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弹性地基上矩形板弯曲的CC型级数解*

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摘　要

本文利用双变量函数的Stockes变换，用CC型级数求弹性地基上矩形板弯曲问题的解析解．以弹性地基上四边自由矩形板中点作用一集中力为例给出数字计算结果．

关键词　弹性地基　矩形板　Stockes变换　解析解

一、引　言

放置在弹性地基上的自由矩形薄板，工程上是经常遇到的．要得到这类矩形薄板弯曲的精确解答不太容易．著作[2]使用叠加法将多种情况叠加以除去沿板边界的剪力及板角点的集中力．我们也曾选取满足全部边界条件和角点条件的挠曲函数，使用能量法来求近似解[4]，但在集中载荷作用下的近似解与著作[2]相差较大．

本文利用文[1]中双变量函数的Stockes变换，用CC型级数求弹性地基上矩形板弯曲问题的解析解．文中给出弹性地基上四边自由矩形板中点受集中力P时的数字计算结果，与著作[2]完全符合．

二、基　本　方　程

在无量纲坐标$\xi=x/a$，$\eta=y/b$中（图1），弹性地基（Winkler模型）上薄板弯曲的基本方程为

$$\frac{\partial^4 W}{\partial \xi^4}+\frac{2}{\beta^2}\frac{\partial^4 W}{\partial \xi^2 \partial \eta^2}+\frac{1}{\beta^4}\frac{\partial^4 W}{\partial \eta^4}+\pi^4 KW=\pi^4 Q(\xi,\eta)\quad(0<\xi,\eta<1)\qquad(2.1)$$

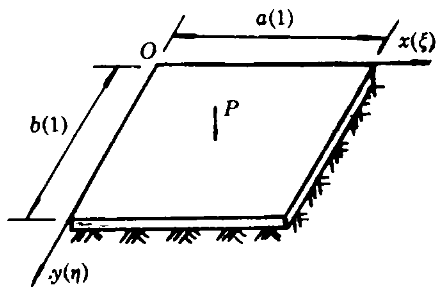

图1　弹性地基上的矩形板

* 1993年12月14日第一次收到．

$$
\left.\begin{aligned}
M_\xi &= -\frac{1}{\pi^2}\left(\frac{\partial^2 W}{\partial \xi^2}+\frac{\mu}{\beta^2}\frac{\partial^2 W}{\partial \eta^2}\right)\\
M_\eta &= -\frac{1}{\pi^2}\left(\frac{\partial^2 W}{\partial \eta^2}+\mu\beta^2\frac{\partial^2 W}{\partial \xi^2}\right)\\
V_\xi &= -\frac{1}{\pi^3}\left(\frac{\partial^3 W}{\partial \xi^3}+\frac{2-\mu}{\beta^2}\frac{\partial^3 W}{\partial \xi\partial \eta^2}\right)\\
V_\eta &= -\frac{1}{\pi^3}\left(\frac{\partial^3 W}{\partial \eta^3}+(2-\mu)\beta^2\frac{\partial^3 W}{\partial \xi^2\partial \eta}\right)\\
R_{\xi\eta} &= \frac{2(1-\mu)}{\pi^2\beta^2}\frac{\partial^2 W}{\partial \xi\partial \eta}
\end{aligned}\right\}\tag{2.2}
$$

这两式中引用了下列无量纲量

$$
\left.\begin{aligned}
&Q=\frac{q}{q_0},\ W=\frac{\pi^4 D}{a^4 q_0}w,\quad \beta=\frac{b}{a},\ K=\frac{ka^4}{\pi^4 D}\\
&M_\xi=\frac{\pi^2}{a^2q_0}M_x,\quad M_\eta=\frac{\pi^2\beta^2}{a^2q_0}M_y,\quad V_\xi=\frac{\pi}{aq_0}V_x\\
&V_\eta=\frac{\pi\beta^2}{aq_0}V_y,\quad R_{\xi\eta}=\frac{\pi^2}{abq_0}R
\end{aligned}\right\}\tag{2.3}
$$

其中，q是分布载荷；q_0是具有载荷量纲的因子；w是板的挠度；D是弯曲刚度；k是弹性地基的系数；M_x，M_y，V_x，V_y和R分别是板内弯矩、等效剪力和角点反力。

板的边界条件可以相应写出。设α是板边界外法线方向，应满足的边界条件是

$$
\left.\begin{aligned}
&\text{固支边界}\qquad W=\partial W/\partial\alpha=0\\
&\text{简支边界}\qquad W=M_\alpha=0\\
&\text{自由边界}\qquad M_\alpha=V_\alpha=0
\end{aligned}\right\}\tag{2.4}
$$

$$
\left.\begin{aligned}
&\text{固支边或简支边的角点}\qquad W=0\\
&\text{两邻边为自由边或固支边的角点}\qquad \partial^2W/\partial\xi\partial\eta=0
\end{aligned}\right\}\tag{2.5}
$$

三、一般解析解

位移W取附录中CC型级数，并把载荷分布函数Q展成CC型级数，设该级数系数为Q_{mn}，代入式(2.1)、(2.2)，经整理得到

$$
\begin{aligned}
&\frac{1}{2}\sum_{n=0}^{\infty}(E_n^1-E_n^0)\cos n\pi\eta+\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}[(-1)^m E_n^1-E_n^0-(-1)^m m^2C_n^1+m^2C_n^0\\
&\quad+m^4A_{mn}]\cos m\pi\xi\cos n\pi\eta+\frac{2}{\beta^2}\Big\{\frac{1}{2}(W''_{00}-W''_{01}-W''_{10}+W''_{11})\\
&\quad+\sum_{m=1}^{\infty}\Big[W''_{00}-W''_{01}-(-1)^mW''_{10}+(-1)^mW''_{11}-\frac{m^2}{2}(D_m^1-D_m^0)\Big]\cos m\pi\xi\\
&\quad+\sum_{n=1}^{\infty}\Big[W''_{00}-W''_{10}-(-1)^nW''_{01}+(-1)^nW''_{11}-\frac{n^2}{2}(C_n^1-C_n^0)\Big]\cos n\pi\eta
\end{aligned}
$$

$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}[2(W_{00}'^{\prime}-(-1)^{m}W_{10}'^{\prime}-(-1)^{n}W_{01}'^{\prime}+(-1)^{m+n}W_{11}'^{\prime})$$

$$-n^{2}((-1)^{m}C_{n}^{1}-C_{n}^{0})-m^{2}((-1)^{n}D_{m}^{1}-D_{m}^{0})+m^{2}n^{2}A_{mn}]\cos m\pi\xi\cos n\pi\eta\Big\}$$

$$+\frac{1}{\beta^{4}}\Big\{\frac{1}{2}\sum_{m=0}^{\infty}(F_{m}^{1}-F_{m}^{0})\cos m\pi\xi+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}[(-1)^{n}F_{m}^{1}-F_{m}^{0}$$

$$-(-1)^{n}n^{2}D_{m}^{1}+n^{2}D_{m}^{0}+n^{4}A_{mn}]\cos m\pi\xi\cos n\pi\eta\Big\}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}KA_{mn}\cos m\pi\xi\cos n\pi\eta=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}Q_{mn}\cos m\pi\xi\cos n\pi\eta \qquad (3.1)$$

和

$$M_{\xi}=-\sum_{n=0}^{\infty}\frac{1}{2}(C_{n}^{1}-C_{n}^{0})\cos n\pi\eta-\frac{\mu}{\beta^{2}}\sum_{m=0}^{\infty}\frac{1}{2}(D_{m}^{1}-D_{m}^{0})\cos m\pi\xi$$

$$-\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}[(-1)^{m}C_{n}^{1}-C_{n}^{0}-m^{2}A_{mn}]\cos n\pi\eta\cos m\pi\xi$$

$$-\frac{\mu}{\beta^{2}}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}[(-1)^{n}D_{m}^{1}-D_{m}^{0}-n^{2}A_{mn}]\cos m\pi\xi\cos n\pi\eta$$

$$0\leqslant\xi,\ \eta\leqslant1 \qquad (3.2)$$

$$M_{\eta}=-\sum_{m=0}^{\infty}\frac{1}{2}(D_{m}^{1}-D_{m}^{0})\cos m\pi\xi-\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}[(-1)^{n}D_{m}^{1}$$

$$-D_{m}^{0}-n^{2}A_{mn}]\cos m\pi\xi\cos n\pi\eta-\mu\beta^{2}\sum_{n=0}^{\infty}\frac{1}{2}(C_{n}^{1}-C_{n}^{0})\cos n\pi\eta$$

$$-\mu\beta^{2}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}[(-1)^{m}C_{n}^{1}-C_{n}^{0}-m^{2}A_{mn}]\cos n\pi\eta\cos m\pi\xi$$

$$0\leqslant\xi,\ \eta\leqslant1 \qquad (3.3)$$

$$V_{\xi}=-\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}[-(-1)^{m}mC_{n}^{1}+mC_{n}^{0}+m^{3}A_{mn}]\cos n\pi\eta\sin m\pi\xi$$

$$-\frac{2-\mu}{\beta^{2}}\sum_{m=1}^{\infty}\frac{1}{2}m(-D_{m}^{1}+D_{m}^{0})\sin m\pi\xi-\frac{2-\mu}{\beta^{2}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}[-(-1)^{n}mD_{m}^{1}$$

$$+mD_{m}^{0}+mn^{2}A_{mn}]\sin m\pi\xi\cos n\pi\eta \qquad 0<\xi<1,\ 0\leqslant\eta\leqslant1 \qquad (3.4)$$

$$(V_{\xi})_{\xi=r}=-\frac{\pi}{2}\sum_{n=0}^{\infty}E_{n}^{r}\cos n\pi\eta-\frac{2-\mu}{\beta^{2}}\ \frac{\pi}{2}\Big\{\frac{1}{2}(W_{r1}'^{\prime}-W_{r0}'^{\prime})$$

$$+\sum_{n=1}^{\infty}[(-1)^{n}W_{r1}'^{\prime}-W_{r0}'^{\prime}-n^{2}C_{n}^{r}]\cos n\pi\eta\Big\} \qquad (r=0,1) \qquad (3.5)$$

$$V_\eta=-\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}[-(-1)^n nD_m^1+nD_m^0+n^3A_{mn}]\cos m\pi\xi\sin n\pi\eta$$

$$-(2-\mu)\beta^2\sum_{n=1}^{\infty}\frac{1}{2}n(-C_n^1+C_n^0)\sin n\pi\eta-(2-\mu)\beta^2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}[-(-1)^m nC_n^1$$

$$+nC_n^0+m^2nA_{mn}]\sin n\pi\eta\cos m\pi\xi \qquad 0\leqslant\xi\leqslant 1,\ 0<\eta<1 \tag{3.6}$$

$$(V_\eta)_{\eta=s}=-\frac{\pi}{2}\sum_{m=0}^{\infty}F_m^s\cos m\pi\xi-(2-\mu)\beta^2\frac{\pi}{2}\left\{\frac{1}{2}(W_{1s}''-W_{0s}'')\right.$$

$$\left.+\sum_{m=1}^{\infty}[(-1)^m W_{1s}''-W_{0s}''-m^2D_m^s]\cos m\pi\xi\right\} \qquad (s=0,1) \tag{3.7}$$

$$(R_{\xi\eta})_{\xi=r,\eta=s}=\frac{(1-\mu)\pi^2}{2}W_{rs}'' \qquad (r=0,1;\ s=0,1) \tag{3.8}$$

式(3.1)～(3.8)中共有9类级数系数：C_n^0，C_n^1，D_m^0，D_m^1，E_n^0，E_n^1，F_m^0，F_m^1，A_{mn}和四个角点未知数W_{00}''，W_{01}''，W_{10}''，W_{11}''，可以通过方程(3.1)，8个边界条件方程(2.4)和4个角点条件方程(2.5)联立求解而确定．

四、算 例

现以弹性地基上四边自由矩形板中点受集中力P为例，说明解题过程．

根据问题的对称性有（取$q_0=P/ab$）

$$W=\sum_{m=0,2,\cdots}^{\infty}\sum_{n=0,2,\cdots}^{\infty}A_{mn}\cos m\pi\xi\cos n\pi\eta \tag{4.1}$$

$$\left.\begin{aligned}&Q_{00}=1,\ Q_{m0}=2(-1)^{m/2} && (m=2,4,6,\cdots)\\&Q_{0n}=2(-1)^{n/2} && (n=2,4,6,\cdots)\\&Q_{mn}=4(-1)^{(m+n)/2} && (m,n=2,4,6,\cdots)\end{aligned}\right\} \tag{4.2}$$

并且

$$-W_0'(\eta)=W_1'(\eta),\quad -W_0^{\cdot}(\xi)=W_1^{\cdot}(\xi)$$

$$-W_0'''(\eta)=W_1'''(\eta),\quad -W_0^{\cdot\cdot\cdot}(\xi)=W_1^{\cdot\cdot\cdot}(\xi)$$

即有

$$-C_n^0=C_n^1=C_n,\quad -D_m^0=D_m^1=D_m,\quad -E_n^0=E_n^1=E_n,\quad -F_m^0=F_m^1=F_m$$

$$(m,n=0,2,4,\cdots) \tag{4.3}$$

由自由角点条件得到

$$W_{00}''=W_{01}''=W_{10}''=W_{11}''=0 \tag{4.4}$$

由边界条件

$$(M_\xi)_{\xi=0,1}=(V_\xi)_{\xi=0,1}=0,\quad (M_\eta)_{\eta=0,1}=(V_\eta)_{\eta=0,1}=0$$

分别得到

$$\left.\begin{aligned}&C_0+\frac{\mu}{\beta^2}D_0+\sum_{m=2,4,\cdots}^{\infty}\left(2C_0+\frac{\mu}{\beta^2}D_m-m^2A_{m0}\right)=0\\&C_n+\frac{\mu}{\beta^2}(2D_0-n^2A_{0n})+\sum_{m=2,4,\cdots}^{\infty}\left[2\left(C_n+\frac{\mu}{\beta^2}D_m\right)-\left(m^2+\frac{\mu n^2}{\beta^2}\right)A_{mn}\right]=0\\&\qquad(n=2,4,6,\cdots)\end{aligned}\right\}\tag{4.5}$$

$$E_0=0,\quad E_n=[(2-\mu)/\beta^2]n^2C_n\qquad(n=2,4,6,\cdots)\tag{4.6}$$

$$\left.\begin{aligned}&D_0+\mu\beta^2C_0+\sum_{n=2,4,\cdots}^{\infty}(2D_0+\mu\beta^2C_n-n^2A_{0n})=0\\&D_m+\mu\beta^2(2C_0-m^2A_{m0})+\sum_{n=2,4,\cdots}^{\infty}[2(\mu\beta^2C_n+D_m)-(\mu\beta^2m^2+n^2)A_{mn}]=0\\&\qquad(m=2,4,6,\cdots)\end{aligned}\right\}\tag{4.7}$$

$$F_0=0,\quad F_m=(2-\mu)\beta^2m^2D_m\qquad(m=2,4,6,\cdots)\tag{4.8}$$

将式(4.1)、(4.2)、(4.3)、(4.4)代入方程式(3.1)，经整理可得

$$\left.\begin{aligned}&A_{00}=\frac{Q_{00}}{K},\quad A_{0n}=\frac{1}{((n^4/\beta^4)+K)}\left(Q_{0n}+\frac{2n^2D_0}{\beta^4}+\frac{\mu n^2}{\beta^2}C_n\right)\qquad(n=2,4,6,\cdots)\\&A_{m0}=\frac{1}{m^4+K}\left(Q_{m0}+2m^2C_0+\frac{\mu m^2}{\beta^2}D_m\right)\qquad(m=2,4,6,\cdots)\\&A_{mn}=\frac{1}{G_{mn}}\left[Q_{mn}+2\left(m^2+\frac{\mu}{\beta^2}n^2\right)C_n+\frac{2}{\beta^2}\left(\mu m^2+\frac{n^2}{\beta^2}\right)D_m\right]\qquad(m,n=2,4,6,\cdots)\end{aligned}\right\}\tag{4.9}$$

其中 $G_{mn}=(m^2+n^2/\beta^2)^2+K$。

将式(4.9)代入式(4.5)、(4.7)，得到无穷联立方程组，截取一定项数可以解出系数C_n，D_m。

数值例子

弹性地基上自由正方形板，集中载荷P作用在板中心，其$\mu=0.167$，$ka^4/D=10^4$。

对于方板，$\beta=1$，且有 $C_i=D_i(i=0,2,4,\cdots)$，将式(4.9)代入式(4.5)、(4.7)，可得相同式子如下：

$$\left.\begin{aligned}&(1+\mu)C_0+\sum_{n=2,4,\cdots}^{\infty}2\left(1-\frac{n^4}{n^4+K}\right)C_0+\sum_{n=2,4,\cdots}^{\infty}\mu\left(1-\frac{n^4}{n^4+K}\right)C_n\\&\quad=\sum_{n=2,4,\cdots}^{\infty}\frac{n^2Q_{0n}}{n^4+K}\\&2\mu\left(1-\frac{m^4}{m^4+K}\right)C_0+\left(1-\frac{\mu^2m^4}{m^4+K}\right)C_m+\sum_{n=2,4,\cdots}^{\infty}2\left[\mu-\frac{(\mu m^2+n^2)(m^2+\mu n^2)}{G_{mn}}\right]C_n\\&\quad+C_m\sum_{n=2,4,\cdots}^{\infty}2\left[1-\frac{(\mu m^2+n^2)^2}{G_{mn}}\right]=\frac{\mu m^2}{m^4+K}Q_{m0}+\sum_{n=2,4,\cdots}^{\infty}\frac{\mu m^2+n^2}{G_{mn}}Q_{mn}\\&\qquad(m=2,4,6,\cdots)\end{aligned}\right\}\tag{4.10}$$

在式(4.10)中截取一定项数可解出$C_i(i=0,2,4,\cdots)$，再代回式(4.9)可求出 $A_{mn}(m,n=0,2,4,\cdots)$，进而可求板内各点位移和内力。

计算结果见表1～3，表中N为所取级数项数。由表可见，位移和内力的计算值都收敛较快，且两者收敛速度一致。地基总反力的计算值均为P，正好与载荷平衡。

表1　　　挠度(w)和弯矩(M_x)的收敛情况

	$N=8$	$N=10$	$N=12$	$N=14$	$N=16$	著作[2]结果
$w_{\xi=0.5,\eta=0.5}$	$0.001230\frac{Pa^2}{D}$	$0.001238\frac{Pa^2}{D}$	$0.001243\frac{Pa^2}{D}$	$0.001245\frac{Pa^2}{D}$	$0.001247\frac{Pa^2}{D}$	$0.00125\frac{Pa^2}{D}$
$M_{x,\xi=0.5,\eta=0.625}$	$0.0391P$	$0.0416P$	$0.0382P$	$0.0365P$	$0.0391P$	$0.041P$

表2　　　板的挠度$w(\times Pa^2/D)$　　($N=20$)

η \ ξ	0.5	0.625	0.75	0.875	1
0.5	0.001249	0.000656	0.000179	0.000001	−0.000057
0.625	0.000656	0.000414	0.000118	−0.000008	−0.000052
0.75	0.000179	0.000118	0.000025	−0.000020	−0.000038
0.875	0.000001	−0.000008	−0.000020	−0.000022	−0.000021
1	−0.000057	−0.000052	−0.000038	−0.000021	−0.000012

表3　　　板的弯矩$M_x(\times P)$　　($N=20$)

η \ ξ	0.5	0.625	0.75	0.875	1
0.5	0.248	−0.013	−0.017	−0.008	0.000
0.625	0.039	−0.001	−0.012	−0.005	0.000
0.75	0.006	0.000	−0.004	−0.002	0.000
0.875	0.000	0.000	−0.001	0.000	0.000
1	0.000	−0.001	0.000	0.001	0.000

附录　CC型级数

$$W=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}\cos m\pi\xi\cos n\pi\eta,\qquad 0\leqslant\xi,\eta\leqslant1$$

$$\frac{\partial W}{\partial\xi}=-\pi\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}mA_{mn}\sin m\pi\xi\cos n\pi\eta,\qquad 0<\xi<1,\ 0\leqslant\eta\leqslant1$$

$$\left(\frac{\partial W}{\partial\xi}\right)_{\xi=r}=\frac{\pi^2}{2}W'_r(\eta)\qquad(r=0,1)$$

$$\frac{\partial W}{\partial\eta}=-\pi\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}nA_{mn}\sin n\pi\eta\cos m\pi\xi,\qquad 0\leqslant\xi\leqslant1,\ 0<\eta<1$$

$$\left(\frac{\partial W}{\partial\eta}\right)_{\eta=s}=\frac{\pi^2}{2}W'_s(\xi)\qquad(s=0,1)$$

$$\frac{\partial^2W}{\partial\xi^2}=\pi^2\left\{\frac{1}{2}\left[W'_1(\eta)-W'_0(\eta)\right]+\sum_{m=1}^{\infty}\left[(-1)^m W'_1(\eta)-W'_0(\eta)\right.\right.$$

$$- m^2 \sum_{n=0}^{\infty} A_{mn}\cos n\pi\eta \Big]\cos m\pi\xi \Big\} \qquad 0\leqslant\xi,\ \eta\leqslant 1$$

$$\frac{\partial^2 W}{\partial\eta^2} = \pi^2\Big\{\frac{1}{2}\Big[W_1^{\cdot}(\xi) - W_0^{\cdot}(\xi)\Big] + \sum_{n=1}^{\infty}\Big[(-1)^n W_1^{\cdot}(\xi) - W_0^{\cdot}(\xi)$$

$$-n^2 \sum_{m=0}^{\infty} A_{mn}\cos m\pi\xi \Big]\cos n\pi\eta \Big\} \qquad 0\leqslant\xi,\ \eta\leqslant 1$$

$$\frac{\partial^2 W}{\partial\xi\partial\eta} = \pi^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} mnA_{mn}\sin m\pi\xi\sin n\pi\eta, \qquad 0<\xi,\ \eta<1$$

$$\left(\frac{\partial^2 W}{\partial\xi\partial\eta}\right)_{\xi=r} = \frac{\pi^2}{2} W_r^{\prime\cdot}(\eta), \left(\frac{\partial^2 W}{\partial\xi\partial\eta}\right)_{\eta=s} = \frac{\pi^2}{2} W_s^{\prime\cdot}(\xi) \qquad (r=0,1;\ s=0,1)$$

$$\frac{\partial^3 W}{\partial\xi^3} = \pi^3 \sum_{m=1}^{\infty}\Big[-(-1)^m mW_1^{\prime}(\eta) + mW_0^{\prime}(\eta) + m^3 \sum_{n=0}^{\infty} A_{mn}\cos n\pi\eta\Big]\sin m\pi\xi$$

$$0<\xi<1,\ 0\leqslant\eta\leqslant 1$$

$$\left(\frac{\partial^3 W}{\partial\xi^3}\right)_{\xi=r} = \frac{\pi^4}{2} W_r^{\prime\prime\prime}(\eta) \qquad (r=0,1)$$

$$\frac{\partial^3 W}{\partial\eta^3} = \pi^3 \sum_{n=1}^{\infty}\Big[-(-1)^n nW_1^{\cdot}(\xi) + nW_0^{\cdot}(\xi) + n^3 \sum_{m=0}^{\infty} A_{mn}\cos m\pi\xi\Big]\sin n\pi\eta$$

$$0\leqslant\xi\leqslant 1,\ 0<\eta<1$$

$$\left(\frac{\partial^3 W}{\partial\eta^3}\right)_{\eta=s} = \frac{\pi^4}{2} W_s^{\cdot\cdot\cdot}(\xi) \qquad (s=0,1)$$

$$\frac{\partial^3 W}{\partial\xi^2\partial\eta} = \pi^2\Big\{\frac{1}{2}\Big[W_1^{\prime\cdot}(\eta) - W_0^{\prime\cdot}(\eta)\Big] + \sum_{m=1}^{\infty}\Big[(-1)^m W_1^{\prime\cdot}(\eta) - W_0^{\prime\cdot}(\eta)$$

$$+ \pi m^2 \sum_{n=1}^{\infty} nA_{mn}\sin n\pi\eta\Big]\cos m\pi\xi\Big\} \qquad 0\leqslant\xi\leqslant 1,\ 0<\eta<1$$

$$\left(\frac{\partial^3 W}{\partial\xi^2\partial\eta}\right)_{\eta=s} = \frac{\pi^2}{2} W_s^{\prime\cdot}(\xi) \qquad (s=0,1)$$

$$\frac{\partial^3 W}{\partial\xi\partial\eta^2} = \pi^2\Big\{\frac{1}{2}\Big[W_1^{\prime\cdot}(\xi) - W_0^{\prime\cdot}(\xi)\Big] + \sum_{n=1}^{\infty}\Big[(-1)^n W_1^{\prime\cdot}(\xi) - W_0^{\prime\cdot}(\xi)$$

$$+ \pi n^2 \sum_{m=1}^{\infty} mA_{mn}\sin m\pi\xi\Big]\cos n\pi\eta\Big\} \qquad 0<\xi<1,\ 0\leqslant\eta\leqslant 1$$

$$\left(\frac{\partial^3 W}{\partial\xi\partial\eta^2}\right)_{\xi=r} = \frac{\pi^2}{2} W_r^{\prime\cdot\cdot}(\eta) \qquad (r=0,1)$$

$$\frac{\partial^4 W}{\partial\xi^4} = \pi^4\Big\{\frac{1}{2}\Big[W_1^{\prime\prime\prime}(\eta) - W_0^{\prime\prime\prime}(\eta)\Big] + \sum_{m=1}^{\infty}\Big[(-1)^m W_1^{\prime\prime\prime}(\eta) - W_0^{\prime\prime\prime}(\eta) - (-1)^m m^2 W_1^{\prime}(\eta)$$

$$+ m^2 W_0^{\prime}(\eta) + m^4 \sum_{n=0}^{\infty} A_{mn}\cos n\pi\eta\Big]\cos m\pi\xi\Big\} \qquad 0\leqslant\xi,\ \eta\leqslant 1$$

$$\frac{\partial^4 W}{\partial \eta^4}=\pi^4\left\{\frac{1}{2}\left[W_1^{\cdots}(\xi)-W_0^{\cdots}(\xi)\right]+\sum_{n=1}^{\infty}\left[(-1)^n W_1^{\cdots}(\xi)-W_0^{\cdots}(\xi)-(-1)^n n^2 W_1^{\cdot}(\xi)\right.\right.$$

$$\left.\left.+n^2 W_0^{\cdot}(\xi)+n^4\sum_{m=0}^{\infty}A_{mn}\cos m\pi\xi\right]\cos n\pi\eta\right\},\qquad 0\leqslant\xi,\ \eta\leqslant 1$$

$$\frac{\partial^4 W}{\partial\xi^2\partial\eta^2}=\pi^2\left\{\frac{1}{2}\left[W_1^{\prime\cdot\cdot}(\eta)-W_0^{\prime\cdot\cdot}(\eta)+W_1^{\prime\prime\cdot}(\xi)-W_0^{\prime\prime\cdot}(\xi)\right]\right.$$

$$+\sum_{m=1}^{\infty}\left[(-1)^m W_1^{\prime\cdot\cdot}(\eta)-W_0^{\prime\cdot\cdot}(\eta)\right]\cos m\pi\xi+\sum_{n=1}^{\infty}\left[(-1)^n W_1^{\prime\prime\cdot}(\xi)\right.$$

$$\left.\left.-W_0^{\prime\prime\cdot}(\xi)\right]\cos n\pi\eta\right\}+\pi^4\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}m^2n^2A_{mn}\cos m\pi\xi\cos n\pi\eta,\qquad 0\leqslant\xi,\ \eta\leqslant 1$$

上面各式中边界值$W_r^{\prime}(\eta)$，$W_s^{\cdot}(\xi)$，$W_r^{\prime\prime\prime}(\eta)$，$W_s^{\cdots}(\xi)$及其导数用以下单级数表示：

$$W_r^{\prime}(\eta)=\sum_{n=0}^{\infty}C_n^r\cos n\pi\eta,\qquad 0\leqslant\eta\leqslant 1,\ r=0,1$$

$$W_s^{\cdot}(\xi)=\sum_{m=0}^{\infty}D_m^s\cos m\pi\xi,\qquad 0\leqslant\xi\leqslant 1,\ s=0,1$$

$$W_r^{\prime\prime\prime}(\eta)=\sum_{n=0}^{\infty}E_n^r\cos n\pi\eta,\qquad 0\leqslant\eta\leqslant 1,\ r=0,1$$

$$W_s^{\cdots}(\xi)=\sum_{m=0}^{\infty}F_m^s\cos m\pi\xi,\qquad 0\leqslant\xi\leqslant 1,\ s=0,1$$

$$W_r^{\prime\cdot}(\eta)=-\pi\sum_{n=1}^{\infty}nC_n^r\sin n\pi\eta,\qquad 0<\eta<1$$

$$\left(W_r^{\prime\cdot}(\eta)\right)_{\eta=s}=\frac{\pi^2}{2}W_{rs}^{\prime\cdot},\qquad r=0,1;\ s=0,1$$

$$W_r^{\prime\cdot\cdot}(\eta)=\pi^2\left\{\frac{1}{2}\left(W_{r1}^{\prime\cdot}-W_{r0}^{\prime\cdot}\right)+\sum_{n=1}^{\infty}\left[(-1)^n W_{r1}^{\prime\cdot}-W_{r0}^{\prime\cdot}-n^2C_n^r\right]\cos n\pi\eta\right\},\qquad 0\leqslant\eta\leqslant 1$$

$$W_s^{\prime\cdot}(\xi)=-\pi\sum_{m=1}^{\infty}mD_m^s\sin m\pi\xi,\qquad 0<\xi<1$$

$$(W_s^{\prime\cdot}(\xi))_{\xi=r}=\frac{\pi^2}{2}W_{rs}^{\prime\cdot},\qquad r=0,1;\ s=0,1$$

$$W_s^{\prime\prime\cdot}(\xi)=\pi^2\left\{\frac{1}{2}\left(W_{1s}^{\prime\cdot}-W_{0s}^{\prime\cdot}\right)+\sum_{m=1}^{\infty}\left[(-1)^m W_{1s}^{\prime\cdot}-W_{0s}^{\prime\cdot}-m^2D_m^s\right]\cos m\pi\xi\right\},\qquad 0\leqslant\xi\leqslant 1$$

其中C_n^r，D_m^s，E_n^r，F_m^s是级数系数。

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CC Series Solution for Bending of Rectangular Plates on Elastic Foundations

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Abstract

The analytical solution for the bending problem of the rectangular plates on an elastic foundation is investigated by using the Stockes' transformation of a double variables function. The numerical results for the rectangular plates with free edges on the elastic foundations under a concentrated force are given in the example.

Key words elastic foundation, rectangular plate, Stockes transformation, analytical solution