^2)\sin^2\theta+Y^2\cos^2\theta}{2[(X^2-\alpha\mu^2D_1)\sin^2\theta+Y^2\cos^2\theta]}\cdot\sin\varphi\right.\\
&\qquad\left.+\frac{T(X+Y)\sin2\theta}{2[(X^2-\alpha^2\mu D_1)\sin^2\theta+Y^2\cos^2\theta]}\cdot\cos\varphi\right\}d\varphi\\
&u_z=u_{z0}+\int_{\varphi_0}^{\varphi}\left\{\frac{X\left[Y^2\cos^2\theta-\alpha^2\mu\dfrac{D_1-D_2}{2}\sin^2\theta\right]}{\alpha^2\mu[(X^2-\alpha^2\mu D_1)\sin^2\theta+Y^2\cos^2\theta}\sin\varphi\right.
\end{aligned}\tag{2.13}
$$

$$+\frac{T\cot\theta[(XY+\alpha^2\mu D_1)\sin^2\theta-Y^2\cos^2\theta]}{\alpha^2\mu[(X^2-\alpha^2\mu D_1)\sin^2\theta+Y^2\cos^2\theta]}\cos\varphi\Big\}d\varphi$$

$$v_x=v_{x0}+\int_{\varphi_0}^{\varphi}\Big\{\frac{Y[X^2-\alpha^2\mu(D_1+D_2)]\sin\theta\cos\theta}{2\alpha^2\mu[(X^2-\alpha^2\mu D_1)\sin^2\theta+Y^2\cos^2\theta]}\sin\varphi$$

$$+\frac{T[(X^2-\alpha^2\mu D_1)\sin^2\theta-Y^2\cos^2\theta]}{\alpha^2\mu[(X^2-\alpha^2\mu D_1)\sin^2\theta+Y^2\cos^2\theta]}\cos\varphi\Big\}d\varphi$$

下面给出两种各向异性塑性特殊情形的一般表达式：

1. $X=Z=\sqrt{3}T$的情形

在这种情形中，$X/Y=2$，于是有

（1） 均匀塑性区

$$\sigma_+=\text{const},\ \varphi=\text{const},\ u_x=\text{const},\ v_x=\text{const} \tag{2.8}$$

（2） 非均匀塑性区

(2.9)形式上保持不变，而(2.10)则变为：

$$\left.\begin{aligned}
A&=3\sin^2\theta\cdot\Big\{\alpha^2\Big[1-\frac{\alpha^2}{2}(1-\nu)\sin^2\theta\Big]-2(1+\nu)\cos^2\theta\Big\}\\
B&=(5-4\nu)\Big(1-\frac{\alpha^2\sin^2\theta}{2}\Big)\alpha^2\sin^2\theta\\
&\quad+(1+\nu)[\alpha^2\sin^2\theta(1+3\sin^2\theta)-2(3\sin^2\theta-1)^2]\\
C&=\sqrt{3}\sin2\theta[\alpha^2\sin^2\theta-2(1+\nu)(3\sin^2\theta-1)]
\end{aligned}\right\} \tag{2.14}$$

式(2.13)变为：

$$\left.\begin{aligned}
\sigma_+&=\sigma_{+0}+\int_{\varphi_0}^{\varphi}\Big\{\frac{[3\alpha^2-10(1+\nu)]\sin^2\theta+2(1+\nu)}{4(1+\nu)+2[6(1+\nu)-(5-4\nu)\alpha^2]\sin^2\theta}\cdot\sin\varphi\\
&\quad+\frac{2\sqrt{3}(1+\nu)\sin2\theta}{2(1+\nu)+[6(1+\nu)-(5-4\nu)\alpha^2]\sin^2\theta}\cdot\cos\varphi\Big\}d\varphi\\
u_x&=u_{x0}+\frac{T}{\alpha^2\mu}\int_{\varphi_0}^{\varphi}\Big\{\frac{2\sqrt{3}(1+\nu)-\sqrt{3}[2(1+\nu)-(1-2\nu)\alpha^2]\sin^2\theta}{2(1+\nu)+[6(1+\nu)-(5-4\nu)\alpha^2]\sin^2\theta}\cdot\sin\varphi\\
&\quad-\frac{\cot\theta\{2(1+\nu)-[6(1+\nu)-(5-4\nu)\alpha^2]\sin^2\theta\}}{2(1+\nu)+[6(1+\nu)-(5-4\nu)\alpha^2]\sin^2\theta}\cos\varphi\Big\}d\varphi\\
v_a&=v_{x0}+\frac{T}{\alpha^2\mu}\int_{\varphi_0}^{\varphi}\Big\{\frac{\sqrt{3}[2(1+\nu)-2(2-\nu)\alpha^2]\sin2\theta}{4(1+\nu)+2[6(1+\nu)-(5-4\nu)\alpha^2]\sin^2\theta}\sin\varphi\\
&\quad-\frac{2(1+\nu)-[10(1+\nu)-(5-4\nu)\alpha^2]\sin^2\theta}{2(1+\nu)+[6(1+\nu)-(5-4\nu)\alpha^2]\sin^2\theta}\cos\varphi\Big\}d\varphi
\end{aligned}\right\}$$

(2.15)

2. $Y=Z=\sqrt{3}T$的情形

在这种情形中，$Y/Z=2$，于是有

（1） 均匀塑性区

$$\sigma_+=\text{const},\ \varphi=\text{const},\ u_x=\text{const},\ v_x=\text{const} \tag{2.8}$$

（2） 非均匀塑性区

(2.9)形式上保持不变，而(2.10)则变为：

$$
\left.\begin{aligned}
A&=3\sin^2\theta\left\{\alpha^2\left[1-\frac{\alpha^2}{2}(1-\nu)\sin^2\theta\right]-2(1+\nu)\cos^2\theta\right\}\\
B&=(5-4\nu)\left(1-\frac{\alpha^2\sin^2\theta}{2}\right)\alpha^2\sin^2\theta\\
&\quad+(1+\nu)[E\alpha^2\sin^2\theta(3\cos^2\theta+1)-2(3\cos^2\theta-1)^2]\\
C&=-\sqrt{3}\sin2\theta[\alpha^2\sin^2\theta+2(1+\nu)(3\cos^2\theta-1)]
\end{aligned}\right\}\tag{2.16}
$$

式(2.13)变为：

$$
\left.\begin{aligned}
\sigma_+&=\sigma_{+0}+\int_{\varphi_0}^{\varphi}\left\{\frac{8(1+\nu)-[3\alpha^2+10(1+\nu)]\sin^2\theta}{2\{8(1+\nu)-[6(1+\nu)+(5-4\nu)\alpha^2]\sin^2\theta\}}\cdot\sin\varphi\right.\\
&\quad\left.+\frac{2\sqrt{3}(1+\nu)\sin2\theta}{8(1+\nu)-[6(1+\nu)+(5-4\nu)\alpha^2]\sin^2\theta}\cdot\cos\varphi\right\}d\varphi\\
u_x&=u_{x0}+\frac{T^2}{\alpha^2\mu}\int_{\varphi_0}^{\varphi}\left\{\frac{\sqrt{3}\{4(1+\nu)-[4(1+\nu)+(2-\nu)\alpha^2]\sin^2\theta\}}{2\{8(1+\nu)-[6(1+\nu)+(5-4\nu)\alpha^2]\sin^2\theta\}}\cdot\sin\varphi\right.\\
&\quad\left.-\frac{\cot\theta\{8(1+\nu)-[12(1+\nu)+(5-4\nu)\alpha^2]\sin^2\theta}{8(1+\nu)-[6(1+\nu)+(5-4\nu)\alpha^2]\sin^2\theta}\cos\varphi\right\}d\varphi\\
v_x&=v_{x0}+\frac{T^2}{\alpha^2\mu}\int_{\varphi_0}^{\varphi}\left\{\frac{\sqrt{3}[(1+\nu)-(1-2\nu)\alpha^2]\sin2\theta}{2\{8(1+\nu)-[6(1+\nu)+(5-4\nu)\alpha^2]\sin^2\theta\}}\sin\varphi\right.\\
&\quad\left.-\frac{8(1+\nu)-[10(1+\nu)-(5-4\nu)\alpha^2]\sin^2\theta}{8(1+\nu)-[6(1+\nu)+(5-4\nu)\alpha^2]\sin^2\theta}\cdot\cos\varphi\right\}d\varphi
\end{aligned}\right\}\tag{2.17}
$$

三、β>2的一般表达式

对于$\beta>2$的情形，在运动坐标系$Oxyz$中，我们有如下的偏微分方程组[1]：

$$
\left.\begin{aligned}
&X\frac{\partial\sigma_+}{\partial x}+X\frac{\partial\sigma_-}{\partial x}+T\frac{\partial\sigma_{xy}}{\partial y}-\rho c^2\frac{\partial u_x}{\partial x}=0\\
&Y\frac{\partial\sigma_+}{\partial y}-Y\frac{\partial\sigma_-}{\partial y}+T\frac{\partial\sigma_{xy}}{\partial x}-\rho c^2\frac{\partial v_x}{\partial x}=0\\
&X\sigma_{xy}\frac{\partial u_x}{\partial x}+T(\beta-2)\sigma_-\frac{\partial u_x}{\partial y}+Y\sigma_{xy}\frac{\partial v_x}{\partial y}+T(\beta-2)\sigma_+\frac{\partial v_x}{\partial x}\\
&\quad-\sigma_{xy}\left(D_1\frac{\partial\sigma_+}{\partial x}+D_2\frac{\partial\sigma_-}{\partial x}\right)-D_4(\beta-2)\sigma_+\frac{\partial\sigma_{xy}}{\partial x}=0\\
&X\sigma_{xy}\frac{\partial u_x}{\partial x}-T(\beta+2)\sigma_-\frac{\partial u_x}{\partial y}-Y\sigma_{xy}\frac{\partial v_y}{\partial y}-T(\beta+2)\sigma_-\frac{\partial v_x}{\partial x}\\
&\quad-\sigma_{xy}\left(D_2\frac{\partial\sigma_+}{\partial x}+D_1\frac{\partial\sigma_-}{\partial x}\right)+D_4(\beta+2)\sigma_-\frac{\partial\sigma_{xy}}{\partial x}=0
\end{aligned}\right\}\tag{3.1}
$$

$\beta>2$的Hill屈服条件为[2]：

$$
-(\beta-2)\sigma_+^2+(\beta+2)\sigma_-^2+\sigma_{xy}^2=1\tag{3.2}
$$

若取[2]

$$
\sigma_+=\frac{1}{\sqrt{\beta-2}}\cdot\text{sh}\varphi\sin\omega,\quad\sigma_-=\frac{1}{\sqrt{\beta+2}}\cdot\text{ch}\varphi\sin\omega,\quad\sigma_{xy}=\cos\omega\tag{3.3}
$$

则式(3.2)恒被满足．ω和φ都只是θ的函数．

同文献[1]一样，将(3.3)代入式(3.1)，并采用如下变换：

$$\frac{\partial}{\partial x}=-\frac{\sin\theta}{r}\frac{d}{d\theta},\quad \frac{\partial}{\partial y}=\frac{\cos\theta}{r}\frac{d}{d\theta} \tag{3.4}$$

我们就得到关于新变量$d\omega/d\theta$，$d\varphi/d\theta$，$du_z/d\theta$和$dv_z/d\theta$的方程组：

$$\left.\begin{aligned}
&\left[X\sin\theta\cos\omega\left(\frac{\mathrm{sh}\varphi}{\sqrt{\beta-2}}+\frac{\mathrm{ch}\varphi}{\sqrt{\beta+2}}\right)+T\cos\theta\sin\omega\right]\frac{d\omega}{d\theta}\\
&\quad+X\sin\theta\sin\omega\left(\frac{\mathrm{ch}\varphi}{\sqrt{\beta-2}}+\frac{\mathrm{sh}\varphi}{\sqrt{\beta+2}}\right)\frac{d\varphi}{d\theta}-\alpha^2\mu\sin\theta\frac{du_z}{d\theta}=0\\
&\left[Y\cos\theta\cos\omega\left(\frac{\mathrm{sh}\varphi}{\sqrt{\beta-2}}-\frac{\mathrm{ch}\varphi}{\sqrt{\beta+2}}\right)+T\sin\theta\sin\omega\right]\frac{d\omega}{d\theta}\\
&\quad+Y\cos\theta\sin\omega\left(\frac{\mathrm{ch}\varphi}{\sqrt{\beta-2}}-\frac{\mathrm{sh}\varphi}{\sqrt{\beta+2}}\right)\frac{d\varphi}{d\theta}+\alpha^2\mu\sin\theta\frac{dv_z}{d\theta}=0\\
&\left[\left(\frac{D_1\mathrm{sh}\varphi}{\sqrt{\beta-2}}+\frac{D_2\mathrm{ch}\varphi}{\sqrt{\beta+2}}\right)\cos^2\omega-\sqrt{\beta-2}\,D_4\mathrm{sh}\varphi\sin^2\omega\right]\sin\theta\frac{d\omega}{d\theta}\\
&\quad+\left[\left(\frac{D_1\mathrm{ch}\varphi}{\sqrt{\beta-2}}+\frac{D_2\mathrm{sh}\varphi}{\sqrt{\beta+2}}\right)\sin\omega\cos\omega\right]\sin\theta\frac{d\varphi}{d\theta}\\
&\quad-(X\sin\theta\cos\omega-T\sqrt{\beta-2}\cos\theta\,\mathrm{sh}\varphi\sin\omega)\frac{du_z}{d\theta}\\
&\quad+(Y\cos\theta\cos\omega-T\sqrt{\beta-2}\sin\theta\,\mathrm{sh}\varphi\sin\omega)\frac{dv_z}{d\theta}=0\\
&\left[\left(\frac{D_2\mathrm{sh}\varphi}{\sqrt{\beta-2}}+\frac{D_3\mathrm{ch}\varphi}{\sqrt{\beta+2}}\right)\cos^2\omega+\sqrt{\beta+2}\,D_4\mathrm{ch}\varphi\sin^2\omega\right]\sin\theta\frac{d\omega}{d\theta}\\
&\quad+\left[\left(\frac{D_2\mathrm{ch}\varphi}{\sqrt{\beta-2}}+\frac{D_3\mathrm{sh}\varphi}{\sqrt{\beta+2}}\right)\sin\omega\cos\omega\right]\sin\theta\frac{d\varphi}{d\theta}\\
&\quad+[X\sin\theta\cos\omega+T\sqrt{\beta+2}\cos\theta\,\mathrm{ch}\varphi\sin\omega]\frac{du_z}{d\theta}\\
&\quad+[Y\cos\theta\cos\omega-T\sqrt{\beta+2}\sin\theta\,\mathrm{sh}\varphi\sin\omega]\frac{dv_z}{d\theta}=0
\end{aligned}\right\} \tag{3.5}$$

根据(3.5)，我们得到下面两种塑性区：

1. **均匀塑性区**($d\omega/d\theta=0$，$d\varphi/d\theta=0$，$du_z/d\theta=0$，$dv_z/d\theta=0$)

在均匀塑性区内，我们有

$$\omega=\mathrm{const},\quad \varphi=\mathrm{const},\quad u_z=\mathrm{const},\quad v_z=\mathrm{const} \tag{3.6}$$

所以，均匀塑性区是均匀应力区．

2. **非均匀塑性区**($d\omega/d\theta\neq 0$，$d\varphi/d\theta\neq 0$，$du_z/d\theta\neq 0$，$dv_z/d\theta\neq 0$)

非均匀塑性区的存在条件是(3.5)的参数行列式为零，即

$$A\sin^2\omega+B\cos^2\omega+C\sin\omega\cos\omega=0 \tag{3.7}$$

其中

$$\left.\begin{aligned}
A=&T^2\alpha^2\sin^2\theta\{\mu(\alpha^2\sin^2\theta-1)[(\beta-2)D_3\mathrm{sh}^2\varphi+(\beta+2)D_1\mathrm{ch}^2\varphi\\
&+\sqrt{\beta^2-4}\,D_2\mathrm{sh}2\varphi]-\sqrt{\beta^2-4}\,(X^2\sin^2\theta-Y^2\cos^2\theta)\mathrm{sh}2\varphi\\
&+(X^2\sin^2\theta+Y^2\cos^2\theta)\cdot[2+\beta(\mathrm{sh}^2\varphi+\mathrm{ch}^2\varphi)]\}\\
&+T^2[\sqrt{\beta+2}\,(X\sin^2\theta-Y\cos^2\theta)\mathrm{ch}\varphi\\
&+\sqrt{\beta-2}\,(X\sin^2\theta+Y\cos^2\theta)\mathrm{sh}\varphi]^2\\
B=&\alpha^2\mu\sin^2\theta\{2D_2(X^2\sin^2\theta-Y^2\cos^2\theta)-(D_1+D_3)\\
&\cdot(X^2\sin^2\theta+Y^2\cos^2\theta)-(D_2^2-D_1D_3)\alpha^2\mu\sin^2\theta\}+X^2Y^2\sin^2(2\theta)\\
C=&T\{\alpha^2\mu\sin^2\theta\,[\sqrt{\beta-2}\,(D_3-D_2)X\mathrm{sh}\varphi+\sqrt{\beta+2}\,(D_1+D_2)Y\mathrm{ch}\varphi\\
&+\sqrt{\beta-2}\,(D_2+D_3)Y\mathrm{sh}\varphi]-2XY\sin2\theta\\
&\cdot[\sqrt{\beta+2}\,(X\sin^2\theta-Y\cos^2\theta)\mathrm{ch}\varphi+\sqrt{\beta-2}\,(X\sin^2\theta+Y\cos^2\theta)\mathrm{sh}\varphi]\}
\end{aligned}\right\}\quad(3.8)$$

由式(3.5)的前三式解出$d\varphi/d\theta$，$du_z/d\theta$和$dv_z/d\theta$，然后积分得到：

$$\left.\begin{aligned}
\varphi&=\varphi_0+\int_{\omega_0}^{\omega}\frac{\Delta_1}{\sin\omega\cdot\Delta}\cdot d\omega\\
u_z&=u_{z0}+\frac{T}{\mu}\int_{\omega_0}^{\omega}\frac{\Delta_2}{\alpha^2\sin\theta\cdot\Delta}\cdot d\omega\\
v_z&=v_{z0}+\frac{T}{\mu}\int_{\omega_0}^{\omega}\frac{\Delta_3}{\alpha^2\sin\theta\cdot\Delta}\cdot d\omega
\end{aligned}\right\}\quad(3.9)$$

其中

$$\begin{aligned}
\Delta=&\Big\{\frac{\mathrm{ch}\varphi}{\sqrt{\beta-2}}[(X^2-\alpha^2\mu D_1)\sin^2\theta+Y^2\cos^2\theta]\\
&+\frac{\mathrm{sh}\varphi}{\sqrt{\beta+2}}[(X^2-\alpha^2\mu D_2)\sin^2\theta-Y^2\cos^2\theta]\Big\}\cos\omega\\
&-T\sqrt{\beta-2}\sin2\theta\Big[\frac{X+Y}{4\sqrt{\beta-2}}\mathrm{sh}2\varphi+\frac{X-Y}{2\cdot\sqrt{\beta+2}}\mathrm{sh}^2\varphi\Big]\sin\omega\\
\Delta_1=&\Big\{\frac{\mathrm{sh}\varphi}{\sqrt{\beta-2}}[(\alpha^2\mu D_1-X^2)\sin^2\theta-Y\cos^2\theta]\\
&+\frac{\mathrm{ch}\varphi}{\sqrt{\beta+2}}[(\alpha^2\mu D_2-X^2)\sin^2\theta+Y^2\cos^2\theta\Big\}\cos^2\omega\\
&+T\sqrt{\beta-2}\cdot\sin2\theta\Big[\frac{X+Y}{2\cdot\sqrt{\beta-2}}(\mathrm{sh}^2\varphi-1)+\frac{X-Y}{4\cdot\sqrt{\beta+2}}\mathrm{sh}2\varphi\Big]\\
&\cdot\sin\omega\cos\omega+\sqrt{\beta-2}\,(T^2-\alpha^2\mu D_4\sin^2\theta)\mathrm{sh}\varphi\cdot\sin^2\omega\\
\Delta_2=&\frac{X\sin\theta}{\sqrt{B^2-4}}[\alpha^2\mu(D_2-D_1)\sin^2\theta-2Y^2\cos^2\theta]\cos^2\omega\\
&+T^2\sqrt{\beta-2}\cdot\sin\theta\Big\{\frac{\mathrm{sh}2\varphi}{2\cdot\sqrt{\beta-2}}[X(1-\alpha^2)\sin^2\theta-Y\cos^2\theta]\\
&+\frac{\mathrm{sh}^2\varphi}{\sqrt{\beta+2}}[X(1-\alpha^2)\sin^2\theta+Y\cos^2\theta]\Big\}\sin^2\omega\\
&-T\cos\theta\Big\{\frac{\mathrm{ch}\omega}{\sqrt{\beta-2}}[(XY+\alpha^2\mu D_1)\sin^2\theta-Y^2\cos^2\theta]
\end{aligned}\quad(3.10)$$

$$+\frac{\mathrm{sh}\varphi}{\sqrt{\beta+2}}[(\alpha^2\mu D_2-XY)\sin^2\theta+Y^2\cos^2\theta]\Big\}\sin\omega\cos\omega$$

$$\Delta_3=\frac{Y\sin\theta\sin2\theta}{2\cdot\sqrt{\beta^2-4}}[2X^2-\alpha^2\mu(D_1+D_2)]\cos^2\omega$$

$$+T^2\sqrt{\beta-2}\cdot\cos\theta\Big\{\frac{\mathrm{sh}2\varphi}{2\cdot\sqrt{\beta-2}}[(X+Y\alpha^2)\sin^2\theta-Y\cos^2\theta]$$

$$+\frac{\mathrm{sh}^2\varphi}{\sqrt{\beta+2}}[(X-Y\alpha^2)\sin^2\theta+Y\cos^2\theta]\Big\}\sin^2\omega$$

$$-T\sin\theta\Big\{\frac{\mathrm{ch}\varphi}{\sqrt{\beta-2}}[(X^2-\alpha^2\mu D_1)\sin^2\theta-XY\cos^2\theta]$$

$$+\frac{\mathrm{sh}\varphi}{\sqrt{\beta+2}}[(X^2-\alpha^2\mu D_2)\sin^2\theta+3XY\cos^2\theta]\Big\}\sin\omega\cos\omega$$

下面给出两种各向异性塑性特殊情形的一般表达式：

1． $X=Z=\sqrt{3}T$ 的情形

在这种情形中，$\beta=X/Y$，于是有

（1） 均匀塑性区

$$\omega=\mathrm{const},\quad \varphi=\mathrm{const},\quad u_x=\mathrm{const},\quad v_x=\mathrm{const} \tag{3.6}$$

（2） 非均匀塑性区

式(3.7)形式上保持不变，而式(3.8)则变为：

$$\left.\begin{aligned}
A=&\alpha^2\sin^2\theta\{(\alpha^2\sin^2\theta-1)[(\beta-2)(1+2\nu\beta+\beta^2)\mathrm{sh}^2\varphi+(2+\beta)(1-2\nu\beta+\beta^2)\mathrm{ch}^2\varphi\\
&-(1-\beta^2)\sqrt{\beta^2-4}\,\mathrm{sh}2\varphi]-\sqrt{\beta^2-4}(\beta^2\sin^2\theta-\cos^2\theta)\mathrm{sh}2\varphi\\
&+(\beta^2\sin^2\theta+\cos^2\theta)[2+\beta(1+2\mathrm{sh}^2\varphi)]\}\\
&+2(1+\nu)[\sqrt{\beta+2}(\beta\sin^2\theta-\cos^2\theta)\mathrm{ch}\varphi+\sqrt{\beta-2}(\beta\sin^2\theta+\cos^2\theta)\mathrm{sh}\varphi]^2\\
B=&6\{\alpha^2\sin^2\theta[(1-\nu)\alpha^2\sin^2\theta-2]+(1+\nu)\sin^2(2\theta)\}\\
C=&2\sqrt{3}\alpha^2\sin^2\theta[(1+\nu)(1+\beta)\sqrt{\beta+2}\,\mathrm{sh}\varphi+(\beta-\nu)\sqrt{\beta+2}\,\mathrm{ch}\varphi]\\
&-4\sqrt{3}(1+\nu)\sin2\theta[\sqrt{\beta+2}(\beta\sin^2\theta-\cos^2\theta)\mathrm{ch}\varphi\\
&+\sqrt{\beta-2}(\beta\sin^2\theta+\cos^2\theta)\mathrm{sh}\varphi]
\end{aligned}\right\} \tag{3.11}$$

式(3.9)形式上保持不变，而式(3.10)则变为：

$$\Delta=\Big\{\Big[\Big(\beta^2-\frac{\alpha^2\mu}{E}(1-2\nu\beta+\beta^2)\Big)\sin^2\theta+\cos^2\theta\Big]\frac{\mathrm{ch}\varphi}{\sqrt{\beta-2}}$$

$$+\Big[\Big(\beta^2-\frac{\alpha^2\mu}{E}(\beta^2-1)\Big)\sin^2\theta-\cos^2\theta\Big]\frac{\mathrm{sh}\varphi}{\sqrt{\beta+2}}\Big\}\cos\omega$$

$$-\frac{\beta}{2\sqrt{3}}\sqrt{\beta-2}\,\sin2\theta\Big[\frac{1+\beta}{2\cdot\sqrt{\beta-2}}\cdot\mathrm{sh}2\varphi-\frac{1-\beta}{\sqrt{\beta+2}}\mathrm{sh}^2\varphi\Big]\sin\omega$$

$$\Delta_1=\Big\{\Big[\Big(\frac{\alpha^2\mu}{E}(1-2\nu\beta+\beta^2)-\beta^2\Big)\sin^2\theta-\cos^2\theta\Big]\frac{\mathrm{sh}\omega}{\sqrt{\beta-2}}$$

$$+\Big[\Big(\frac{\alpha^2\mu}{E}(\beta^2-1)-\beta^2\Big)\sin^2\theta+\cos^2\theta\Big]\frac{\mathrm{ch}\omega}{\sqrt{\beta+2}}\Big\}\cos^2\omega$$

$$+\frac{\beta\sqrt{\beta-2}}{2\sqrt{3}}\sin2\theta\left[\frac{1+\beta}{\sqrt{\beta-2}}(\text{sh}^2\varphi-1)+\frac{1-\beta}{2\cdot\sqrt{\beta+2}}\text{sh}2\varphi\right]\sin\omega\cos\omega$$

$$+\frac{\beta^2}{3}\sqrt{\beta-2}(1-\alpha^2\sin^2\theta)\text{sh}\varphi\cdot\sin^2\omega$$

$$\Delta_2=\frac{2\sqrt{3}\sin\theta}{\beta\sqrt{\beta^2-4}}\left\{\left[\frac{\alpha^2\mu}{E}(\nu\beta-1)\sin^2\theta-\cos^2\theta\right]\right\}\cos^2\omega \tag{3.12}$$

$$+\frac{\beta\sqrt{\beta-2}}{2\sqrt{3}}\sin\theta\left\{[\beta(1-\alpha^2)\sin^2\theta-\cos^2\theta]\frac{\text{sh}2\varphi}{\sqrt{\beta-2}}\right.$$

$$\left.+[\beta(1-\alpha^2)\sin^2\theta+\cos^2\theta]\frac{2\text{sh}^2\varphi}{\sqrt{\beta+2}}\right\}\sin^2\omega$$

$$-\cos\theta\left\{\left[\left(\beta+\frac{\alpha^2\mu}{E}(1-2\nu\beta+\beta^2)\sin^2\theta-\cos^2\theta\right)\right]\frac{\text{ch}\varphi}{\sqrt{\beta-2}}\right.$$

$$\left.-\left[\left(\frac{\alpha^2\mu}{E}(\beta^2-1)-\beta^2\right)\sin^2\theta+\cos^2\theta\right]\frac{\text{sh}\varphi}{\sqrt{\beta+2}}\right\}\sin\omega\cos\omega$$

$$\Delta_3=\frac{\sqrt{3}\sin\theta\sin2\theta}{\sqrt{\beta^2-4}}\left[\beta-\frac{\alpha^2\mu}{E}(\beta-\nu)\right]\cos^2\omega$$

$$+\frac{\beta\sqrt{\beta-2}}{2\sqrt{3}}\cos\theta\left\{[(\beta+\alpha^2)\sin^2\theta-\cos^2\theta]\frac{\text{sh}2\varphi}{\sqrt{\beta-2}}\right.$$

$$\left.+[(\beta-\alpha^2)\sin^2\theta+\cos^2\theta]\frac{2\text{sh}^2\varphi}{\sqrt{\beta+2}}\right\}\sin^2\omega$$

$$-\sin\theta\left\{\left[\left((\beta^2-\frac{\alpha^2\mu}{E}(1-2\nu\beta+\beta^2)\right)\sin^2\theta-\beta\text{cns}^2\theta\right]\frac{\text{ch}\varphi}{\sqrt{\beta-2}}\right.$$

$$\left.+\left[\left(\beta^2+\frac{\alpha^2\mu}{E}(1-\beta^2)\right)\sin^2\theta+3\beta\cos^2\theta\right]\frac{\text{sh}\varphi}{\sqrt{\beta+2}}\right\}\sin\omega\cos\omega$$

2. $Y=Z=\sqrt{3}T$ 的情形

在这种情形中，$\beta=Y/Z$，于是有

(1) 均匀塑性区

$$\omega=\text{const},\ \varphi=\text{const},\ u_x=\text{const},\ v_x=\text{const} \tag{3.6}$$

(2) 非均匀塑性区

式(3.7)形式上保持不变，而式(3.8)则变为：

$$\left.\begin{aligned}
A=&\alpha^2\sin^2\theta\{(\alpha^2\sin^2\theta-1)[(\beta-2)(1+2\nu\beta+\beta^2)\text{sh}^2\varphi+(\beta+2)\\
&\cdot(1-2\nu\beta+\beta^2)\text{ch}^2\varphi+(1-\beta^2)\sqrt{\beta^2-4}\,\text{sh}2\varphi]-\sqrt{\beta^2-4}\,(\sin^2\theta\\
&-\beta^2\cos^2\theta)\text{sh}2\varphi+(\sin^2\theta+\beta^2\cos^2\theta)[2+\beta(1+\text{sh}^2\varphi)]\}+2(1+\nu)\\
&\cdot[\sqrt{\beta+2}\,(\sin^2\theta-\beta\cos^2\theta)\text{ch}\varphi+\sqrt{\beta-2}\,(\sin^2\theta+\beta\cos^2\theta)\text{sh}\varphi]^2\\
B=&-6\{\alpha^2\sin^2\theta[2+(1-\nu)\alpha^2\sin^2\theta]+(1+\nu)\sin^2(2\theta)\}\\
C=&\,2\sqrt{3}\alpha^2\sin^2\theta[(1+\nu)(1+\beta)\sqrt{\beta-2}\,\text{sh}\varphi+(1-\nu\beta)\sqrt{\beta+2}\,\text{ch}\varphi]\\
&-4\sqrt{3}(1+\nu)\sin2\theta[\sqrt{\beta+2}\,(\sin^2\theta-\beta\cos^2\theta)\text{ch}\varphi\\
&+\sqrt{\beta-2}\,(\sin^2\theta+\beta\cos^2\theta)\text{sh}\varphi]
\end{aligned}\right\} \tag{3.13}$$

式(3.9)形式上保持不变，而式(3.10)则变为：

$$\Delta=\left\{\left[\left(1-\frac{\alpha^2\mu}{E}(1-2\nu\beta+\beta^2)\right)\sin^2\theta+\beta^2\cos^2\theta\right]\frac{\mathrm{ch}\varphi}{\sqrt{\beta-2}}\right.$$
$$\left.+\left[\left(1-\frac{\alpha^2\mu}{E}(1-\beta^2)\right)\sin^2\theta-\beta^2\cos^2\theta\right]\frac{\mathrm{sh}\varphi}{\sqrt{\beta+2}}\right\}\cos\omega$$
$$-\frac{\beta\sqrt{\beta-2}}{2\sqrt{3}}\sin2\theta\left[\frac{1+\beta}{2\cdot\sqrt{\beta-2}}\mathrm{sh}2\varphi+\frac{1-\beta}{\sqrt{\beta+2}}\mathrm{sh}^2\varphi\right]\sin\omega$$

$$\Delta_1=\left\{\left[\left(\frac{\alpha^2\mu}{E}(1-2\nu\beta+\beta^2)-1\right)\sin^2\theta-\beta^2\cos^2\theta\right]\frac{\mathrm{sh}\varphi}{\sqrt{\beta-2}}\right.$$
$$\left.+\left[\left(\frac{\alpha^2\mu}{E}(1-\beta^2)-1\right)\sin^2\theta+\beta^2\cos^2\theta\right]\frac{\mathrm{sh}\varphi}{\sqrt{\beta+2}}\right\}\cos^2\omega$$
$$+\frac{\beta\sqrt{\beta-2}}{2\sqrt{3}}\sin2\theta\left[\frac{1+\beta}{\sqrt{\beta-2}}(\mathrm{sh}^2\varphi-1)+\frac{1-\beta}{2\cdot\sqrt{\beta+2}}\mathrm{sh}2\varphi\right]\sin\omega\cos\omega$$
$$+\frac{\beta\sqrt{\beta-2}}{3}(1-\alpha^2\sin^2\theta)\mathrm{sh}\varphi\sin^2\omega$$

$$\Delta_2=\frac{2\sqrt{3}\sin\theta}{\sqrt{\beta^2-4}}\left\{\frac{\alpha^2\mu}{E}(\nu-\beta)\sin^2\theta-\beta\cos^2\theta\right\}\cos^2\omega \tag{3.14}$$
$$+\frac{\beta\sqrt{\beta-2}}{2\sqrt{3}}\sin\theta\left\{[(1-\alpha^2)\sin^2\theta-\beta\cos^2\theta]\frac{\mathrm{sh}2\varphi}{\sqrt{\beta-2}}\right.$$
$$\left.+[(1-\alpha^2)\sin^2\theta+\beta\cos^2\theta]\frac{2\mathrm{sh}^2\varphi}{\sqrt{\beta+2}}\right\}\sin^2\omega$$
$$-\cos\theta\left\{\left[\left(\beta+\frac{\alpha^2\mu}{E}(1-2\nu\beta+\beta^2)\right)\sin^2\theta-\beta^2\cos^2\theta\right]\frac{\mathrm{ch}\varphi}{\sqrt{\beta-2}}\right.$$
$$\left.+\left[\left(\frac{\alpha^2\mu}{E}(1-\beta^2)-\beta\right)\sin^2\theta+\beta^2\cos^2\theta\right]\frac{\mathrm{sh}\varphi}{\sqrt{\beta+2}}\right\}\sin\omega\cos\omega$$

$$\Delta_3=\frac{\sqrt{3}\sin\theta\sin2\theta}{\sqrt{\beta^2-4}}\left[2-\frac{\alpha^2\mu}{E}(1-\nu\beta)\right]\cos^2\omega$$
$$+\frac{\beta\sqrt{\beta-2}}{2\sqrt{2}}\cos\theta\left\{[(1+\alpha^2\beta)\sin^2\theta-\beta\cos^2\theta]\frac{\mathrm{sh}2\varphi}{\sqrt{\beta-2}}\right.$$
$$\left.+[(1-\beta\alpha^2)\sin^2\theta+\beta\cos^2\theta]\frac{2\mathrm{sh}^2\varphi}{\sqrt{\beta+2}}\right\}\sin^2\omega$$
$$-\sin\theta\left\{\left[\left(1-\frac{\alpha^2\mu}{E}(1-2\nu\beta+\beta^2)\right)\sin^2\theta-\beta\cos^2\theta\right]\frac{\mathrm{ch}\varphi}{\sqrt{\beta-2}}\right.$$
$$\left.+\left[\left(1-\frac{\alpha^2\mu}{E}(1-\beta^2)\right)\sin^2\theta+3\beta\cos^2\theta\right]\frac{\mathrm{sh}\varphi}{\sqrt{\beta+2}}\right\}\sin\omega\cos\omega$$

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A Supplementary Study of Anisotropic Plastic Fields at a Rapidly Propagating Plane-Stress Crack-Tip (I)

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Abstract

The results in ref.[1] are not suitable for the case of $\beta \geqslant 2$. For this. reason, by using the methods in ref. [1] and ref. [2], we derive the general expressions of anisotropic plastic tilds at a rapidly propagating plane-stress crack-tip for both the cases of $\beta=2$ and $\beta>2$.

Key words rapid propagation, plane stress, crack-tip, anisotropic plastic fields, plastic zone, general expressions