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广义重调和算子及其在薄板弯曲中的应用*

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摘 要

本文用δ-函数具体构造出广义重调和算子，建立相应的二次泛函表达式，并将其应用于弹性薄板的弯曲问题．结果表明，当自变量函数为广义函数时，变分泛函中的自变量函数自然就允许某种程度的不连续性，用Lagrange 乘子法所得的修正变分原理实际上是文中给出的变分原理的特殊形式．

关键词 δ-函数 广义导数 薄板 变分（数学）

求解线性微分方程的有限元法，指的是将算子方程化为与之等价的变分原理，再通过分段（片）多项式来逼近的数值方法．对于因剖分插值而导致二次泛函中的自变量函数的某种程度的不连续性，Fraeijs de Veubeke (1974)[1] J. T. Oden 和 J. N. Reddy (1976)[2]，胡海昌(1981)[3]曾先后指出，变分原理中的自变量函数应理解为广义函数．为了避开广义函数的概念，传统的方法是将经典的算子方程化为等价的变分原理，再利用 Lagrange 乘子法将变分原理进行修正[4][5]．再一种做法是采用某种变换以避开广义函数，并认为修正的广义变分原理是广义变分原理为便于有限元法应用而不涉及广义函数的特殊形式，并不是广义变分原理的进一步推广[3]．因此，本文的目的在于直接从广义微分方程式出发，建立相应的变分原理，其中的自变量函数就是广义函数．至于广义微分方程的概念在[2][6]中有所论述，本文则是用 [7] 中提出的方法从间断的的场变量函数出发，逐段（片）予以定义．本文从广义导数的概念出发，具体给出作用于广义函数的广义重调和微分算子的数学模式，而不引入更多的抽象函数空间，以期便于工程应用．文中还给出了二次泛函形式并用之于薄板的弯曲．

一、广义重调和算子

设二维闭域$\bar{\Omega}=\Omega\cup\partial\Omega$，而$\partial\Omega=\Gamma_0\cup C_i$，其中$\Gamma_0$为$\Omega$域的外层边界；$C_i$为$\Omega$内部因场变量函数某种程度的不连续性而形成的围线．又设$W=W(x, y)$是定义在Ω内的广义函数，它不存在通常意义下的微分，但具有广义微分，即有

$$\langle D_{xxxx}W,\ \phi\rangle=\langle W,\ \phi_{xxxx}\rangle,\qquad \forall\phi(x,y)\in C_0^\infty(\Omega)\tag{1.1}$$

其中$\phi_{xxxx}=\partial^4\phi(x,y)/\partial x^4$，而$D_{xxxx}$应是对变量$x$的四阶广义偏导数．由于$W(x, y)$除了在$C_i$

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上外，是分片连续可微的函数，因此，通过分部积分并求其和后，可以有

$$\langle W,\phi_{xxxx}\rangle=\langle W_{xxxx},\phi\rangle+\langle[W_{xxx}]l,\phi\rangle_{C_i}-\langle[W_{xx}]l,\phi_x\rangle_{C_i}$$
$$+\langle[W_x]l,\phi_{xx}\rangle_{C_i}-\langle[W]l,\phi_{xxx}\rangle_{C_i} \tag{1.2}$$

其中已顾及到$\langle\cdot,\cdot\rangle_{\Gamma_0}=0$，而$\langle\cdot,\cdot\rangle_{\Gamma_0}$表示沿边界$\Gamma_0$构成的内积，$l=\cos(n,x)$是沿$C_i$曲线外向法线$n$的$x$方向余弦．而且其中命

$$\left.\begin{aligned}&[W]=W(x_i^+,\ y_i^+)-W(x_i^-,\ y_i^-)\equiv W(C_i^+)-W(C_i^-)\\&[W_x]\equiv\frac{\partial W(C_i^+)}{\partial x}-\frac{\partial W(C_i^-)}{\partial x},\quad[W_{xx}]\equiv\frac{\partial^2W(C_i^+)}{\partial x^2}-\frac{\partial^2W(C_i^-)}{\partial x^2}\\&\cdots\cdots\end{aligned}\right\} \tag{1.3}$$

利用Dirac-δ函数的性质，(1.2)又可以表示为

$$\langle D_{xxxx}W,\phi\rangle=\langle W_{xxxx},\phi\rangle+\langle[W_{xxx}]l\delta(x-x_i)\delta(y-y_i),\phi\rangle$$
$$+\left\langle[W_{xx}]\,l\frac{\partial\delta(x-x_i)\delta(y-y_i)}{\partial x},\ \phi\right\rangle$$
$$+\left\langle[W_x]\,l\frac{\partial^2\delta(x-x_i)\delta(y-y_i)}{\partial x^2},\ \phi\right\rangle$$
$$+\left\langle[W]\,l\frac{\partial^3\delta(x-x_i)\delta(y-y_i)}{\partial x^3},\ \phi\right\rangle \tag{1.4}$$

其中点$(x_i,\ y_i)$表示间断线C_i上的点．于是可得

$$D_{xxxx}W=W_{xxxx}+[W_{xxx}]_{C_i}l\delta(x-x_i)\delta(y-y_i)$$
$$+[W_{xx}]_{C_i}l\frac{\partial\delta(x-x_i)\delta(y-y_i)}{\partial x}+[W_x]_{C_i}l\frac{\partial^2\delta(x-x_i)\delta(y-y_i)}{\partial x^2}$$
$$+[W]_{C_i}l\frac{\partial^3\delta(x-x_i)\delta(y-y_i)}{\partial x^3} \tag{1.5}$$

其中$[\cdot]_{C_i}$表示沿C_i的间断值（见(1.3)式）

同理可求得$D_{yyyy}W$，$D_{xyxy}W$和$D_{yxyx}W$的相应表达式．从而可得广义重调和算子为

$$\Delta_\delta^2W=\Delta^2W+\{[W_{xxx}]_{C_i}l+[W_{xyx}]_{C_i}m+[W_{yxy}]_{C_i}l$$
$$+[W_{yyy}]_{C_i}m\}\delta(x-x_i)\delta(y-y_i)+\{[W_{xx}]_{C_i}l$$
$$+[W_{yx}]_{C_i}m\}\frac{\partial\delta(x-x_i)\delta(y-y_i)}{\partial x}+\{[W_{yy}]_{C_i}m$$
$$+[W_{xy}]_{C_i}l\}\frac{\partial\delta(x-x_i)\delta(y-y_i)}{\partial y}+[W_x]_{C_i}l\frac{\partial^2\delta(x-x_i)\delta(y-y_i)}{\partial x^2}$$
$$+[W_y]_{C_i}m\frac{\partial^2\delta(x-x_i)\delta(y-y_i)}{\partial y^2}+[W]_{C_i}m\frac{\partial^3\delta(x-x_i)\delta(y-y_i)}{\partial x\partial y\partial x}$$
$$+[W]_{C_i}l\frac{\partial^3\delta(x-x_i)\delta(y-y_i)}{\partial y\partial x\partial y}+[W]_{C_i}l\frac{\partial^3\delta(x-x_i)\delta(y-y_i)}{\partial x^3}$$
$$+[W]_{C_i}m\frac{\partial^3\delta(x-x_i)\delta(y-y_i)}{\partial y^3}+\{[W_y]_{C_i}l+[W_x]_{C_i}m\}$$
$$\cdot\frac{\partial^2\delta(x-x_i)\delta(y-y_i)}{\partial x\partial y} \tag{1.6}$$

其中：$l=\cos(n,x)$，$m=\cos(n,y)$分别是沿C_i外向法线的x，y方向余弦；

$$\Delta_\delta^2=D_{xxxx}+D_{xyxy}+D_{yxyx}+D_{yyyy}$$

$$\Delta^2=\frac{\partial^4}{\partial x^4}+\frac{\partial^4}{\partial x\partial y\partial x\partial y}+\frac{\partial^4}{\partial y\partial x\partial y\partial x}+\frac{\partial^4}{\partial y^4}$$

我们称Δ_δ^2为广义重调和算子。

如果用n，s表示沿C_i的法向和切向坐标，则由坐标(x,y)与(n,s)之间的变换关系，可得

$$\{[W_{xxx}]_{Ci}l+[W_{xyx}]_{Ci}m+[W_{yxy}]_{Ci}l+[W_{yyy}]_{Ci}m\}\delta(x-x_i)\delta(y-y_i)$$
$$=\frac{\partial[\Delta W]_{Ci}}{\partial n}\delta(x-x_i)\delta(y-y_i) \qquad (1.7)$$

(1.7)中的 $\Delta W=\dfrac{\partial^2 W}{\partial x^2}+\dfrac{\partial^2 W}{\partial y^2}$

而 $$\{[W_{xx}]_{Ci}l+[W_{xy}]_{Ci}m\}\frac{\partial\delta(x-x_i)\delta(y-y_i)}{\partial x}$$
$$+\{[W_{xy}]_{Ci}l+[W_{yy}]_{Ci}m\}\frac{\partial\delta(x-x_i)\delta(y-y_i)}{\partial y}$$
$$=\left[\frac{\partial^2 W}{\partial n^2}\right]_{C_i}\frac{\partial\delta(x-x_i)\delta(y-y_i)}{\partial n}+\left[\frac{\partial^2 W}{\partial n\partial s}\right]_{C_i}\frac{\partial\delta(x-x_i)\delta(y-y_i)}{\partial s} \qquad (1.8)$$

(1.8) 中的 $$\left[\frac{\partial^2 W}{\partial n^2}\right]_{C_i}\equiv\frac{\partial^2 W(C_i^+)}{\partial n^2}-\frac{\partial^2 W(C_i^-)}{\partial n^2} \qquad (1.9)$$

$$\left[\frac{\partial^2 W}{\partial n\partial s}\right]_{C_i}\equiv\frac{\partial^2 W(C_i^+)}{\partial n\partial s}-\frac{\partial^2 W(C_i^-)}{\partial n\partial s} \qquad (1.10)$$

又

$$[W]_{Ci}l\frac{\partial}{\partial x}\left\{\frac{\partial^2\delta(x-x_i)\delta(y-y_i)}{\partial x^2}+\frac{\partial^2\delta(x-x_i)\delta(y-y_i)}{\partial y^2}\right\}$$
$$+[W]_{Ci}m\frac{\partial}{\partial y}\left\{\frac{\partial^2\delta(x-x_i)\delta(y-y_i)}{\partial x^2}+\frac{\partial^2\delta(x-x_i)\delta(y-y_i)}{\partial y^2}\right\}$$
$$=[W]_{Ci}\frac{\partial}{\partial n}(\Delta\delta(x-x_i)\ \delta(y-y_i)) \qquad (1.11)$$

以及

$$[W_x]_{Ci}\left\{l\frac{\partial^2\delta(x-x_i)\delta(y-y_i)}{\partial x^2}+m\frac{\partial^2\delta(x-x_i)\delta(y-y_i)}{\partial x\partial y}\right\}$$
$$+[W_y]_{Ci}\left\{m\frac{\partial^2\delta(x-x_i)\delta(y-y_i)}{\partial y^2}+l\frac{\partial^2\delta(x-x_i)\delta(y-y_i)}{\partial y\partial x}\right\}$$
$$=[W_n]_{Ci}\frac{\partial^2}{\partial n^2}\delta(x-x_i)\delta(y-y_i)+[W_s]_{Ci}\frac{\partial^2}{\partial n\partial s}\delta(x-x_i)\delta(y-y_i) \qquad (1.12)$$

利用(1.7)，(1.8)，(1.11)和(1.12)，广义重调和算子又可表示为

$$\Delta_\delta^2 W=\Delta^2 W+\left\{\frac{\partial[\Delta W]_{Ci}}{\partial n}\delta(x-x_i)\delta(y-y_i)\right.$$

$$+\left[\frac{\partial^2 W}{\partial n^2}\right]_{C_i}\frac{\partial\delta(x-x_i)\delta(y-y_i)}{\partial n}+\left[\frac{\partial^2 W}{\partial n\partial s}\right]_{C_i}\frac{\partial\delta(x-x_i)\delta(y-y_i)}{\partial s}$$

$$+[W_n]_{C_i}\frac{\partial^2}{\partial n^2}\delta(x-x_i)\delta(y-y_i)+[W_s]_{C_i}\frac{\partial^2}{\partial n\partial s}\delta(x-x_i)\delta(y-y_i)$$

$$+[W]_{C_i}\frac{\partial}{\partial n}(\Delta\delta(x-x_i)\delta(y-y_i))\Big\} \tag{1.13}$$

二、广义重调和算子的二次泛函表达式

对应于(1.13)的二次泛函为

$$\langle\Delta_\delta^2 W,\ W\rangle=\langle\Delta^2 W,\ W\rangle+\langle\{\cdots\},\ W\rangle \tag{2.1}$$

而

$$\langle\Delta^2 W,W\rangle=\iint_\Omega\left[\left(\frac{\partial^2 W}{\partial x^2}\right)^2+2\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2+\left(\frac{\partial^2 W}{\partial y^2}\right)^2\right]d\Omega$$

$$+\int_{\Gamma_0}W\left[\frac{\partial}{\partial n}(\Delta W)+\frac{\partial}{\partial s}\left(\frac{\partial^2 W}{\partial n\partial s}\right)\right]d\Gamma-\int_{\Gamma_0}\frac{\partial W}{\partial n}\cdot\frac{\partial^2 W}{\partial n^2}d\Gamma \tag{2.2}$$

又由于有

$$\left\langle\frac{\partial[\Delta W]_{C_i}}{\partial n}\delta(x-x_i)\delta(y-y_i),\ W\right\rangle=\left\langle\frac{\partial[\Delta W]}{\partial n},\ W\right\rangle_{C_i} \tag{2.3}$$

$$\left\langle\left[\frac{\partial^2 W}{\partial n^2}\right]_{C_i}\frac{\partial\delta(x-x_i)\delta(y-y_i)}{\partial n},\ W\right\rangle=-\left\langle\left[\frac{\partial^2 W}{\partial n^2}\right],\ \frac{\partial W}{\partial n}\right\rangle_{C_i} \tag{2.4}$$

$$\left\langle\left[\frac{\partial^2 W}{\partial n\partial s}\right]_{C_i}\frac{\partial}{\partial s}\delta(x-x_i)\delta(y-y_i),\ W\right\rangle=-\left\langle\left[\frac{\partial^2 W}{\partial n\partial s}\right],\ \frac{\partial W}{\partial s}\right\rangle_{C_i} \tag{2.5}$$

$$\left\langle[W_s]_{C_i}\frac{\partial^2}{\partial n\partial s}\delta(x-x_i)\delta(y-y_i),\ W\right\rangle=\left\langle[W_s],\ \frac{\partial^2 W}{\partial n\partial s}\right\rangle_{C_i} \tag{2.6}$$

$$\left\langle[W_n]_{C_i}\frac{\partial^2}{\partial n^2}\delta(x-x_i)\delta(y-y_i),\ W\right\rangle=\left\langle[W_n],\ \frac{\partial^2 W}{\partial n^2}\right\rangle_{C_i} \tag{2.7}$$

$$\left\langle[W]_{C_i}\frac{\partial}{\partial n}(\Delta\delta(x-x_i)\delta(y-y_i)),\ W\right\rangle=-\left\langle[W],\ \frac{\partial(\Delta W)}{\partial n}\right\rangle_{C_i} \tag{2.8}$$

其中 $W_n=\frac{\partial W}{\partial n}$, $W_s=\frac{\partial W}{\partial s}$

将(2.3)～(2.8)代入(2.1)中，得

$$\langle\Delta_\delta^2 W,\ W\rangle=\langle\Delta^2 W,\ W\rangle+\left\langle\left[\frac{\partial\Delta W}{\partial n}\right],\ W\right\rangle_{C_i}-\left\langle\left[\frac{\partial^2 W}{\partial n^2}\right],\ \frac{\partial W}{\partial n}\right\rangle_{C_i}$$

$$-\left\langle\left[\frac{\partial^2 W}{\partial n\partial s}\right],\ \frac{\partial W}{\partial s}\right\rangle_{C_i}+\left\langle\left[\frac{\partial W}{\partial n}\right],\ \frac{\partial^2 W}{\partial n^2}\right\rangle_{C_i}$$

$$+\left\langle\left[\frac{\partial W}{\partial s}\right],\ \frac{\partial^2 W}{\partial n\partial s}\right\rangle_{C_i}-\left\langle[W],\ \frac{\partial\Delta W}{\partial n}\right\rangle_{C_i} \tag{2.9}$$

又因为

$$-\left\langle\left[\frac{\partial^2 W}{\partial n\partial s}\right],\ \frac{\partial W}{\partial s}\right\rangle_{C_i}=-\oint_{C_i}\left[\frac{\partial^2 W}{\partial n\partial s}\right]\cdot\frac{\partial W}{\partial s}d\Gamma=-\left\{\left[\frac{\partial^2 W}{\partial n\partial s}\right]\cdot W\right\}$$

$$+\oint_{C_i}\frac{\partial}{\partial s}\left[\frac{\partial^2 W}{\partial n\partial s}\right]\cdot W d\Gamma=\left\langle\frac{\partial}{\partial s}\left[\frac{\partial^2 W}{\partial n\partial s}\right],\ W\right\rangle_{C_i} \tag{2.10}$$

其中$\{[\partial^2 W/\partial n\partial s],W\}=0$。这是因为沿闭曲线$C_i$一周后又回到了原始出发点。

同理又有

$$\left\langle\left[\frac{\partial W}{\partial s}\right],\ \frac{\partial^2 W}{\partial n\partial s}\right\rangle_{C_i}=-\left\langle[W],\ \frac{\partial^3 W}{\partial n\partial s^2}\right\rangle_{C_i} \tag{2.11}$$

于是可得

$$\langle\Delta_0^2 W,\ W\rangle=\langle\Delta^2 W,\ W\rangle+\left\langle\frac{\partial}{\partial n}[\Delta W]+\frac{\partial}{\partial s}\left[\frac{\partial^2 W}{\partial n\partial s}\right],\ W\right\rangle_{C_i}$$

$$-\left\langle\left[\frac{\partial^2 W}{\partial n^2}\right],\ \frac{\partial W}{\partial n}\right\rangle_{C_i}-\left\langle[W],\ \frac{\partial^3 W}{\partial s^2\partial n}+\frac{\partial(\Delta W)}{\partial n}\right\rangle_{C_i}$$

$$+\left\langle\left[\frac{\partial W}{\partial n}\right],\ \frac{\partial^2 W}{\partial n^2}\right\rangle_{C_i} \tag{2.12}$$

此即广义重调和算子的二次泛函表达式。

三、在薄板弯曲中的应用

由(2.12)所表示的二次泛函尚不宜用于弹性薄板的弯曲，原因是没有顾及 Poisson 系数ν的影响。由弹性薄板理论可知

$$\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}+p(x,\ y)=0 \tag{3.1}$$

而

$$\left.\begin{aligned}M_x&=-D\left(\frac{\partial^2 W}{\partial x^2}+\nu\frac{\partial^2 W}{\partial y^2}\right)\\M_y&=-D\left(\frac{\partial^2 W}{\partial y^2}+\nu\frac{\partial^2 W}{\partial x^2}\right)\\M_{xy}&=-D(1-\nu)\frac{\partial^2 W}{\partial x\partial y}\end{aligned}\right\} \tag{3.2}$$

以及

$$Q_x=-D\frac{\partial}{\partial x}(\Delta W),\quad Q_y=-D\frac{\partial}{\partial y}(\Delta W) \tag{3.3}$$

将(3.2)代入(3.1)，可得

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 W}{\partial x^2}+\nu\frac{\partial^2 W}{\partial y^2}\right)+2\frac{\partial^2}{\partial x\partial y}(1-\nu)\frac{\partial^2 W}{\partial x\partial y}+\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 W}{\partial y^2}+\nu\frac{\partial^2 W}{\partial x^2}\right)=\frac{p(x,y)}{D} \tag{3.4}$$

若令(3.4)的左边$=\Delta^2 W$，则有

$$\langle\Delta^2 W,\ W\rangle=\iint_\Omega\left\{\frac{\partial^2 W}{\partial x^2}\left(\frac{\partial^2 W}{\partial x^2}+\nu\frac{\partial^2 W}{\partial y^2}\right)+2(1-\nu)\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2\right.$$

$$+\frac{\partial^2 W}{\partial y^2}\left(\frac{\partial^2 W}{\partial y^2}+\nu\frac{\partial^2 W}{\partial x^2}\right)\Bigg\}d\Omega+\int_{\Gamma_0}WNWd\Gamma-\int_{\Gamma_0}\frac{\partial W}{\partial n}MWd\Gamma \tag{3.5}$$

其中 $$NW=\frac{\partial(\Delta W)}{\partial n}+(1-\nu)\frac{\partial}{\partial s}\left(\frac{\partial^2 W}{\partial n\partial s}\right) \tag{3.6}$$

$$MW=\nu\Delta W+(1-\nu)\frac{\partial^2 W}{\partial n^2} \tag{3.7}$$

可见只有当(3.5)中的 $\nu=0$ 时，才能得出 (2.2)．事实上，若令 (3.2)中的 $\nu=0$，然后将它们代入(3.1)，也可得到 $D\Delta^2 W=p(x,y)$．

再有当边界为曲线时，则有[4]

$$\left.\begin{aligned}M_n&=-D\left\{\frac{\partial^2 W}{\partial n^2}+\nu\left(\frac{\partial^2 W}{\partial s^2}+\frac{1}{\rho}\frac{\partial W}{\partial n}\right)\right\}\\M_s&=-D\left\{\nu\frac{\partial^2 W}{\partial n^2}+\left(\frac{\partial^2 W}{\partial s^2}+\frac{1}{\rho}\frac{\partial W}{\partial n}\right)\right\}\\M_{ns}&=-D\left\{(1-\nu)\left(\frac{\partial^2 W}{\partial n\partial s}-\frac{1}{\rho}\frac{\partial W}{\partial s}\right)\right\}\end{aligned}\right\} \tag{3.8}$$

及

$$H_n=Q_n+\frac{\partial M_{ns}}{\partial s}=-D\Bigg\{\frac{\partial(\Delta W)}{\partial n}+(1-\nu)\frac{\partial^3 W}{\partial n\partial s^2}$$
$$-(1-\nu)\frac{\partial}{\partial s}\left(\frac{1}{\rho}\cdot\frac{\partial W}{\partial s}\right)\Bigg\} \tag{3.9}$$

其中 ρ 是边界线的曲率半径．如果 C_i 是直线（在有限元分割时多用直线)，且令 $\nu=0$ 及 $D=1$，则得

$$\frac{\partial^2 W}{\partial n^2}=-M_n,\quad \frac{\partial(\Delta W)}{\partial n}+\frac{\partial}{\partial s}\left(\frac{\partial^2 W}{\partial n\partial s}\right)=-H_n \tag{3.10}$$

再将(3.10)代入(2.12)，得

$$\langle\Delta_\delta^2 W,W\rangle=\langle\Delta^2 W,W\rangle+\left\langle[M_n],\frac{\partial W}{\partial n}\right\rangle_{C_i}$$
$$-\langle[H_n],W\rangle_{C_i}+\langle[W],H_n\rangle_{C_i}-\left\langle\left[\frac{\partial W}{\partial n}\right],M_n\right\rangle_{C_i} \tag{3.11}$$

应该提及，(3.11)虽然是在 $\nu=0$，$D=1$ 的条件下推出的，但是，对于$D>1$，$\nu\neq0$时，也同样适用，只不过此时应该用(3.5)来取代(2.2)，用(3.8) 和 (3.9)来取代(3.10)就是了．如是，我们就把广义重调和算子应用于弹性薄板的弯曲问题．

值得注意的是，(3.11)表明沿C_i线不仅挠度和转角允许间断，而且弯矩和剪力也允许间断，所以说给出的是混合变分原理，具有一般性．

如果沿C_i线无线布载荷和线布力矩，那末沿C_i线内力应该连续，从而有

$$[H_n]=[M_n]=0 \tag{3.12}$$

将(3.12)代入(3.11)，可得

$$\langle\Delta_\delta^2 W,W\rangle=\langle\Delta^2 W,W\rangle+\langle[W],H_n\rangle_{C_i}-\left\langle\left[\frac{\partial W}{\partial n}\right],W_n\right\rangle_{C_i} \tag{3.13}$$

当W，$\partial W/\partial n$ 在多条线上发生间断时，(3.13)仍然成立，这时只要把所有间断线上的$\langle\cdot,\cdot\rangle_{C_i}$

总和起来便可以了。

在(3.12)成立的前提下得到的(3.13)与专著[3]中用一种变换的方法得出的结果是完全一致的。当用有限元法解薄板的弯曲时，选用的分片多项式函数沿 C_i 上连续还是易于实现的，此时有 $[W]_{Ci}=0$。但是，要使 $\partial W/\partial n$ 在 C_i 也连续则不易办到，因此 $[\partial W/\partial n]_{Ci}\neq 0$，从而有

$$\langle \Delta_0^2 W, W\rangle = \langle \Delta^2 W, W\rangle - \langle [\partial W/\partial n], M_n\rangle_{Ci} \tag{3.14}$$

上述结果也与[5]、[8]中用 Lagrange 乘子法对变分原理进行修正后得到的结果相一致。但是，这种修正却是以广义函数作为自变量函数时的某种特殊形式而已。而且，引用 Lagrange乘子对变分原理进行修正时，必须把自变量函数的某种程度的不连续性人为地作为约束条件来处理，而从广义函数来看，这种不连续性则是必然的结果。再有，基于广义函数导出的变分原理与样条逼近有机结合，样条函数有什么样的不连续性，变分原理中就出现相应的间断项，两者处于统一的框架之中。

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Generalized Biharmonic Operator and Its Application to the Bending of Elastic Thin Plates

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Abstract

In this paper, δ-function is used to construct the generalized biharmonic operators, the corresponding quadratic function is presented, and the latter is applied to the bending of elastic thin plates. The result shows that when the arguments in the variational functional are generalized functions, discontinuity to some degree is allowed, and the modified variational principle by using the Lagrange multipliers is merely a special form of the result mentioned above.

Key words δ-function, generalized derivative, thin plate, variation(mathematics)