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圆筒受余弦分布压力之解及其k→0的极限

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摘　要

本文得到一个新的应力函数，用此解答了圆筒受余弦分布压力的问题，并为解决圆筒在轴向受任意分布荷载作用的空间轴对称问题打下了基础．根据求出的解答，取压力沿轴向不变化时的极限，就导出了厚壁圆筒的Lamè公式．

关键词　圆筒　余弦压力　分析解

一、边界条件

考察圆筒受余弦分布压力的情况(见图1)．采用柱坐标边界条件可表示为

当$r=a$时，
$$\left.\begin{aligned}&\sigma_r=-q_1\cos\frac{\pi z}{L}\\&\tau_{rz}=0\end{aligned}\right\}\tag{1.1}$$

当$r=b$时，
$$\left.\begin{aligned}&\sigma_r=-q_2\cos\frac{\pi z}{L}\\&T_{rz}=0\end{aligned}\right\}\tag{1.2}$$

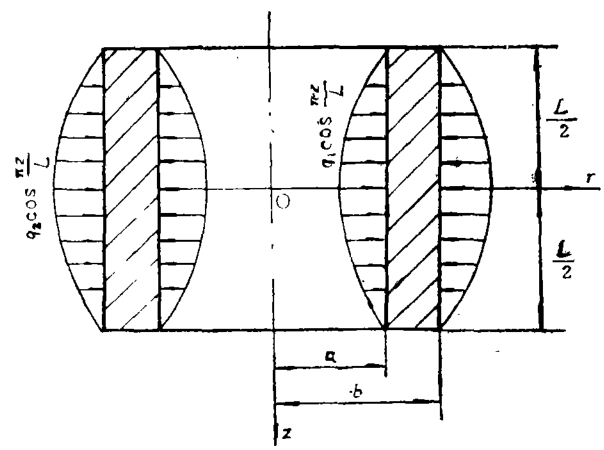

图 1

当 $z=\pm\frac{L}{2}$ 时，

$$\left.\begin{array}{l}\sigma_z=0\\ \tau_{rz}=0\end{array}\right\} \tag{1.3}$$

(1.3)式不必严格满足．按Saint-Venant原理[1]，端部条件可放松，只要应力总和不变．由于是轴对称问题，剪应力在端面上的积分 $\int_{S_0}\tau_{rz}ds=0$ 是自然满足的．

二、新应力函数

空间轴对称问题的应力函数满足双调和方程[2]

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}\right)=0 \tag{2.1}$$

也可简记为 $\nabla^2(\nabla^2\phi)=0$．设(2.1)式的解为

$$\phi=f(r)\sin kz \tag{2.2}$$

将(2.2)代入(2.1)式，就得到确定 $f(r)$ 的下列四阶常微分方程

$$\frac{d^4f}{dr^4}+\frac{2}{r}\frac{d^3f}{dr^3}-\left(\frac{1}{r^2}+2k^2\right)\frac{d^2f}{dr^2}+\left(\frac{1}{r^3}-\frac{2k^2}{r}\right)\frac{df}{dr}+k^4f=0 \tag{2.3}$$

取这方程的解为如下形式的级数

$$f(r)=\sum_{i=0}^{\infty}C_ir^{P+i},\quad C_0\neq 0 \tag{2.4}$$

将此级数代入方程(2.3)，得系数间的下列关系

$$P^2(P-2)^2C_0=0,$$
$$(P+1)^2(P-1)^2C_1=0,$$
$$(P+2)^2P^2C_2-2k^2P^2C_0=0,$$
$$(P+3)^2(P+1)^2C_3-2k^2(P+1)^2C_1=0,$$
$$C_{i+4}=\frac{2k^2(P+i+2)^2C_{i+2}-k^4C_i}{(P+i+4)^2(P+i+2)^2}\qquad(i=0,1,2,3,\cdots)$$

设 $C_0=1$，则有

$$C_{2i-1}=0\qquad(i=1,2,3,\cdots) \tag{2.5}$$

$$C_{2i}=\frac{(i+1)k^{2i}}{(P+2i)^2[P+2(i-1)]^2[P+2(i-2)]^2[P+2(i-3)]^2\cdots(P+2)^2}\qquad(i=1,2,3,\cdots) \tag{2.6}$$

将(2.5)、(2.6)代入(2.4)式，得

$$f(r)=r^P\left[1+\frac{2k^2}{(P+2)^2}r^2+\frac{3k^4}{(P+4)^2(P+2)^2}r^4+\frac{4k^6}{(P+6)^2(P+4)^2(P+2)^2}r^6+\cdots\right] \tag{2.7}$$

当 $P=0$ 时，得一解

$$f_1(r)=\sum_{i=0}^{\infty}\frac{(i+1)k^{2i}r^{2i}}{4^i\cdot(i!)^2} \tag{2.8}$$

当$P=2$时，得另一解

$$f_2(r)=\sum_{i=1}^{\infty}\frac{ik^{2(i-1)}\cdot r^{2i}}{4^{i-1}(i!)^2} \tag{2.9}$$

关于(2.7)式对P求偏导，得

$$\frac{\partial f}{\partial P}=f(r)r^P\ln r$$

$$-2r^P\sum_{i=1}^{\infty}\frac{(i+1)k^{2i}\left(\frac{1}{P+2}+\frac{1}{P+4}+\frac{1}{P+6}+\cdots+\frac{1}{P+2i}\right)r^{2i}}{(P+2i)^2[P+2(i-1)]^2[P+2(i-2)]^2[P+2(i-3)]^2\cdots(P+2)^2} \tag{2.10}$$

当$P=0$时，得一解

$$f_3(r)=f_1(r)\ln r-\sum_{i=1}^{\infty}\frac{(i+1)k^{2i}r^{2i}}{4^i(i!)^2}\sum_{m=1}^{i}\frac{1}{m} \tag{2.11}$$

当$P=2$时，得另一解

$$f_4(r)=f_2(r)\ln r-\sum_{i=1}^{\infty}\frac{(i+1)k^{2i}r^{2i+2}}{4^i[(i+1)!]^2}\sum_{m=1}^{i}\frac{1}{m+1} \tag{2.12}$$

根据级数收敛性的比值判定法知，以上求出的$f_1(r)$，$f_2(r)$，$f_3(r)$和$f_4(r)$在定义域$[a, b]$上收敛。显然，它们彼此线性无关。因此，得(2.3)的通解

$$f(r)=A_1f_1(r)+A_2f_2(r)+A_3f_3(r)+A_4f_4(r) \tag{2.13}$$

应力函数为

$$\phi=\sin kz[A_1f_1(r)+A_2f_2(r)+A_3f_3(r)+A_4f_4(r)] \tag{2.14}$$

上式即为所求新应力函数。

三、余弦分布压力之解

将(2.14)式代入应力分量公式[2]

$$\left.\begin{aligned}
\sigma_r&=\frac{\partial}{\partial z}\left(\mu\nabla^2-\frac{\partial^2}{\partial r^2}\right)\phi\\
\sigma_\theta&=\frac{\partial}{\partial z}\left(\mu\nabla^2-\frac{1}{r}\frac{\partial}{\partial r}\right)\phi\\
\sigma_z&=\frac{\partial}{\partial z}\left[(2-\mu)\nabla^2-\frac{\partial^2}{\partial z^2}\right]\phi\\
\tau_{rz}&=\frac{\partial}{\partial r}\left[(1-\mu)\nabla^2-\frac{\partial^2}{\partial z^2}\right]\phi
\end{aligned}\right\} \tag{3.1}$$

经运算和整理求得应力分量的表达式

$$\left.\begin{aligned}
\sigma_r&=k\cos kz[A_1W_{11}(r)+A_2W_{12}(r)+A_3W_{13}(r)+A_4W_{14}(r)]\\
\sigma_\theta&=k\cos kz[A_1W_{21}(r)+A_2W_{22}(r)+A_3W_{23}(r)+A_4W_{24}(r)]\\
\sigma_z&=k\cos kz[A_1W_{31}(r)+A_2W_{32}(r)+A_3W_{33}(r)+A_4W_{34}(r)]\\
\tau_{rz}&=\sin kz[A_1W_{41}(r)+A_2W_{42}(r)+A_3W_{43}(r)+A_4W_{44}(r)]
\end{aligned}\right\} \tag{3.2}$$

其中$W_{11}(r),\cdots,W_{44}(r)$是$r$的函数，含有$f_1(r)$、…、$f_4(r)$及其三阶以下导数．它们的计算公式在文献[3]中给出．作者用Fortran语言编制了程序，利用该程序，很容易算出r为任一值时的$W_{11}(r)$，…，$W_{44}(r)$的值．

对于图1所示受载情况，在(3.2)式中取

$$k=\frac{\pi}{L},$$

注意到σ_z，τ_{rz}在两端已自然满足边界条件，故只将边界条件(1.1)、(1.2)式代入(3.2)式．经运算可解出

$$\left.\begin{aligned}A_1&=\frac{1}{k\cdot\Delta}(-q_1\Delta_{11}+q_2\Delta_{12})\\A_2&=\frac{1}{k\cdot\Delta}(q_1\Delta_{21}+q_2\Delta_{22})\\A_3&=\frac{1}{k\cdot\Delta}(-q_1\Delta_{31}+q_2\Delta_{22})\\A_4&=\frac{1}{k\cdot\Delta}(q_1\Delta_{41}-q_2\Delta_{42})\end{aligned}\right\}\tag{3.3}$$

其中

$$\Delta=\begin{vmatrix}W_{11}(a)&W_{12}(a)&W_{13}(a)&W_{14}(a)\\W_{11}(b)&W_{12}(b)&W_{13}(b)&W_{14}(b)\\W_{41}(a)&W_{42}(a)&W_{43}(a)&W_{44}(a)\\W_{41}(b)&W_{42}(b)&W_{43}(b)&W_{44}(b)\end{vmatrix},$$

$$\Delta_{11}=\begin{vmatrix}W_{12}(b)&W_{13}(b)&W_{14}(b)\\W_{42}(a)&W_{43}(a)&W_{44}(a)\\W_{42}(b)&W_{43}(b)&W_{44}(b)\end{vmatrix},\quad\Delta_{12}=\begin{vmatrix}W_{12}(a)&W_{13}(a)&W_{14}(a)\\W_{42}(a)&W_{43}(a)&W_{44}(a)\\W_{42}(b)&W_{43}(b)&W_{44}(b)\end{vmatrix},$$

$$\Delta_{21}=\begin{vmatrix}W_{11}(b)&W_{13}(b)&W_{14}(b)\\W_{41}(a)&W_{43}(a)&W_{44}(a)\\W_{41}(b)&W_{43}(b)&W_{44}(b)\end{vmatrix},\Delta_{22}=-\begin{vmatrix}W_{11}(a)&W_{13}(a)&W_{14}(a)\\W_{41}(b)&W_{43}(b)&W_{44}(b)\\W_{41}(a)&W_{43}(a)&W_{44}(a)\end{vmatrix},$$

$$\Delta_{31}=\begin{vmatrix}W_{11}(b)&W_{12}(b)&W_{14}(b)\\W_{41}(a)&W_{42}(a)&W_{44}(a)\\W_{41}(b)&W_{42}(b)&W_{44}(b)\end{vmatrix},\quad\Delta_{32}=\begin{vmatrix}W_{11}(a)&W_{12}(a)&W_{14}(a)\\W_{41}(a)&W_{42}(a)&W_{44}(a)\\W_{41}(b)&W_{42}(b)&W_{44}(b)\end{vmatrix},$$

$$\Delta_{41}=\begin{vmatrix}W_{11}(b)&W_{12}(b)&W_{13}(b)\\W_{41}(a)&W_{42}(a)&W_{43}(a)\\W_{41}(b)&W_{42}(b)&W_{43}(b)\end{vmatrix},\quad\Delta_{42}=\begin{vmatrix}W_{11}(a)&W_{12}(a)&W_{13}(a)\\W_{41}(a)&W_{42}(a)&W_{43}(a)\\W_{41}(b)&W_{42}(b)&W_{43}(b)\end{vmatrix}.$$

将(3.3)代入(3.2)式，得

$$\begin{aligned}\sigma_r=\frac{\cos kz}{\Delta}[&(-q_1\Delta_{11}+q_2\Delta_{12})W_{11}(r)+(q_1\Delta_{21}-q_2\Delta_{22})W_{12}(r)\\&+(-q_1\Delta_{31}+q_2\Delta_{32})W_{13}(r)+(q_1\Delta_{41}-q_2\Delta_{42})W_{14}(r)]\end{aligned}\tag{3.4}$$

$$\sigma_\theta=\frac{\cos kz}{\Delta}[(-q_1\Delta_{11}+q_2\Delta_{12})W_{21}(r)+(q_1\Delta_{21}-q_2\Delta_{22})W_{22}(r)$$
$$+(-q_1\Delta_{31}+q_2\Delta_{32})W_{23}(r)+(q_1\Delta_{41}-q_2\Delta_{42})W_{24}(r)] \tag{3.5}$$

$$\sigma_z=\frac{\cos kz}{\Delta}[(-q_1\Delta_{11}+q_2\Delta_{12})W_{31}(r)+(q_1\Delta_{21}-q_2\Delta_{22})W_{32}(r)$$
$$+(-q_1\Delta_{31}+q_2\Delta_{32})W_{33}(r)+(q_1\Delta_{41}-q_2\Delta_{42})W_{34}(r)] \tag{3.6}$$

$$\tau_{rz}=\frac{\sin kz}{k\cdot\Delta}[(-q_1\Delta_{11}+q_2\Delta_{12})W_{41}(r)+(q_1\Delta_{21}-q_2\Delta_{22})W_{42}(r)$$
$$+(-q_1\Delta_{31}+q_2\Delta_{32})W_{43}(r)+(q_1\Delta_{41}-q_2\Delta_{42})W_{44}(r)] \tag{3.7}$$

上面四个公式即为圆筒受余弦分布压力之解．这种解法为求解圆筒的任意空间轴对称变形问题打下了基础．该问题将在另一篇文章中讨论．

四、k→0 的极限情况

当$k\to 0$时，由(2.8)，(2.9)，(2.11)，(2.12)得

$$f_1(r)=1,\ f_1'(r)=0,\ f_1''(r)=0,\ f_1^{(3)}(r)=0,$$
$$f_2(r)=r^2,\ f_2(r)=2r,\ f_2''(r)=2,\ f_2^{(3)}(r)=0,$$
$$f_3(r)=\ln r,\ f_3'(r)=\frac{1}{r},\quad f_3''(r)=-\frac{1}{r^2},\quad f_3^{(3)}(r)=\frac{2}{r^3},$$
$$f_4(r)=r^2\ln r,\ f_4'(r)=2r\ln r+r,\ f_4''(r)=2\ln r+3,\ f_4^{(3)}=\frac{2}{r},$$

由$W_{11}(r)$，…，$W_{44}(r)$的计算公式，得

$$\left.\begin{array}{l}
W_{11}(r)=0,\ W_{12}(r)=2(2\mu-1)\\
W_{13}(r)=\frac{1}{r^2},\ W_{14}(r)=4\mu(\ln r+1)-(2\ln r+3)\\
W_{21}(r)=0,\ W_{22}(r)=2(2\mu-1)\\
W_{23}(r)=-\frac{1}{r^2},\ W_{24}(r)=4\mu(\ln r+1)-(2\ln r+1)\\
W_{31}(r)=0,\ W_{32}(r)=4(2-\mu)\\
W_{33}(r)=0,\ W_{34}(r)=4(\ln r+1)(2-\mu)\\
W_{41}(r)=0,\ W_{42}(r)=0\\
W_{43}(r)=0,\ W_{44}(r)=\frac{4}{r}(1-\mu)
\end{array}\right\} \tag{4.1}$$

将(4.1)同边界条件(1.1)、(1.2)式一起代入(3.2)式，得

$$\left.\begin{array}{l}
2(2\mu-1)A_2+\frac{1}{a^2}A_3+[4\mu(\ln a+1)-(2\ln a+3)]A_4=-\frac{q_1}{k}\\
2(2\mu-1)A_2+\frac{1}{b^2}A_3+[4\mu(\ln b+1)-(2\ln b+3)]A_4=-\frac{q_2}{k}\\
\frac{4}{a}(1-\mu)A_4=0,\ \frac{4}{b}(1-\mu)A_4=0
\end{array}\right\} \tag{4.2}$$

由(4.2)式解得

$$A_3=\frac{b^2a^2(q_2-q_1)}{(b^2-a^2)k},\ A_2=\frac{q_1a^2-q_2b^2}{(b^2-a^2)\cdot k\cdot 2(2\mu-1)},\ A_4=0 \tag{4.3}$$

将(4.1)、(4.3)代入(3.2)式，得

$$\left.\begin{aligned}\sigma_r&=\cos kz\left[\frac{q_1a^2-q_2b^2}{b^2-a^2}+\frac{b^2a^2(q_2-q_1)}{(b^2-a^2)r^2}\right]\\ \sigma_\theta&=\cos kz\left[-\frac{a^2b^2(q_2-q_1)}{(b^2-a^2)\cdot r^2}+\frac{a^2q_1-b^2q_2}{b^2-a^2}\right]\\ \sigma_z&=0\\ \tau_{rz}&=0\end{aligned}\right\} \tag{4.4}$$

当$k\to 0$时，即$L\to\infty$，圆筒无限长，此时，压力可看做沿轴向不变化，则(4.4)式成为

$$\left.\begin{aligned}\sigma_r&=\frac{q_1a^2-q_2b^2}{b^2-a^2}+\frac{b^2a^2(q_2-q_1)}{(b^2-a^2)\cdot r^2}\\ \sigma_\theta&=\frac{q_1a^2-q_2b^2}{b^2-a^2}-\frac{b^2a^2(q_2-q_1)}{(b^2-a^2)\cdot r^2}\end{aligned}\right\} \tag{4.5}$$

(4.5)式即为著名的Lamè公式.

对于有限长圆筒，可设$k=\pi/L_a$，L_a为一假想的长度值．令$L_a\to\infty$，压力仍可看做沿轴向不变化，同样可导出Lamè公式(4.5)．

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A Solution of Cosine Pressures on a Hollow Cylinder and the Limit When k→0

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Abstract

A new stress function is found in this paper and then the problems of cosine pressures on a hollow cylinder are solved with the new stress function, which provides the basis for the solution of the problems of space symmetrical deformation of a hollow cylinder. When the pressures do not vary in the axial direction, that is, when $k\to 0$, the Lamè formulae can be deduced.

Key words hollow cylinder, cosine pressure, analytic solution