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推广的KdV方程特征问题解的存在唯一性*

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摘 要

推广的 KdV 方程$u_t+\alpha uu_x+\mu u_{x3}+\varepsilon u_{x5}=0$[1]是典型的可积方程。它先后在研究冷等离子体中磁声波的传播[2]，传输线中孤立波[3]和分层流体中界面孤立波[4]时导出。本文对推广的KdV方程的特征问题，在 Riemann 函数的基础上，设计一恰当结构，并由此化特征问题为一与之等价的积分微分方程。而该积分微分方程对应的映射E是列自身的映射[5]，依不动点原理，积分微分方程有唯一的正则解，即推广的KdV方程的特征问题有唯一解，且由积分微分方程序列所得的迭代解于$\bar{\Omega}$上一致收敛。

关键词 Riemann函数 结构 积分微分方程 不动点 一致收敛

一、设 计 结 构

在推广的KdV方程的特征问题

$$(T)\quad\begin{cases}u_t+\alpha uu_x+\mu u_{x^3}+\varepsilon u_{x^5}=0 & ((x,t)\in\Omega)\\ u(x,0)=h(x)\\ u(a,t)=f(t)\\ u_x(a,t)=g(t)\\ u_{x^2}(a,t)=\varphi(t)\\ u_{x^3}(a,t)=\psi(t)\\ u_{x^4}(a,t)=s(t)\end{cases}$$

中，$\bar{\Omega}=\{(x,t):a\leqslant x\leqslant b,\ 0\leqslant t\leqslant T_0,\ a,b,\ T_0\in R^+\}$，$h(x)\in C^5[a,b]$，$f(t)$，$g(t)$，$\varphi(t)$，$\psi(t)$，$s(t)\in C^1[0,T_0]$，$h(a)=f(0)$，$h'(a)=g(0)$，$h''(a)=\varphi(0)$，$h'''(a)=\psi(0)$，$h^{(4)}(a)=s(0)$　$(a,\mu,\varepsilon\in R)$

对推广的KdV方程．若$u\in C^{5,1}[[a,b]\times[0,T_0]]$，则方程可表成算子形式

$$\begin{aligned}\mathscr{L}(u)&=\frac{1}{2}u_{x^5t}+u_{x^5}\\&=-\frac{1}{\varepsilon}\left[u_t+\alpha uu_x+\mu u_{x^3}-\frac{\varepsilon}{2}u_{x^5t}\right]\\&=F(u,\ u_x,\ u_t,\ u_{x^3},\ u_{x^5t})\end{aligned}\qquad(1.1)$$

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于是特征问题(T)化为

$$(T)'\quad\begin{cases}\mathscr{L}(u)=F(u,\ u_x,\ u_{x^3},\ u_{x^5t})\\ u(x,0)=h(x)\\ u(a,t)=f(t)\\ u_x(a,t)=g(t)\\ u_{x^2}(a,t)=\varphi(t)\\ u_{x^3}(a,t)=\psi(t)\\ u_{x^4}(a,t)=s(t)\end{cases}$$

引入方程$\mathscr{L}(u)=0$的 Riemann 函数的概念．经计算算子$\mathscr{L}(u)$的共轭算子为

$$\mathscr{L}^*(V)=\frac{1}{2}V_{x^5t}-V_{x^5} \tag{1.2}$$

对$\forall\,(x_0,t_0)\in\bar{\Omega}$，问题

$$(R)\quad\begin{cases}\mathscr{L}^*(V)=0\\ V(x_0,\ t;\ x_0,\ t_0)=0\\ V(x,\ t_0;\ x_0,\ t_0)=(x-x_0)^4\\ V_x(x_0,\ t;\ x_0,\ t_0)=0\\ V_x(x,\ t_0;\ x_0,\ t_0)=4(x-x_0)^3\\ V_{x^2}(x_0,\ t;\ x_0,\ t_0)=0\\ V_{x^2}(x,\ t_0;\ x_0,\ t_0)=12(x-x_0)^2\\ V_{x^3}(x_0,\ t;\ x_0,\ t_0)=0\\ V_{x^3}(x,\ t_0;\ x_0,\ t_0)=4!(x-x_0)\\ V_{x^4}(x_0,\ t;\ x_0,\ t_0)=4!\exp[t-t_0]\\ V_{x^4}(x,\ t_0;\ x_0,\ t_0)=4!\end{cases}$$

的解$V(x,t;\ x_0,\ t_0)=(x-x_0)^4\exp[t-t_0]$称为方程$\mathscr{L}(u)=0$的 Riemann 函数.设

$$C_0^6(\bar{\Omega})=\{u:u,\ u_x,\ u_t,\ u_{x^3},\ u_{x^5t}\in C(\bar{\Omega})\} \tag{1.3}$$

并于$C_0^6(\bar{\Omega})$上定义范数

$$\|u\|_\lambda=\frac{\varepsilon}{4L_0}\max\{(|u_t|+|uu_x|+|u_{x^3}|+|u_{x^5t}|)\exp[-\lambda(x_0+t_0)]\} \tag{1.4}$$

其中 $L_0=\max\left\{1,\ \alpha,\ \mu,\frac{\varepsilon}{2}\right\}$，$\lambda$为充分大的正常数．可知$C_0^6(\bar{\Omega})$为 Banach 空间.

对任给的$\bar{u}\in C_0^6(\bar{\Omega})$，讨论辅助问题

$$(T)''\quad\begin{cases}\mathscr{L}(u)=F(\bar{u},\ \bar{u}_x,\ \bar{u}_t,\ \bar{u}_{x^3},\ \bar{u}_{x^5t})\\ u(x,0)=h(x)\\ u(a,t)=f(t)\\ u_x(a,t)=g(t)\\ u_{x^2}(a,t)=\varphi(t)\\ u_{x^3}(a,t)=\psi(t)\\ u_{x^4}(a,t)=s(t)\end{cases}$$

设计结构

$$V\mathscr{L}(u)-u\mathscr{L}^*(V)=\frac{\partial}{\partial x}\left(\frac{1}{2}Vu_{x^4t}+Vu_{x^4}-V_xu_{x^3}+2V_{x^2}u_{x^2}+V_{x^2}u_{x^2t}-V_xu_{x^3t}\right)$$

$$-\left(2V_{x^3}u_{x^2}+V_{x^2}u_{x^3}+V_{x^3}u_{x^2t}-\frac{1}{2}V_{x}u_{x^4t}\right)$$
$$=VF(\bar{u},\ \bar{u}_x,\ \bar{u}_t,\ \bar{u}_{x^3},\ \bar{u}_{x^5t}) \tag{1.5}$$

设$\bar{D}=\{(x,t):\ a\leqslant x\leqslant x_0\leqslant b,\ 0\leqslant t\leqslant t_0\leqslant T_0\}$ (1.6)

将(1.5)两端在$\bar{D}$上积分，且利用定解条件，得

$$\begin{aligned}
&[u(x_0,t_0)-f(t_0)]-\exp[-t_0][h(x_0)-h(a)]\\
&+(a-x_0)[g(t_0)-\exp[-t_0]h'(a)]\\
&+(a-x_0)\int_0^{t_0}\exp[t-t_0]g(t)dt-\frac{1}{2}(a-x_0)^2[\varphi(t_0)-\exp[-t_0]\varphi(0)]\\
&-\frac{1}{2}(a-x_0)^2\int_0^{t_0}\exp[t-t_0]\varphi(t)\,dt+\frac{1}{6}(a-x_0)^3[\psi(t_0)-\exp[-t_0]\psi(0)]\\
&+\frac{1}{6}(a-x_0)^3\int_0^{t_0}\exp[-t-t_0]\psi(t)dt-\int_0^{t_0}\exp[t-t_0]f(t)\,dt\\
&+\int_0^{t_0}\exp[t-t_0]u(x_0,t)dt-\frac{1}{24}(a-x_0)^4[s(t_0)-\exp[-t_0]s(0)]\\
&-\frac{1}{24}(a-x_0)^4\int_0^{t_0}\exp[t-t_0]s(t)dt\\
&=\frac{1}{12}\int_0^{t_0}\int_a^{x_0}\exp[t-t_0](x-x_0)^4F(\bar{u},\bar{u}_x,\bar{u}_t,\bar{u}_{x^3},\bar{u}_{x^5t})\,dxdt
\end{aligned} \tag{1.7}$$

(1.7)两端关于t_0求导，得

$$\begin{aligned}
&u_t(x_0,t_0)-f'(t_0)+\exp[-t_0][h(x_0)-h(a)]\\
&+(a-x_0)[g'(t_0)+\exp[-t_0]h'(a)]\\
&+(a-x_0)g(t_0)-(a-x_0)\int_0^{t_0}\exp[t-t_0]g(t)dt\\
&-\frac{1}{2}(a-x_0)^2[\varphi'(t_0)+\exp[-t_0]\varphi(0)]\\
&-\frac{1}{2}(a-x_0)^2\varphi(t_0)+\frac{1}{2}(a-x_0)^2\int_0^{t_0}\exp[t-t_0]\varphi(t)dt\\
&+\frac{1}{6}(a-x_0)^3[\psi'(t_0)+\exp[-t_0]\psi(0)]\\
&+\frac{1}{6}(a-x_0)^3\psi(t_0)-\frac{1}{6}(a-x_0)^3\int_0^{t_0}\exp[t-t_0]\psi(t)dt\\
&-f(t_0)+\int_0^{t_0}\exp[t-t_0]f(t)dt+u(x_0,\ t_0)-\int_0^{t_0}\exp[t-t_0]u(x_0,t)dt\\
&-\frac{1}{24}(a-x_0)^4[s'(t_0)+\exp[-t_0]s(0)]\\
&-\frac{1}{24}(a-x_0)^4s(t_0)+\frac{1}{24}(a-x_0)^4\int_0^{t_0}\exp[t-t_0]s(t)dt\\
&=\frac{1}{12}\int_a^{x_0}(x-x_0)^4F(\bar{u},\ \bar{u}_x,\ \bar{u}_t,\ \bar{u}_{x^3},\ \bar{u}_{x^5t})|_{t=t_0}dx
\end{aligned}$$

$$-\frac{1}{12}\int_0^{t_0}\int_a^{x_0}\exp[t-t_0](x-x_0)^4F(\bar{u},\bar{u}_x,\bar{u}_t,\bar{u}_{x^3},\bar{u}_{x^3t})\,dxdt \tag{1.8}$$

(1.7)+(1.8)且经整理，得

$$\begin{aligned}
&u_t(x_0,t_0)+2u(x_0,t_0)\\
&\quad=f'(t_0)+2f(t_0)-(a-x_0)[g'(t_0)+2g(t_0)]+\frac{1}{2}(a-x_0)^2[\varphi'(t_0)+2\varphi(t_0)]\\
&\quad-\frac{1}{12}(a-x_0)^3[\psi'(t_0)+2\psi(t_0)]+\frac{1}{24}(a-x_0)^4[s'(t_0)+2s(t_0)]\\
&\quad+\frac{1}{12}\int_a^{x_0}(x-x_0)^4F(\bar{u},\bar{u}_x,\bar{u}_t,\bar{u}_{x^3},\bar{u}_{x^3t})\,|_{t=t_0}dx
\end{aligned} \tag{1.9}$$

解方程(1.9)，得

$$\begin{aligned}
u(x_0,t_0)=\exp[-2t_0]\Big\{&h(x_0)+f(t_0)\exp[2t_0]\\
&-f(0)-(a-x_0)[g(t_0)\exp[2t_0]-g(0)]\\
&+\frac{1}{2}(a-x_0)^2[\varphi(t_0)\exp[2t_0]-\varphi(0)]\\
&-\frac{1}{12}(a-x_0)^3[\psi(t_0)\exp[2t_0]-\psi(0)]\\
&+\frac{1}{24}(a-x_0)^4[s(t_0)\exp[2t_0]-s(0)]\\
&+\frac{1}{12}\int_0^{t_0}\int_a^{x_0}\exp[2t](x-x_0)^4F(\bar{u},\bar{u}_x,\bar{u}_t,\bar{u}_{x^3},\bar{u}_{x^3t})\,dxdt\Big\}
\end{aligned} \tag{1.10}$$

上述结果表明．若辅助问题 $(T)''$ 有正则解u，则必满足公式 (1.10)，反之，由公式(1.10)给出的函数u，不难验证．它必为问题$(T)''$的正则解．

于是有以下引理．

引理 特征问题$(T)'$有正则解的充分必要条件是积分微分方程

$$\begin{aligned}
u(x_0,t_0)=\exp[-2t_0]\Big\{&h(x_0)+f(t_0)\exp[2t_0]\\
&-f(0)-(a-x_0)[g(t_0)\exp[2t_0]-g(0)]\\
&+\frac{1}{2}(a-x_0)^2[\varphi(t_0)\exp[2t_0]-\varphi(0)]\\
&-\frac{1}{6}(a-x_0)^3[\psi(t_0)\exp[2t_0]-\psi(0)]\\
&+\frac{1}{24}(a-x_0)^4[s(t_0)\exp[2t_0]-s(0)]\\
&+\frac{1}{12}\int_0^{t_0}\int_a^{x_0}\exp[2t](x-x_0)^4F(u,u_x,u_t,u_{x^3},u_{x^3t})\,dxdt\Big\}
\end{aligned} \tag{1.11}$$

有解；若有解，则它们的解相同．

二、解的存在唯一性

在$C_0^6(\bar{\Omega})$上定义映射

$$[Eu](x_0,t_0)=\exp[-2t_0]\Big\{h(x_0)+f(t_0)\exp[2t_0]$$
$$-f(0)-(a-x_0)[g(t_0)\exp[2t_0]-g(0)]$$
$$+\frac{1}{2}(a-x_0)^2[\varphi(t_0)\exp[2t_0]-\varphi(0)]$$
$$-\frac{1}{6}(a-x_0)^3[\psi(t_0)\exp[2t_0]-\psi(0)]$$
$$+\frac{1}{24}(a-x_0)^4[s(t_0)\exp[2t_0]-s(0)]$$
$$+\frac{1}{12}\int_0^{t_0}\int_a^{x_0}\exp[2t](x-x_0)^4F(u,u_x,u_t,u_{x^3},u_{x^5t})dxdt\Big\} \tag{2.1}$$

若证明了E为由$C_0^6(\bar{\Omega})$到自身的映射，则特征问题(T')就化成了不动点问题.

定理 特征问题$(T)'$有唯一的正则解$u^*\in C_0^6(\Omega)$，且对给的$u_0\in C_0^6(\bar{\Omega})$，序列

$$u_n(x_0,t_0)=\exp[-2t_0]\Big\{h(x_0)+f(t_0)\exp[2t_0]$$
$$-f(0)-(a-x_0)[g(t_0)\exp[2t_0]-g(0)]$$
$$+\frac{1}{2}(a-x_0)^2[\varphi(t_0)\exp[2t_0]-\varphi(0)]$$
$$-\frac{1}{6}(a-x_0)^3[\psi(t_0)\exp[2t_0]-\psi(0)]$$
$$+\frac{1}{24}(a-x_0)^4[s(t_0)\exp[2t_0]-s(0)]$$
$$+\frac{1}{12}\int_0^{t_0}\int_a^{x_0}\exp[2t](x-x_0)^4F(u_{n-1},u_{n-1\ x},u_{n-1\ t},u_{n-1\ x^3},u_{n-1\ x^5t})dxdt)\Big\} \tag{2.2}$$

于$\bar{\Omega}$上一致收敛于u^*.

证明 证明E为由$C_0^6(\bar{\Omega})$到自身的映射.

对$\forall u,\bar{u}\in C_0^6(\bar{\Omega})$. 设

$$M=\max_{(x_0,t_0)\in\bar{\Omega}}\left\{4\left|\int_0^{t_0}\int_a^{x_0}(x-x_0)^4\exp[2(t-t_0)]F(u,u_x,u_t,u_{x^3},u_{x^5t})dxdt\right|\right\}$$

$$N=\max_{(x_0,t_0)\in\bar{\Omega}}\left\{4\left|\int_0^{t_0}\int_a^{x_0}(x-x_0)^3\exp[2(t-t_0)]F(\bar{u},\bar{u}_x,\bar{u}_t,\bar{u}_{x^3},\bar{u}_{x^5t})dxdt\right|\right\}$$

$$R(x_0,t_0)=\exp[-2t_0]\{h(x_0)+f(t_0)\exp[2t_0]$$
$$-f(0)-(a-x_0)[g(t_0)\exp[2t_0]-g(0)]$$
$$+\frac{1}{2}(a-x_0)^2[\varphi(t_0)\exp[2t_0]-\varphi(0)]$$
$$-\frac{1}{6}(a-x_0)^3[\psi(t_0)\exp[2t_0]-\psi(0)]$$
$$+\frac{1}{24}(a-x_0)^4[s(t_0)\exp[2t_0]-s(0)]$$

$$G(x_0,t_0)=\exp[-2t_0]\{h'(x_0)+[g(t_0)\exp[2t_0]-g(0)]$$
$$-(a-x_0)[\varphi(t_0)\exp[2t_0]-\varphi(0)]$$

$$+\frac{1}{2}(a-x_0)^2[\psi(t_0)\exp[2t_0]-\psi(0)]$$

$$-\frac{1}{6}(a-x_0)^3[s(t_0)\exp[2t_0]-s(0)]$$

$$\tilde{R}=\max_{(x_0,t_0)\in\bar{\Omega}}|R(x_0,t_0)|,\quad \tilde{G}=\max_{(x_0,t_0)\in\bar{\Omega}}|G(x_0,t_0)|$$

于是

$$|[Eu]_t-[E\bar{u}]_t|\leqslant\frac{(b-a)^4}{4\lambda}\left(\frac{1}{12}+\frac{1}{6\lambda}\right)\|u-\bar{u}\|_\lambda\exp[\lambda(x_0+t_0)]$$

$$|[Eu][Eu]_x-[E\bar{u}][E\bar{u}]_x|\leqslant\frac{1}{48\lambda^2}[(a-b)^4\tilde{G}+(b-a)^3\tilde{R}$$
$$+(b-a)^3M+(b-a)^4N]\cdot\|u-\bar{u}\|_\lambda\exp[\lambda(x_0+t_0)]$$

$$|[Eu]_{x^3}-[E\bar{u}]_{x^3}|\leqslant\frac{1}{2\lambda^2}(b-a)\|u-\bar{u}\|_\lambda\exp[\lambda(x_0+t_0)]$$

$$|[Eu]_{x^5t}-[E\bar{u}]_{x^5t}|\leqslant\frac{1}{2\lambda}\|u-\bar{u}\|_\lambda\exp[\lambda(x_0+t_0)]$$
$$+\frac{1}{2}\|u-\bar{u}\|_\lambda\exp[\lambda(x_0+t_0)]$$

依(1.4)，则 $\|Eu-E\bar{u}\|_\lambda\leqslant\left(\frac{1}{2}+\frac{I}{\lambda}\right)\|u-\bar{u}\|_\lambda$

其中

$$I=\frac{1}{4}\left\{(b-a)^4\left(\frac{1}{12}+\frac{1}{6\lambda}\right)+\frac{1}{12\lambda}[(a-b)^4\tilde{G}+(b-a)^3\tilde{R}+(b-a)^3M\right.$$
$$\left.+(b-a)^4N]+\frac{2}{\lambda}(b-a)+2\right\}$$

选择λ，使$\frac{I}{\lambda}<\frac{1}{2}$，则$E$为由$C_0^{\circ}(\bar{\Omega})$到自身的映射．依不动点原理．可知映射$E$有唯一的不动点$u^*\in C_0^{\circ}(\bar{\Omega})$，即积分微分方程(1.11)有唯一正则解$u^*$．依引理．特征问题$(T)'$有唯一正则解$u^*$．

令

$$M_{n+p-1}=\max_{(x_0,t_0)\in\bar{\Omega}}\left\{4\left|\int_0^{t_0}\int_a^{x_0}\exp[2(t-t_0)](x-x_0)^4\right.\right.$$
$$\left.\left.\cdot F(u_{n+p-1},u_{n+p-1\ x},u_{n+p-1\ t},u_{n+p-1\ x^3},u_{n+p-1\ x^5t})dxdt\right|\right\}$$

$$N_{n-1}=\max_{(x_0,t_0)\in\bar{\Omega}}\left\{4\left|\int_0^{t_0}\int_a^{x_0}\exp[2(t-t_0)](x-x_0)^3\right.\right.$$
$$\left.\left.\cdot F(u_{n-1},u_{n-1\ x},u_{n-1\ t},u_{n-1\ x^3},u_{n-1\ x^5t})dxdt\right|\right\}$$

同于上述证明，有

$$\|u_{n+p}-u_n\|_\lambda\leqslant\left(\frac{1}{2}+\frac{I_{n-1}}{\lambda}\right)\|u_{n+p-1}-u_{n-1}\|_\lambda$$
$$\leqslant\left(\frac{1}{2}+\frac{I_{n-1}}{\lambda}\right)\left(\frac{1}{2}+\frac{I_{n-2}}{\lambda}\right)\|u_{n+p-2}-u_{n-2}\|_\lambda$$
$$\leqslant\cdots\cdots\cdots\cdots$$

$$\leqslant \prod_{k=1}^{n}\left(\frac{1}{2}+\frac{I_{n-k}}{\lambda}\right)\|u_p-u_0\|_\lambda$$

选择λ，使 $\frac{I_{n-k}}{\lambda}<\frac{1}{2}\ (k=1,2,3,\cdots,n)$，于是

$$\prod_{k=1}^{n}\left(\frac{1}{2}+\frac{I_{n-k}}{\lambda}\right)\xrightarrow[n\to\infty]{}0$$

即 $\|u_{n+p}-u_n\|_\lambda\xrightarrow[n\to\infty]{}0$，依 Cauchy 准则．则由积分微分方程(2.2)所确定的特征问题$(T)'$的迭代解于$\bar{\Omega}$上一致收敛于u^*．

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The Uniqueness and Existence of Solution of the Characteristic Problem on the Generalized KdV Equation

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Abstract

The generalized KDV equation, $u_t+\alpha uu_x+\mu u_{x^3}+\varepsilon u_{x^5}=0$, is a typical integrable equation. It is derived by studying the disseminations of magnet sound wave in cold plasma[2], the isolated wave in transmission line[3], and the isolated wave in the boundary surface of the divided layer fluid [4]. For the characteristic problem of the generalized KdV equation, this paper, on the basis of Riemann function, designs a suitable structure, then changes the characteristic problem into an equivalent integral and differential equation whose corresponding mapping E is a mapping to itself[5]. According to the principle of fixed point, the above integral and differential equation has a unique regular solution, so the characteristic problem of the generalized KdV equation has a unique solution. The iterative solution derived from the integral-differential equation sequence is uniformly convergent in $\bar{\Omega}$.

Key words Riemann function, structure, integral and differential equation, fixed point, uniformly convergent