应用数学和力学，第 15 卷第 8 期（1994年 8 月）
Applied Mathematics and Mechanics

应用数学和力学编委会编
重 庆 出 版 社 出 版

圆柱型正交各向异性圆形薄板的非线性非对称弯曲问题(Ⅰ)*

秦圣立　　黄家寅

（曲阜师范大学物理系，1994年 2 月 1 日收到）

摘　　要

本文利用“两变量法”[1]研究了圆柱型正交各向异性圆形薄板在非均布横向载荷作用下的非线性非对称弯曲问题，并得到在周边为可移夹支条件下的本问题的一致有效渐近解。

关键词　正交各向异性圆板　非对称弯曲　两变量法

一、引　　言

由于复合材料板、壳的广泛应用，关于它们的弯曲问题和屈曲问题越来越引起人们的注意。文献[2]研究了复合材料非均匀圆柱正交各向异性圆板弯曲问题。文献[2]中的问题属于线性轴对称弯曲问题，它的控制方程是一个三阶的线性常微分方程，文中求得了这一问题的精确解。本文利用“两变量法”[1]和“混合摄动法”[3]研究圆柱型正交各向异性圆形薄板的非线性非轴对称弯曲问题，并求得了这一问题的一致有效渐近解。最后作为实例考察了文献[2]中的一种情况，我们将所得到的渐近解与精确解进行比较，其结果基本一致。由于篇幅所限，我们将分为两篇文章讨论。

二、圆柱型正交各向异性圆形薄板的基本方程和边界条件

圆柱型正交各向异性圆形薄板的挠度函数$W(r,\theta)$和应力函数$\Phi(r,\theta)$满足以下的方程[4]：

$$\left.\begin{aligned}&\frac{D_1}{h}W^{\cdot\cdot\cdot\cdot}+2\frac{D_3}{h}W^{\cdot\cdot\prime\prime}+\frac{D_2}{h}W^{\prime\prime\prime\prime}=L(W,\Phi)+\frac{q(r,\theta)}{h}\\&\delta_1'\Phi^{\cdot\cdot\cdot\cdot}+2\delta_3'\Phi^{\cdot\cdot\prime\prime}+\delta_2'\Phi^{\prime\prime\prime\prime}=-\frac{1}{2}L(W,W)\end{aligned}\right\}\tag{2.1}$$

其中

$$W^{\cdot\cdot\cdot\cdot}=\frac{\partial^4W}{\partial r^4},\quad W^{\cdot\cdot\prime\prime}=\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial W}{\partial\theta}$$

---

* 江福汝推荐。

$$W''''=\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W$$

$$L(W,\Phi)=\frac{\partial^2 W}{\partial r^2}\left(\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}\right)+\left(\frac{1}{r}\frac{\partial W}{\partial r}+\frac{1}{r^2}\frac{\partial^2 W}{\partial\theta^2}\right)\frac{\partial^2\Phi}{\partial r^2}$$

$$-2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial W}{\partial\theta}\right)$$

其中，

$$D_1=\frac{E_r h^3}{12(1-\mu_r\mu_\theta)},\qquad D_2=\frac{E_\theta h^3}{12(1-\mu_r\mu_\theta)}$$

$$D_3=D_1\mu_\theta+2D_k,\qquad D_k=\frac{h^3}{12}G$$

$$\delta_1'=\frac{1}{E_\theta},\ \delta_2'=\frac{1}{E_r},\ 2\delta_3'=\frac{1}{G}-2\frac{\mu_r}{E_r}$$

h是圆形薄板的厚度；E_r，E_θ分别为径向r和环向θ的杨氏模量；μ_r，μ_θ分别为径向r和环向θ的泊松比；G为圆板的剪切模量；D_1，D_2分别为径向和环向的抗弯刚度；D_3是折合刚度；D_k是抗扭刚度．

假设圆板的半径为C，且周边为可移夹支，则边界条件为：

$$\left.\begin{aligned}&W(r,\theta)|_{r=C}=0,\ W_{,r}(r,\theta)|_{r=C}=0\\&\left[\frac{1}{r}\Phi_{,r}+\frac{1}{r^2}\Phi_{,\theta\theta}\right]\Big|_{r=C}=0,\ \left[\frac{1}{r}\Phi_{,r}+\frac{1}{r^2}\Phi_{,\theta\theta}\right]\Big|_{r=0}\text{为有限值}\\&\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right]\Big|_{r=C}=0,\ \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right]\Big|_{r=0}\text{为有限值}\\&W;\ W_{,r},\quad\text{在}r=0\text{处取有限值}\end{aligned}\right\}\tag{2.2}$$

我们引入如下的无量纲变量[5]：

$$\tilde W=\frac{W}{C},\quad \tilde r=\frac{r}{C},\quad \tilde\Phi=\frac{\Phi}{E_rC^2},\quad \tilde q=\frac{Cq}{hE_r}$$

并将方程(2.1)和边界条件(2.2)无量纲化(略去符号"～")，则得：

$$\left.\begin{aligned}&\varepsilon_1^2W''''+\varepsilon_2^2W''''+\varepsilon_3^2W''''=L(W,\Phi)+q(r,\theta)\\&\Phi''''+\delta_1\Phi''''+\delta_2\Phi''''=-\frac{1}{2}\delta_2L(W,W)\end{aligned}\right\}\tag{2.3}$$

若假定$G<E_\theta<E_r$，则其中

$$\varepsilon_1^2=\frac{h^2}{12(1-\mu_r\mu_\theta)C^2}<<1$$

$$\varepsilon_2^2=2\left(\varepsilon_2^2\mu_\theta+\frac{1}{6}\frac{G}{E_r}\frac{h^2}{C^2}\right)<<1$$

$$\varepsilon_3^2=\frac{E_\theta}{E_r}\varepsilon_1^2=\delta_2\varepsilon_1^2<<1$$

$$\delta_1=E_\theta\left(\frac{1}{G}-2\frac{\mu_r}{E_r}\right),\quad \delta_2=\frac{E_\theta}{E_r}<1$$

无量纲化的边界条件为：

$$\left.\begin{aligned}
&W(r,\theta)|_{r=1}=0,\quad \left.\frac{\partial W(r,\theta)}{\partial r}\right|_{r=1}=0\\
&\left[\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}\right]\bigg|_{r=0}\text{取有限值}\\
&\left[\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}\right]\bigg|_{r=1}=0\\
&\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right]\bigg|_{r=0}\text{取有限值},\quad \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right]\bigg|_{r=1}=0\\
&W,\ \frac{\partial W}{\partial r},\quad \text{在}r=0\text{处取有限值}
\end{aligned}\right\}\tag{2.4}$$

三、微分算子展开

为了得到递推方程和递推边界条件，首先将微分算子展开，为此，我们在 $r=1$ 的邻域内引入如下的变量[5]：

$$\xi=\frac{u(r,\theta)}{\varepsilon_1^p},\ \eta=r,\ \theta=\theta\tag{3.1}$$

把对r,θ的偏导数换为对ξ，η，θ的偏导数，并把ξ，η，θ视为独立变量，即

$$\frac{\partial}{\partial r}=\frac{\partial}{\partial\xi}\frac{\partial\xi}{\partial r}+\frac{\partial}{\partial\eta}\frac{\partial\eta}{\partial r}=\varepsilon_1^{-p}\left(u_{,r}\frac{\partial}{\partial\xi}+\varepsilon_1^p\frac{\partial}{\partial\eta}\right)\tag{3.2}$$

$$\frac{\partial}{\partial\theta}=\frac{\partial}{\partial\xi}\frac{\partial\xi}{\partial\theta}+\frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial\theta}=\varepsilon_1^{-p}\left(u_{,\theta}\frac{\partial}{\partial\xi}+\varepsilon_1^p\frac{\partial}{\partial\theta}\right)\tag{3.3}$$

…………………………………………

$$\frac{\partial^2}{\partial r^2}=\varepsilon_1^{-2p}\left\{u_{,r}^2\frac{\partial^2}{\partial\xi^2}+\varepsilon_1^p\left(2u_{,r}\frac{\partial^2}{\partial\xi\partial\eta}+u_{,rr}\frac{\partial}{\partial\xi}\right)+\varepsilon_1^{2p}\frac{\partial^2}{\partial\eta^2}\right\}\tag{3.4}$$

$$\frac{\partial^2}{\partial\theta^2}=\varepsilon_1^{-2p}\left\{u_{,\theta}^2\frac{\partial^2}{\partial\xi^2}+\varepsilon_1^p\left(2u_{,\theta}\frac{\partial^2}{\partial\xi\partial\theta}+u_{,\theta\theta}\frac{\partial}{\partial\xi}\right)+\varepsilon_1^{2p}\frac{\partial^2}{\partial\theta^2}\right\}\tag{3.5}$$

$$\begin{aligned}\frac{\partial^3}{\partial r^3}=\varepsilon_1^{-3p}\bigg\{&u_{,r}^3\frac{\partial^3}{\partial\xi^3}+\varepsilon_1^p\left(3u_{,r}^2\frac{\partial^3}{\partial\xi^2\partial\eta}+3u_{,r}u_{,rr}\frac{\partial^2}{\partial\xi^2}\right)\\
&+\varepsilon_1^{2p}\left[3u_{,r}\frac{\partial^3}{\partial\xi\partial\eta^2}+3u_{,rr}\frac{\partial^2}{\partial\xi\partial\eta}+u_{,rrr}\frac{\partial}{\partial\xi}\right]+\varepsilon_1^{3p}\frac{\partial^3}{\partial\eta^3}\bigg\}\end{aligned}\tag{3.6}$$

$$\begin{aligned}\frac{\partial^4}{\partial r^4}=\varepsilon_1^{-4p}\bigg\{&u_{,r}^4\frac{\partial^4}{\partial\xi^4}+\varepsilon_1^p\left[4u_{,r}^3\frac{\partial^4}{\partial\eta\partial\xi^3}+6u_{,r}^2u_{,rr}\frac{\partial^3}{\partial\xi^3}\right]\\
&+\varepsilon_1^{2p}\left[6u_{,r}^2\frac{\partial^4}{\partial\xi^2\partial\eta^2}+12u_{,r}u_{,rr}\frac{\partial^3}{\partial\eta\partial\xi^2}+(3u_{,rr}^2+4u_{,r}u_{,rrr})\frac{\partial^2}{\partial\xi^2}\right]\\
&+\varepsilon_1^{3p}\left[4u_{,r}\frac{\partial^4}{\partial\xi\partial\eta^3}+6u_{,rr}\frac{\partial^3}{\partial\xi\partial\eta^2}+4u_{,rrr}\frac{\partial^2}{\partial\xi\partial\eta}+u_{,rrrr}\frac{\partial}{\partial\xi}\right]\\
&+\varepsilon_1^{4p}\frac{\partial^4}{\partial\eta^4}\bigg\}\end{aligned}\tag{3.7}$$

四、递推方程和递推边界条件

假设挠度函数$W(r,\theta)$和应力函数$\Phi(r,\theta)$对ε_1为N阶和对ε_2为M阶的渐近展开式为：

$$W(r,\theta,\varepsilon_1,\varepsilon_2)=\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}W_{nm}(r,\theta)$$

$$+\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\alpha_1)p}\varepsilon_2^{(m+\alpha_2)p}\nu_{nm}(\xi,\eta,\theta) \tag{4.1}$$

$$\Phi(r,\theta,\varepsilon_1,\varepsilon_2)=\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}\varphi_{nm}(r,\theta)$$

$$+\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\beta_1)p}\varepsilon_2^{(m+\beta_2)p}\psi_{nm}(\xi,\eta,\theta) \tag{4.2}$$

把微分算子(3.2)～(3.7)和挠度函数和应力函数的展开式(4.1)和(4.2)代入偏微分方程组(2.3)，并注意到边界层型函数ν_{nm}和ψ_{nm}的性质，则得：

$$\Big\{\varepsilon_1^2\Big[\frac{\partial^4}{\partial r^4}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}W_{nm}\Big)$$

$$+\delta_2\Big(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big)\Big(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}W_{nm}\Big)\Big]$$

$$+\varepsilon_2^2\Big[\Big(\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}W_{nm}\Big)\Big]$$

$$-\Big[\frac{\partial^2}{\partial r^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}W_{nm}\Big)\Big(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big)$$

$$\cdot\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}\varphi_{nm}\Big)$$

$$+\Big(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}W_{nm}\Big)$$

$$\cdot\frac{\partial^2}{\partial r^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}\varphi_{nm}\Big)$$

$$-2\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial}{\partial\theta}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}\varphi_{nm}\Big)\Big(\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}\Big)$$

$$\cdot\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}W_{nm}\right)\Big]\Big\}$$

$$+\Big\{\varepsilon_1^2\Big[\varepsilon_1^{-4p}\left(\sum_{i=0}^{4}\varepsilon_1^{ip}D_i\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\alpha_1)p}\varepsilon_2^{(m+\alpha_2)p}\nu_{nm}\right)$$

$$+\delta_2\Big(\frac{1}{\eta}\varepsilon_1^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\right)$$

$$+\frac{1}{\eta^2}\varepsilon_1^{-2p}\left(\sum_{i=0}^{2}\varepsilon_1^{ip}B_i'\right)\Big)\Big(\frac{1}{\eta}\varepsilon_1^{-p}\cdot\left(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\right)$$

$$+\frac{1}{\eta^2}\varepsilon_1^{-2p}\left(\sum_{i=0}^{2}\varepsilon_1^{ip}B_i'\right)\Big)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\alpha_1)p}\varepsilon_2^{(m+\alpha_2)p}\nu_{nm}\right)\Big]$$

$$+\varepsilon_2^2\Big[\varepsilon_1^{-4p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\right)\frac{1}{\eta}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}B_i\right)\left(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\right)$$

$$\cdot\frac{1}{\eta}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}B_i\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\alpha_1)p}\varepsilon_2^{(m+\alpha_2)p}\nu_{nm}\right)\Big]$$

$$-\Big[\varepsilon_1^{-2p}\left(\sum_{i=0}^{2}\varepsilon_1^{ip}A_i'\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}W_{nm}\right)\cdot\Big(\frac{1}{\eta}\varepsilon_1^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\right)$$

$$+\frac{1}{\eta^2}\varepsilon_1^{-2p}\left(\sum_{i=0}^{2}\varepsilon_1^{ip}B_i'\right)\Big)\cdot\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\beta_1)p}\varepsilon_2^{(m+\beta_2)p}\psi_{nm}\right)$$

$$+\Big(\frac{1}{\eta}\varepsilon_1^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\right)$$

$$+\frac{1}{\eta^2}\varepsilon_1^{-2p}\left(\sum_{i=0}^{2}\varepsilon_1^{ip}B_i'\right)\Big)\cdot\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}W_{nm}\right)$$

$$\cdot\Big(\varepsilon_1^{-2p}\left(\sum_{i=0}^{2}\varepsilon_1^{ip}A_i'\right)\Big)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\beta_1)p}\varepsilon_2^{(m+\beta_2)p}\psi_{nm}\right)$$

$$-2\varepsilon_1^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\right)\Big(\frac{1}{\eta}\varepsilon_1^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}B_i\right)\Big)$$

$$\cdot\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\beta_1)p}\varepsilon_2^{(m+\beta_2)p}\psi_{nm}\right)\Big(\varepsilon_1^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\right)\Big)$$

$$\cdot\Big(\frac{1}{\eta}\varepsilon_1^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}B_i\right)\Big)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}W_{nm}\right)\Big]$$

$$
\begin{aligned}
&-\Big[\varepsilon_1^{-2p}\Big(\sum_{i=0}^{2}\varepsilon_1^{ip}A_i'\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\alpha_1)p}\varepsilon_2^{(m+\alpha_2)p}\nu_{nm}\Big)\\
&\cdot\Big(\frac{1}{\eta}\varepsilon_1^{-p}\Big(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\Big)\\
&+\frac{1}{\eta^2}\varepsilon_1^{-2p}\Big(\sum_{i=0}^{2}\varepsilon_1^{ip}B_i'\Big)\Big)\cdot\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}\varphi_{nm}\Big)\\
&+\Big(\frac{1}{\eta}\varepsilon_1^{-p}\Big(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\Big)\\
&+\frac{1}{\eta^2}\varepsilon_1^{-2p}\Big(\sum_{i=0}^{2}\varepsilon_1^{ip}B_i'\Big)\Big)\cdot\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\alpha_1)p}\varepsilon_2^{(m+\alpha_2)p}\nu_{nm}\Big)\\
&\cdot\Big(\varepsilon_1^{-2p}\Big(\sum_{i=0}^{2}\varepsilon_1^{ip}A_i'\Big)\Big)\cdot\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}\varphi_{nm}\Big)\\
&-2\varepsilon_1^{-p}\Big(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\Big)\Big(\frac{1}{\eta}\varepsilon_1^{-p}\Big(\sum_{i=0}^{1}\varepsilon_1^{ip}B_i\Big)\Big)\\
&\cdot\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}\varphi_{nm}\Big)\Big(\varepsilon_1^{-p}\Big(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\Big)\Big)\\
&\cdot\Big(\frac{1}{\eta}\varepsilon_1^{-p}\Big(\sum_{i=0}^{1}\varepsilon_1^{ip}B_i\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\alpha_1)p}\varepsilon_2^{(m+\alpha_2)p}\nu_{nm}\Big)\Big)\Big]\\
&-\Big[\varepsilon_1^{-2p}\Big(\sum_{i=0}^{2}\varepsilon_1^{ip}A_i'\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\alpha_1)p}\varepsilon_2^{(m+\alpha_2)p}\nu_{nm}\Big)\\
&\cdot\Big(\frac{1}{\eta}\varepsilon_1^{-p}\Big(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\Big)+\frac{1}{\eta^2}\varepsilon_1^{-2p}\Big(\sum_{i=0}^{2}\varepsilon_1^{ip}B_i'\Big)\Big)\\
&\cdot\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\beta_1)p}\varepsilon_2^{(m+\beta_2)p}\psi_{nm}\Big)\Big]\\
&+\Big(\frac{1}{\eta}\varepsilon_1^{-p}\Big(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\Big)+\frac{1}{\eta^2}\varepsilon_1^{-2p}\Big(\sum_{i=0}^{2}\varepsilon_1^{ip}B_i'\Big)\Big)\\
&\cdot\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\alpha_1)p}\varepsilon_2^{(m+\alpha_2)p}\nu_{nm}\Big)\\
&\cdot\Big[\varepsilon_1^{-2p}\Big(\sum_{i=0}^{2}\varepsilon_1^{ip}A_i'\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\beta_1)p}\varepsilon_2^{(m+\beta_2)p}\psi_{nm}\Big)
\end{aligned}
$$

$$-2\varepsilon_1^{-p}\Big(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\Big)\Big(\frac{1}{\eta}\varepsilon_1^{-p}\Big(\sum_{i=0}^{1}\varepsilon_1^{ip}B_i\Big)\Big)$$

$$\cdot\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\beta_1)p}\varepsilon_2^{(m+\beta_2)p}\psi_{nm}\Big)$$

$$\cdot\Big(\varepsilon_1^{-p}\Big(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\Big)\Big)\Big(\frac{1}{\eta}\varepsilon_1^{-p}\sum_{i=0}^{1}\varepsilon_1^{ip}B_i\Big)$$

$$\cdot\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\alpha_1)p}\varepsilon_2^{(m+\alpha_2)p}\nu_{nm}\Big)\Big]\Big\}=q(r,\theta) \tag{4.3}$$

$$\Big\{\frac{\partial^4}{\partial r^4}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}\varphi_{nm}\Big)+\delta_1\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}$$

$$\cdot\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}\varphi_{nm}\Big)$$

$$+\delta_2\Big[\Big(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big)\Big(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big)\cdot\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}\varphi_{nm}\Big)$$

$$+\frac{\partial^2}{\partial r^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}W_{nm}\Big)\Big(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big)$$

$$\cdot\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}W_{nm}\Big)$$

$$-\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial}{\partial\theta}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}W_{nm}\Big)$$

$$\cdot\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial}{\partial\theta}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}W_{nm}\Big)\Big]\Big\}$$

$$+\Big\{\varepsilon_1^{-4p}\Big(\sum_{i=0}^{4}\varepsilon_1^{ip}D_i\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\beta_1)p}\varepsilon_2^{(m+\beta_2)p}\psi_{nm}\Big)$$

$$+\delta_1\Big(\varepsilon_1^{-p}\Big(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\Big)\Big)\Big(\frac{1}{\eta}\varepsilon_1^{-p}\Big(\sum_{i=0}^{1}\varepsilon_1^{-ip}B_i\Big)\Big)$$

$$\cdot\Big(\varepsilon_1^{-p}\Big(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\Big)\Big)\Big(\frac{1}{\eta}\varepsilon_1^{-p}\Big(\sum_{i=0}^{1}\varepsilon_1^{-p}B_i\Big)\Big)$$

$$\cdot\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\beta_1)p}\varepsilon_2^{(m+\beta_2)p}\psi_{nm}\Big)+\delta_2\Big[\Big(\frac{1}{\eta}\varepsilon_1^{-p}\Big(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\Big)$$

$$+\frac{1}{\eta^2}\varepsilon_1^{-2p}\left(\sum_{i=0}^{2}\varepsilon_1^{ip}B_i'\right)\Bigg)$$

$$\cdot\left(\frac{1}{\eta}\varepsilon_1^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\right)+\frac{1}{\eta^2}\varepsilon_1^{-2p}\left(\sum_{i=0}^{2}\varepsilon_1^{ip}B_i'\right)\right)$$

$$\cdot\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\beta_1)p}\varepsilon_2^{(m+\beta_2)p}\psi_{nm}\right)+\varepsilon_1^{-2p}\left(\sum_{i=0}^{2}\varepsilon_1^{ip}A_i'\right)$$

$$\cdot\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}W_{nm}\right)\cdot\Bigg(\frac{1}{\eta}\varepsilon_1^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\right)$$

$$+\frac{1}{\eta^2}\varepsilon_1^{-2p}\left(\sum_{i=0}^{2}\varepsilon_1^{ip}B_i'\right)\Bigg)\cdot\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\alpha_1)p}\varepsilon_2^{(m+\alpha_2)p}\nu_{nm}\right)$$

$$+\varepsilon_1^{-2p}\left(\sum_{i=0}^{2}\varepsilon_1^{ip}A_i'\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\alpha_1)p}\varepsilon_2^{(m+\alpha_2)p}\nu_{nm}\right)$$

$$\cdot\left(\frac{1}{\eta}\varepsilon_1^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\right)+\frac{1}{\eta^2}\varepsilon_1^{-2p}\left(\sum_{i=0}^{2}\varepsilon_1^{ip}B_i'\right)\right)$$

$$\cdot\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}W_{nm}\right)-2\varepsilon_1^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\right)\left(\frac{1}{\eta}\varepsilon_1^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}B_i\right)\right)$$

$$\cdot\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}W_{nm}\right)\cdot\left(\varepsilon_1^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\right)\right)$$

$$\cdot\left(\frac{1}{\eta}\varepsilon_1^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}B_i\right)\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\alpha_1)p}\varepsilon_2^{(m+\alpha_2)p}\nu_{nm}\right)$$

$$+\varepsilon_1^{-2p}\left(\sum_{i=0}^{2}\varepsilon_1^{ip}A_i'\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\alpha_1)p}\varepsilon_2^{(m+\alpha_2)p}\nu_{nm}\right)$$

$$\cdot\Bigg(\frac{1}{\eta}\varepsilon_1^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\right)$$

$$+\frac{1}{\eta^2}\varepsilon_1^{-2p}\left(\sum_{i=0}^{2}\varepsilon_1^{ip}B_i'\right)\Bigg)\cdot\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\alpha_1)p}\varepsilon_2^{(m+\alpha_2)p}\nu_{nm}\right)$$

$$-\varepsilon_1^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\right)\left(\frac{1}{\eta}\varepsilon_1^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}B_i\right)\right)$$

$$\cdot\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\alpha_1)p}\varepsilon_2^{(m+\alpha_2)p}\nu_{nm}\right)$$

$$\cdot \varepsilon_2^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\right)\left(\frac{1}{\eta}\varepsilon_1^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}B_i\right)\right)$$

$$\cdot\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\alpha_1)p}\varepsilon_2^{(m+\alpha_2)p}\nu_{nm}\right)\Big]\Big\}=0 \tag{4.4}$$

把微分算子(3.2)～(3.7)和挠度函数$W(r,\theta)$以及应力函数$\Phi(r,\theta)$的展开式(4.1)和(4.2)代入边界条件(2.4)，则得：

$$\left[\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}W_{nm}\right)\Big|_{r=1}+\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\alpha_1)p}\varepsilon_2^{(m+\alpha_2)p}\nu_{nm}\right)\Big|_{\eta=1}\right]=0 \tag{4.5}$$

$$\left[\frac{\partial}{\partial r}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}W_{nm}\right)\Big|_{r=1}+\left(\varepsilon_1^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\right)\right.\right.$$

$$\left.\left.\cdot\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\alpha_1)p}\varepsilon_2^{(m+\alpha_2)p}\nu_{mn}\right)\right)\Big|_{\eta=1}\right]=0 \tag{4.6}$$

$$\left[\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}\varphi_{nm}\right)\Big|_{r=1}\right.$$

$$+\left(\frac{1}{\eta}\varepsilon_1^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\right)+\frac{1}{\eta^2}\varepsilon_1^{-2p}\left(\sum_{i=0}^{2}\varepsilon_1^{ip}B_i'\right)\right)$$

$$\left.\cdot\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\beta_1)p}\varepsilon_2^{(m+\beta_2)p}\psi_{nm}\right)\Big|_{\eta=1}\right]=0 \tag{4.7}$$

$$\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}\varphi_{nm}\right)\Big|_{r=1}\right.$$

$$+\varepsilon_1^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\right)\left(\frac{1}{\eta}\varepsilon_1^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}B_i\right)\right)$$

$$\left.\cdot\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\beta_1)p}\varepsilon_2^{(m+\beta_2)p}\psi_{nm}\right)\Big|_{\eta=1}\right]=0 \tag{4.8}$$

$$\left[\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}\varphi_{nm}\right)+\left(\frac{1}{\eta}\varepsilon_1^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\right)\right.\right.$$

$$\left.\left.+\frac{1}{\eta^2}\varepsilon_1^{-2p}\left(\sum_{i=0}^{2}\varepsilon_1^{ip}B_i'\right)\right)\cdot\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\beta_1)p}\varepsilon_2^{(m+\beta_2)p}\psi_{nm}\right)\right]\Big|_{r=0}\text{取有限值} \tag{4.9}$$

$$\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}\varphi_{nm}\right)\right.$$

$$+\varepsilon_1^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\right)\left(\frac{1}{\eta}\varepsilon_1^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}B_i\right)\right)$$

$$\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\beta_1)p}\varepsilon_2^{(m+\beta_2)p}\psi_{nm}\right)\Bigg]\Bigg|_{r=0}\text{取有限值} \tag{4.10}$$

$$\left.\begin{array}{l}\left[\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}W_{mn}\right)+\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\alpha_1)p}\varepsilon_2^{(m+\alpha_2)p}\nu_{nm}\right)\right]\\ \left[\frac{\partial}{\partial r}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{np}\varepsilon_2^{mp}W_{mn}\right)\right.\\ \qquad\left.+\varepsilon_1^{-p}\left(\sum_{i=0}^{1}\varepsilon_1^{ip}A_i\right)\cdot\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{(n+\alpha_1)p}\varepsilon_2^{(m+\alpha_2)p}\nu_{nm}\right)\right]\end{array}\right\} \tag{4.11}$$

在$r=0$处(4.11)为有限值．其中

$$A_0=u_{,r}\frac{\partial}{\partial\xi},\quad A_1=\frac{\partial}{\partial\eta};\quad B_0=u_{,\theta}\frac{\partial}{\partial\xi},\quad B_1=\frac{\partial}{\partial\theta}$$

$$A'_0=u^2_{,r}\frac{\partial^2}{\partial\xi^2},\quad A'_1=2u_{,r}\frac{\partial^2}{\partial\xi\partial\eta}+u_{,rr}\frac{\partial}{\partial\xi},\quad A'_2=\frac{\partial^2}{\partial\eta^2}$$

$$B'_0=u^2_{,\theta}\frac{\partial^2}{\partial\xi^2},\quad B'_1=2u_{,\theta}\frac{\partial^2}{\partial\xi\partial\eta}+u_{,\theta\theta}\frac{\partial}{\partial\xi},\quad B'_2=\frac{\partial^2}{\partial\theta^2}$$

$$D_0=u^4_{,r}\frac{\partial^4}{\partial\xi^4},\qquad D_1=4u^3_{,r}\frac{\partial^4}{\partial\xi^3\partial\eta}+6u^2_{,r}u_{,rr}\frac{\partial^3}{\partial\xi^3}$$

$$D_2=6u^2_{,r}\frac{\partial^4}{\partial\xi^2\partial\eta^2}+12u_{,r}u_{,rr}\frac{\partial^3}{\partial\xi^2\partial\eta}+(3u^2_{,rr}+4u_{,r}u_{,rrr})\frac{\partial^2}{\partial\xi^2}$$

$$D_3=4u_{,r}\frac{\partial^4}{\partial\xi\partial\eta^3}+6u_{,rr}\frac{\partial^3}{\partial\xi\partial\eta^2}+4u_{,rrr}\frac{\partial^2}{\partial\xi\partial\eta}+u_{,rrrr}\frac{\partial}{\partial\xi}$$

$$D_4=\frac{\partial^4}{\partial\eta^4}$$

为了得到递推方程和递推边界条件，首先确定α_1，α_2，β_1和β_2的值．由边界条件 (4.6) 和(4.7)比较ε_1的最低次幂项知，应取$\alpha_1=1$，$\beta_1=2$，而比较ε_2的最低次幂项知，应取$\alpha_2=\beta_2=0$，再由方程(4.3)的第二个大括号中，比较ε_1 的最低次幂项可知，应取 $\alpha_1=2$；而由方程(4.4)的第二个大括号中，比较ε_2 的最低次幂项可知，应取 $\beta_1=4$，所以我们取 $\alpha_1=2$，$\beta_1=4$，$\alpha_2=\beta_2=0$，把$\alpha_1=2$，$\beta_1=4$，$\alpha_2=\beta_2=0$代入方程(4.3)，并比较 ε_1 的最低次幂项，可知应取$p=1$．

我们把$\alpha_1=2$，$\beta_1=4$，$\alpha_2=\beta_2=0$ 和 $p=1$代入方程 (4.3)、(4.4) 和边界条件 (4.5)～(4.11)，并比较 $\varepsilon_1\varepsilon_2$ 的同次幂项的系数，由方程(4.3) 和 (4.4)的第一个大括号，比较 $\varepsilon_1^0\varepsilon_2^0$ 同次幂项的系数，得递推方程和递推边界条件：

$$\frac{\partial^2W_{00}}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{00}+\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{00}\frac{\partial^2}{\partial r^2}\varphi_{00}$$

$$-2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\varphi_{00}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}W_{00}\right)=-q(r,\theta) \tag{4.12}$$

$$\begin{aligned}&\frac{\partial^4\varphi_{00}}{\partial r^4}+\delta_1\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi_{00}}{\partial\theta}\right)\\&\quad+\delta_2\Big[\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{00}\\&\quad+\frac{\partial^2W_{00}}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{00}\\&\quad-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)W_{00}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)W_{00}\Big]=0\end{aligned} \tag{4.13}$$

$$W_{00}|_{r=1}=0 \tag{4.14}$$

$$\left.\frac{\partial W_{00}}{\partial r}\right|_{r=1}=0 \tag{4.15}$$

$$\left.\left[\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{00}\right]\right|_{r=1}=0 \tag{4.16}$$

$$\left.\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\varphi_{00}\right]\right|_{r=1}=0 \tag{4.17}$$

和

$$\left[\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{00}\right] \tag{4.18}$$

$$\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\varphi_{00}\right] \tag{4.19}$$

$$W_{00};\ \frac{\partial W_{00}}{\partial r} \tag{4.20}$$

在$r=0$处(4.18)～(4.20)均为有限值。

对应于$\varepsilon_1^1\varepsilon_2^0$幂次项的方程和边界条件为：

$$\begin{aligned}&\frac{\partial^2W_{10}}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{00}\ \frac{\partial^2W_{00}}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{10}\\&\quad+\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{10}\frac{\partial^2\varphi_{00}}{\partial r^2}+\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{00}\frac{\partial^2}{\partial r^2}\varphi_{10}\\&\quad-2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)W_{10}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\varphi_{00}\\&\quad-2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)W_{00}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\varphi_{10}=0\end{aligned} \tag{4.21}$$

$$\begin{aligned}&\frac{\partial^4\varphi_{10}}{\partial r^4}+\delta_1\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial\varphi_{10}}{\partial\theta}+\delta_2\Big[\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\\&\quad\cdot\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{10}+\frac{\partial^2W_{10}}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{00}\\&\quad+\frac{\partial^2W_{00}}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{10}-2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)W_{10}\end{aligned}$$

$$\cdot\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)W_{00}\Big]=0 \tag{4.22}$$

$$W_{10}|_{r=1}=0 \tag{4.23}$$

$$\left.\frac{\partial W_{10}}{\partial r}\right|_{r=1}+A_0\nu_{00}|_{r=1}=0 \tag{4.24}$$

$$\left.\left[\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{10}\right]\right|_{r=1}=0 \tag{4.25}$$

$$\left.\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\varphi_{10}\right]\right|_{r=1}=0 \tag{4.26}$$

和
$$\left[\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{10}\right] \tag{4.27}$$

$$\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\varphi_{10}\right] \tag{4.28}$$

$$W_{10};\quad \frac{\partial W_{10}}{\partial r} \tag{4.29}$$

在$r=0$处(4.27)～(4.29)均为有限值。

对应于$\varepsilon_1^0\varepsilon_2^1$幂次项的方程和边界条件为：

$$\begin{aligned}&\frac{\partial^2 W_{01}}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{00}+\frac{\partial^2 W_{00}}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{01}\\&+\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{01}\frac{\partial^2}{\partial r^2}\varphi_{00}+\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{00}\frac{\partial^2}{\partial r^2}\varphi_{01}\\&-2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\varphi_{01}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)W_{00}\\&-2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\varphi_{00}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)W_{01}=0\end{aligned} \tag{4.30}$$

$$\begin{aligned}&\frac{\partial^4}{\partial r^4}\varphi_{01}+\delta_1\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}\varphi_{01}+\delta_2\Big[\Big(\frac{1}{r}\frac{\partial}{\partial r}\\&+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big)\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{01}+\frac{\partial^2}{\partial r^2}W_{01}\Big(\frac{1}{r}\frac{\partial}{\partial r}\\&+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big)W_{00}+\frac{\partial^2}{\partial r^2}W_{00}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{01}\\&-2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)W_{00}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)W_{01}\Big]=0\end{aligned} \tag{4.31}$$

$$W_{01}|_{r=1}=0 \tag{4.32}$$

$$\left.\frac{\partial W_{01}}{\partial r}\right|_{r=1}=0 \tag{4.33}$$

$$\left.\left[\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{01}\right]\right|_{r=1}=0 \tag{4.34}$$

$$\left.\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\varphi_{01}\right]\right|_{r=1}=0 \tag{4.35}$$

和
$$\left[\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{01}\right] \tag{4.36}$$

$$\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\varphi_{01}\right)\right] \tag{4.37}$$

$$W_{01};\quad \frac{\partial W_{01}}{\partial r} \tag{4.38}$$

在$r=0$处，(4.36)～(4.38)均取有限值。

………………………………

对应于$\varepsilon_1^n\varepsilon_2^m$幂次项的方程和边界条件为：

$$\begin{aligned}&\frac{\partial^4}{\partial r^4}W_{(n-2)m}+\delta_2\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{(n-2)m}\\&+\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}W_{n(m-2)}-\sum_{i=0}^{n}\sum_{j=0}^{m}\left(\frac{\partial^2}{\partial r^2}W_{ij}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}\right.\\&\left.+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{(n-i)(m-j)}+\sum_{i=0}^{n}\sum_{j=0}^{m}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{ij}\frac{\partial^2}{\partial r^2}\varphi_{(n-i)(m-j)}\\&-2\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\varphi_{ij}\left(\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}\right)W_{(n-i)(m-j)}=0\end{aligned} \tag{4.39}$$

$$\begin{aligned}&\frac{\partial^4}{\partial r^4}\varphi_{nm}+\delta_1\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}\varphi_{nm}+\delta_2\left[\left(\frac{1}{r}\frac{\partial}{\partial r}\right.\right.\\&\left.+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{nm}+\sum_{i=0}^{n}\sum_{j=0}^{m}\left(\frac{\partial^2}{\partial r^2}W_{ij}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}\right.\\&\left.+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_{(n-i)(m-j)}-\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)W_{ij}\\&\left.\cdot\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)W_{(n-i)(m-j)}\right]=0\end{aligned} \tag{4.40}$$

$$[W_{nm}|_{r=1}+\nu_{(n-2)m}|_{\eta=1}]=0 \tag{4.41}$$

$$\left[\frac{\partial}{\partial r}W_{nm}|_{r=1}+(A_0\nu_{(n-1)m}+A_1\nu_{(n-2)m})|_{\eta=1}\right]=0 \tag{4.42}$$

$$\begin{aligned}&\left[\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{nm}|_{r=1}+\left(\frac{1}{\eta}A_0\varphi_{(n-3)m}+\frac{1}{\eta}A_1\psi_{(n-4)m}\right.\right.\\&\left.\left.+\frac{1}{\eta^2}B_0'\psi_{(n-2)m}+\frac{1}{\eta^2}B_1'\psi_{(n-3)m}+\frac{1}{\eta^2}B_2'\psi_{(n-4)m}\right)\Big|_{\eta=1}\right]=0\end{aligned} \tag{4.43}$$

$$\begin{aligned}&\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\varphi_{nm}|_{r=1}+\left(A_0\left(\frac{1}{\eta}B_0\right)\right)\psi_{(n-2)m}+A_0\left(\frac{1}{\eta}B_1\right)\psi_{(n-3)m}\right.\\&\left.+A_1\left(\frac{1}{\eta}B_0\right)\psi_{(n-3)m}+A_1\left(\frac{1}{\eta}B_1\right)\psi_{(n-4)m}\right)\Big|_{\eta=1}\Big]=0\end{aligned} \tag{4.44}$$

和 $$\left[\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi_{nm}+\left(\frac{1}{\eta}A_0\psi_{(n-3)m}+\frac{1}{\eta}A_1\psi_{(n-4)m}\right.\right.$$
$$\left.\left.+\frac{1}{\eta^2}B_0'\psi_{(n-2)m}+\frac{1}{\eta^2}B_1'\psi_{(n-3)m}+\frac{1}{\eta^2}B_2'\psi_{(n-4)m}\right)\right] \tag{4.45}$$

$$\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\varphi_{nm}+A_0\left(\frac{1}{\eta}B_0\right)\psi_{(n-2)m}+A_0\left(\frac{1}{\eta}B_1\right)\psi_{(n-3)m}\right.$$
$$\left.+A_1\left(\frac{1}{\eta}B_0\right)\psi_{(n-3)m}+A_1\left(\frac{1}{\eta}B_1\right)\psi_{(n-4)m}\right] \tag{4.46}$$

$$(W_{nm}+\nu_{(n-2)m}) \tag{4.47}$$

$$\left(\frac{\partial}{\partial r}W_{nm}+A_0\nu_{(n-2)m}+A_1\nu_{(n-1)m}\right) \tag{4.48}$$

在$r=0$处，(4.45)～(4.48)均为有限值。

由方程(4.3)和(4.4)的第二个大括号，比较e_1e_2的同幂次项的系数，并注意到函数W_{nm}和φ_{nm}仅是r，θ的函数，可得边界层校正项的递推方程：

$$D_0\nu_{00}+\delta_2\frac{1}{\eta^2}B_0'\frac{1}{\eta^2}B_0'\nu_{00}-\left[A_0'\nu_{00}\left(\frac{1}{\eta}A_1+\frac{1}{\eta^2}B_2'\right)\varphi_{00}\right.$$
$$\left.+\frac{1}{\eta^2}B_0'\nu_{00}A_2'\varphi_{00}-2A_1\frac{1}{\eta}B_1\varphi_{00}A_0\frac{1}{\eta}B_0\nu_{00}\right]=0 \tag{4.49}$$

$$D_0\psi_{00}+\delta_1A_0\frac{1}{\eta}B_0A_0\frac{1}{\eta}B_0\psi_{00}$$
$$+\delta_2\left[\frac{1}{\eta^2}B_0'\frac{1}{\eta^2}B_0'\psi_{00}+A_2'W_{00}\frac{1}{\eta^2}B_0'\nu_{00}\right.$$
$$+A_0'\nu_{00}\left(\frac{1}{\eta}A_1+\frac{1}{\eta^2}B_2'\right)W_{00}-2A_1\frac{1}{\eta}B_1W_{00}A_0\frac{1}{\eta}B_0\nu_{00}$$
$$+A_0'\nu_{00}\frac{1}{\eta^2}B_0'\nu_{00}$$
$$\left.-A_0\frac{1}{\eta}B_0\nu_{00}A_0\frac{1}{\eta}B_0\nu_{00}\right]=0 \tag{4.50}$$

……………………………………

$$\sum_{i=0}^{4}D_i\nu_{(n-i)m}$$
$$+\delta_2\left[\sum_{i=0}^{1}\sum_{j=0}^{1}\frac{1}{\eta}A_i\frac{1}{\eta}A_j\nu_{(n-i-j-2)m}\right.$$
$$+\sum_{i=0}^{1}\sum_{j=0}^{2}\frac{1}{\eta}A_i\frac{1}{\eta^2}B_j'\nu_{(n-i-j-1)m}$$
$$+\sum_{i=0}^{2}\sum_{j=0}^{1}\frac{1}{\eta^2}B_i'\frac{1}{\eta}A_j\nu_{(n-i-j-1)m}$$

$$+\sum_{i=0}^{2}\sum_{j=0}^{2}\frac{1}{\eta^2}B'_i\frac{1}{\eta^2}B'_j\nu_{(n-i-j)m}\Big]$$

$$+\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1}\sum_{l=0}^{1}A_i\frac{1}{\eta}B_jA_k\frac{1}{\eta}B_l\nu_{(n-i-j-k-l+2)(m-2)}$$

$$-\Big[\sum_{i=0}^{1}\sum_{r=0}^{n}\sum_{s=0}^{m}A'_2W_{(n-i-r-3)(m-s)}\frac{1}{\eta}A_i\psi_{rs}$$

$$+\sum_{i=0}^{2}\sum_{r=0}^{n}\sum_{s=0}^{m}A'_2W_{(n-i-r-2)(m-s)}\frac{1}{\eta^2}B'_i\psi_{sr}$$

$$-2\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{r=0}^{n}\sum_{s=0}^{m}A_i\frac{1}{\eta}B_j\psi_{(n-i-j-r-2)(m-s)}A_1\frac{1}{\eta}B_1W_{rs}$$

$$+\sum_{i=0}^{2}\sum_{r=0}^{n}\sum_{s=0}^{m}\Big(\frac{1}{\eta}A_1+\frac{1}{\eta^2}B'_2\Big)W_{(n-i-r-2)(m-s)}A'_i\psi_{rs}\Big]$$

$$-\Big[\sum_{i=0}^{2}\sum_{r=0}^{n}\sum_{s=0}^{m}A'_i\nu_{(n-i-r)(m-s)}\Big(\frac{1}{\eta}A_1+\frac{1}{\eta^2}B'_2\Big)\varphi_{rs}$$

$$+\sum_{i=0}^{1}\sum_{r=0}^{n}\sum_{s=0}^{m}\frac{1}{\eta}A_i\nu_{(n-i-r-1)(m-s)}A'_2\varphi_{rs}$$

$$+\sum_{i=0}^{2}\sum_{r=0}^{n}\sum_{s=0}^{m}\frac{1}{\eta^2}B'_i\nu_{(n-i-r)(m-s)}A'_2\varphi_{rs}$$

$$-2\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{r=0}^{n}\sum_{s=0}^{m}A_1\frac{1}{\eta}B_1\varphi_{(n-i-j-r)(m-s)}A_i\frac{1}{\eta}B_j\nu_{rs}\Big]$$

$$-\Big[\sum_{i=0}^{2}\sum_{j=0}^{1}\sum_{r=0}^{n}\sum_{s=0}^{m}A'_i\nu_{(n-i-j-r-3)(m-s)}\frac{1}{\eta}A_j\psi_{rs}$$

$$+\sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{r=0}^{n}\sum_{s=0}^{m}A'_i\nu_{(n-i-j-r-2)(m-s)}\frac{1}{\eta^2}B'_j\psi_{rs}$$

$$+\sum_{i=0}^{1}\sum_{j=0}^{2}\sum_{r=0}^{n}\sum_{s=0}^{m}\frac{1}{\eta}A_i\nu_{(n-i-j-r-3)(m-s)}A'_j\psi_{rs}$$

$$+\sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{r=0}^{n}\sum_{s=0}^{m}\frac{1}{\eta^2}B'_i\nu_{(n-i-j-r-2)(m-s)}A'_j\psi_{rs}$$

$$+\sum_{i=0}^{2}\sum_{j=0}^{2}\frac{1}{\eta^2}B_i'\ \frac{1}{\eta^2}B_j'\nu_{(n-i-j)m}\Big]$$

$$+\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1}\sum_{l=0}^{1}A_i\frac{1}{\eta}B_jA_k\frac{1}{\eta}B_l\nu_{(n-i-j-k-l+2)(m-2)}$$

$$-\Big[\sum_{i=0}^{1}\sum_{r=0}^{n}\sum_{s=0}^{m}A_2'W_{(n-i-r-3)(m-s)}\frac{1}{\eta}A_i\psi_{rs}$$

$$+\sum_{i=0}^{2}\sum_{r=0}^{n}\sum_{s=0}^{m}A_2'W_{(n-i-r-2)(m-s)}\frac{1}{\eta^2}B_i'\psi_{sr}$$

$$-2\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{r=0}^{n}\sum_{s=0}^{m}A_i\frac{1}{\eta}B_j\psi_{(n-i-j-r-2)(m-s)}A_1\frac{1}{\eta}B_1W_{rs}$$

$$+\sum_{i=0}^{2}\sum_{r=0}^{n}\sum_{s=0}^{m}\left(\frac{1}{\eta}A_1+\frac{1}{\eta^2}B_2'\right)W_{(n-i-r-2)(m-s)}A_i'\psi_{rs}\Big]$$

$$-\Big[\sum_{i=0}^{2}\sum_{r=0}^{n}\sum_{s=0}^{m}A_i'\nu_{(n-i-r)(m-s)}\left(\frac{1}{\eta}A_1+\frac{1}{\eta^2}B_2'\right)\varphi_{rs}$$

$$+\sum_{i=0}^{1}\sum_{r=0}^{n}\sum_{s=0}^{m}\frac{1}{\eta}A_i\nu_{(n-i-r-1)(m-s)}A_2'\varphi_{rs}$$

$$+\sum_{i=0}^{2}\sum_{r=0}^{n}\sum_{s=0}^{m}\frac{1}{\eta^2}B_i'\nu_{(n-i-r)(m-s)}A_2'\varphi_{rs}$$

$$-2\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{r=0}^{n}\sum_{s=0}^{m}A_1\frac{1}{\eta}B_1\varphi_{(n-i-j-r)(m-s)}A_i\frac{1}{\eta}B_j\nu_{rs}\Big]$$

$$-\Big[\sum_{i=0}^{2}\sum_{j=0}^{1}\sum_{r=0}^{n}\sum_{s=0}^{m}A_i'\nu_{(n-i-j-r-3)(m-s)}\frac{1}{\eta}A_j\psi_{rs}$$

$$+\sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{r=0}^{n}\sum_{s=0}^{m}A_i'\nu_{(n-i-j-r-2)(m-s)}\frac{1}{\eta^2}B_j'\psi_{rs}$$

$$+\sum_{i=0}^{1}\sum_{j=0}^{2}\sum_{r=0}^{n}\sum_{s=0}^{m}\frac{1}{\eta}A_i\nu_{(n-i-j-r-3)(m-s)}A_j'\psi_{rs}$$

$$+\sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{r=0}^{n}\sum_{s=0}^{m}\frac{1}{\eta^2}B_i'\nu_{(n-i-j-r-2)(m-s)}A_j'\psi_{rs}$$