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求平面弹性问题的更普遍的位移型解*

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摘　要

本文得到了平面弹性问题的更普遍的位移型解答．文献[1]所得到的位移通解，只是本文的一个特殊情况．和文献[1]相比较，本文的通解中含有较多的任意常数因而可以满足更多的边界条件．

关键词　平面弹性问题　位移解答　双调和方程

一、引　言

对于平面弹性问题的位移型解答，人们总希望能像应力解那样，能获得一个位移函数的控制方程，其阶数不是过高，并能给出由位移函数求应力及位移的简便公式．

最近，王林生[1]和曾又林[2]各自得到了平面弹性问题的位移通解，他们均把问题归结为求解双调和方程．文献[2]的结果还适用于正交各向异性的情形．本文仅研究理想弹性体的平面问题，获得了更加普遍的位移型解答．文献[1]的结果只是本文的一个特殊情况．和文献[1]相比较，本文的通解中含有更多的任意常数，可满足更多的边界条件，因而可解决更多的实际问题．

二、基本方程及一组位移解答

求平面弹性问题的位移型解答的基本方程是[1]

1．**平衡微分方程**

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0 \tag{2.1}$$

$$\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{xy}}{\partial x}=0 \tag{2.2}$$

2．**用应力和位移表示的几何方程**（此处仅考虑了平面应力问题）

$$\frac{\partial u}{\partial x}=\frac{1}{E}(\sigma_x-\mu\sigma_y) \tag{2.3}$$

$$\frac{\partial v}{\partial y}=\frac{1}{E}(\sigma_y-\mu\sigma_x) \tag{2.4}$$

$$\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=\frac{1}{E}\cdot 2(1+\mu)\tau_{xy} \tag{2.5}$$

* 薛大为推荐．

把式(2.3)～(2.5)代入式(2.2)及(2.1)可得

$$\begin{cases} -\dfrac{1+\mu}{1-\mu}\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{2}{1-\mu}\cdot\dfrac{\partial^2 v}{\partial y^2}+\dfrac{\partial^2 v}{\partial x^2} & (2.6)\\ \dfrac{\partial^2 u}{\partial x^2}+\dfrac{(1-\mu)}{2}\dfrac{\partial^2 u}{\partial y^2}=-\dfrac{1+\mu}{2}\dfrac{\partial^2 v}{\partial x\partial y} & (2.7)\end{cases}$$

我们发现，如果令

$$u=D\frac{\partial^3\phi}{\partial x^3}+D_1\frac{\partial^3\phi}{\partial x\partial y^2} \tag{2.8}$$

$$v=K\frac{\partial^3\phi}{\partial y^3}+K_1\frac{\partial^3\phi}{\partial x^2\partial y} \tag{2.9}$$

则可使式(2.6)变为（适当的选取任意常数D，D_1，K及K_1的数值）

$$\frac{\partial}{\partial y}\nabla^4\phi=0$$

即

$$\frac{\partial^5\phi}{\partial x^4\partial y}+2\frac{\partial^5\phi}{\partial x^2\partial y^3}+\frac{\partial^5\phi}{\partial y^5}=0 \tag{2.10}$$

将式(2.8)及(2.9)代入式(2.3)及(2.4)可求得

$$\sigma_x=A\frac{\partial^4\phi}{\partial x^4}+A_1\frac{\partial^4\phi}{\partial x^2\partial y^2}+A_2\frac{\partial^4\phi}{\partial y^4} \tag{2.11}$$

$$\sigma_y=B\frac{\partial^4\phi}{\partial x^4}+B_1\frac{\partial^4\phi}{\partial x^2\partial y^2}+B_2\frac{\partial^4\phi}{\partial y^4} \tag{2.12}$$

$$A=ED+B\mu,\quad A_1=ED_1+B_1\mu,\quad A_2=B_2\mu \tag{2.13}$$

$$\left.\begin{aligned} &B=E\mu D/(1-\mu^2)\\ &B_1=\frac{E(\mu D_1+K_1)}{1-\mu^2}\\ &B_2=EK/(1-\mu^2)\end{aligned}\right\} \tag{2.14}$$

估计到式(2.11)，为了使式(2.1)变为

$$\frac{\partial}{\partial x}\nabla^4\phi=0 \tag{2.15}$$

我们试探令

$$\tau_{xy}=C\frac{\partial^4\phi}{\partial x^3\partial y}+C_1\frac{\partial^4\phi}{\partial x\partial y^3} \tag{2.16}$$

将式(2.11)及(2.16)代入式(2.1)得

$$\frac{\partial}{\partial x}\left[A\frac{\partial^4\phi}{\partial x^4}+(A_1+C)\frac{\partial^4\phi}{\partial x^2\partial y^2}+(A_2+C_1)\frac{\partial^4\phi}{\partial y^4}\right]=0 \tag{2.17}$$

比较式(2.15)及(2.17)，我们令

$$A=m,\quad A_2+C_1=m,\quad A_1+C=2m \tag{2.18}$$

式中 m为任意常数．将(2.12)及(2.16)代入式(2.2)得

$$\frac{\partial}{\partial y}\left[(B+C)\frac{\partial^4\phi}{\partial x^4}+(B_1+C_1)\frac{\partial^4\phi}{\partial x^2\partial y^2}+B_2\frac{\partial^4\phi}{\partial y^4}\right]=0$$

为了使它成为

$$\frac{\partial}{\partial y}\nabla^4\phi=0 \tag{2.19}$$

我们令 $B_2=a$，$B+C=a$，$B_1+C_1=2a$ (2.20)

式中 a为任意常数。于是，我们有式(2.13)，(2.14)，(2.18)及(2.20)等12个方程，可决定A，A_1，A_2，B，B_1，B_2，C，C_1,D,D_1,K及K_1等12个任意常数（把m及a作为参数），再代入式(2.12)，(2.11)，(2.16)，(2.8)及(2.9)最后得

$$\left.\begin{aligned}
&\sigma_x=m\frac{\partial^4\phi}{\partial x^4}+[(2+\mu)m-a]\frac{\partial^4\phi}{\partial x^2\partial y^2}+a\mu\frac{\partial^4\phi}{\partial y^4}\\
&\sigma_y=m\mu\frac{\partial^4\phi}{\partial x^4}+(2a-m+2\mu)\frac{\partial^4\phi}{\partial x^2\partial y^2}+a\frac{\partial^4\phi}{\partial y^4}\\
&\tau_{xy}=(a-m\mu)\frac{\partial^4\phi}{\partial x^3\partial y}+(m-a\mu)\frac{\partial^4\phi}{\partial x\partial y^3}\\
&u=\frac{1}{E}m(1-\mu^2)\frac{\partial^3\phi}{\partial x^3}+\frac{1}{E}[-a\mu^2+2(m-a)\mu+2m-a]\frac{\partial^3\phi}{\partial x\partial y^2}\\
&v=\frac{1}{E}\cdot a(1-\mu^2)\frac{\partial^3\phi}{\partial y^3}+\frac{1}{E}[-m\mu^2+2(a-m)\mu+2a-m]\frac{\partial^3\phi}{\partial x^2\partial y}\\
&\frac{\partial^4\phi}{\partial x^4}+2\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial y^4}=\text{常数}
\end{aligned}\right\}$$

(2.21a,b,c,d,e,f)

不难验证，式(2.21)满足全部基本方程(2.1)～(2.5)，其中式（2.1）给出$\partial(\nabla^4\phi)/\partial x=0$，式(2.2)给出$\partial(\nabla^4\phi)/\partial y=0$。

对于多连体，还要考虑位移单值条件[1]。

由于双调和方程在数学上研究得比较成熟，我们通过变量代换，把式（2.21f）变换成为双调和方程。

式（2.21f）为

$$\nabla^4\phi=\beta \tag{2.22}$$

式中β为任意常数。令

$$\phi=\varphi+\frac{\beta}{56}(x^4+y^4+x^2y^2) \tag{2.23}$$

式中$\varphi(x,y)$为任意函数，则式（2.21）变换成为

$$\left.\begin{aligned}
&\sigma_x=m\frac{\partial^4\varphi}{x^4}+[(2+\mu)m-a]\frac{\partial^4\varphi}{\partial x^2\partial y^2}+a\mu\frac{\partial^4\varphi}{\partial y^4}+\frac{\beta}{14}[8m+\mu(6a+m)-a]\\
&\sigma_y=m\mu\frac{\partial^4\varphi}{\partial x^4}+[2a-m+a\mu]\frac{\partial^4\varphi}{\partial x^2\partial y^2}+a\frac{\partial^4\varphi}{\partial y^4}+\frac{\beta}{14}[8a+\mu(6m+a)-m]\\
&\tau_{xy}=(a-m\mu)\frac{\partial^4\varphi}{\partial x^3\partial y}+(m-a\mu)\frac{\partial^4\varphi}{\partial x\partial y^3}\\
&u=\frac{1}{E}\cdot m(1-\mu^2)\frac{\partial^3\varphi}{\partial x^3}+\frac{1}{E}[-a\mu^2+2(m-a)\mu+2m-a]\frac{\partial^3\varphi}{\partial x\partial y^2}\\
&\qquad+\frac{\beta}{14E}[-\mu^2(6m+a)+2\mu(m-a)+8m-a]x\\
&v=\frac{1}{E}\cdot a(1-\mu^2)\frac{\partial^3\varphi}{\partial y^3}+\frac{1}{E}[-m\mu^2+2(a-m)\mu+2a-m]\frac{\partial^3\varphi}{\partial x^2\partial y}\\
&\qquad+\frac{\beta}{14E}[-\mu^2(6a+m)+2\mu(a-m)+8a-m]y\\
&\nabla^4\varphi=0
\end{aligned}\right\} \tag{2.24}$$

式中 a，m及β均为任意常数。

于是，我们用式（2.24）取代了式(2.21)。

在文献[2]中，如果令$\varphi=\partial\varphi_1/\partial x$，则文献[2]中的式（4.6）变为

$$\left.\begin{aligned}&\sigma_x=G\left(-\frac{\partial^4\varphi_1}{\partial x^2\partial y^2}+\mu\frac{\partial^4\varphi_1}{\partial y^4}\right)\\&\sigma_y=G\left[(2+\mu)-\frac{\partial^4\varphi_1}{\partial x^2\partial y^2}+\frac{\partial^4\varphi}{\partial y^4}\right]\\&\tau_{xy}=G\left(\frac{\partial^4\varphi_1}{\partial x^3\partial y}-\mu\frac{\partial^4\varphi_1}{\partial x\partial y^3}\right)\\&u=-\frac{1}{2}(1+\mu)\frac{\partial^3\varphi_1}{\partial x\partial y^2}\\&v=\frac{\partial^3\varphi_1}{\partial x^2\partial y}+\frac{1}{2}(1-\mu)\frac{\partial^3\varphi_1}{\partial y^3}\\&\nabla^4\varphi_1=\beta_1(x)\end{aligned}\right\}\tag{2.25}$$

式中$\beta_1(x)$为x的任意函数。

如果令$\beta=m=0$，$a=G=E/2(1+\mu)$及$\beta_1(x)=0$，则式(2.24)和（2.25）完全一致（式（2.24）中的φ相当于式(2.25)中的φ_1）。

三、第二组位移解

如果令

$$u=D_2\partial^3\phi/\partial y^3+D_3\partial^3\phi/\partial x^2\partial y \tag{3.1}$$

$$v=K_2\partial^3\phi/\partial x^3+K_3\partial^3\phi/\partial x\partial y^2 \tag{3.2}$$

并适当选取任意常数D_2，D_3，K_2及K_3的数值，可使式(2.7)成为

$$\frac{\partial}{\partial y}\nabla^4\phi=0 \tag{3.3}$$

将式（3.1）及（3.2）代入式（2.3）及（2.4）可得

$$\sigma_x=A_3\frac{\partial^4\phi}{\partial x\partial y^3}+A_4\frac{\partial^4\phi}{\partial x^3\partial y} \tag{3.4}$$

$$\sigma_y=B_3\frac{\partial^4\phi}{\partial x\partial y^3}+B_4\frac{\partial^4\phi}{\partial x^3\partial y} \tag{3.5}$$

$$A_3=ED_2+\mu B_3\,,\quad A_4=ED_3+\mu B_4 \tag{3.6}$$

$$B_3=\frac{E}{1-\mu^2}(\mu D_2+K_3),\quad B_4=\frac{E}{1-\mu^2}(\mu D_3+K_2) \tag{3.7}$$

估计到式(3.5)，要使得式（2.2）导致

$$\frac{\partial}{\partial x}\nabla^4\phi=0 \tag{3.8}$$

我们试探令

$$\tau_{xy}=C_2\frac{\partial^4\phi}{\partial x^4}+C_3\frac{\partial^4\phi}{\partial x^2\partial y^2} \tag{3.9}$$

将式（3.5）及（3.9）代入式（2.2）得

$$\frac{\partial}{\partial x}\left[B_3\frac{\partial^4\phi}{\partial y^4}+(B_4+C_3)\frac{\partial^4\phi}{\partial x^2\partial y^2}+C_2\frac{\partial^4\phi}{\partial x^4}\right]=0 \tag{3.10}$$

比较式（3.8）及（3.10），我们令

$$B_3=b,\ B_4+C_3=2b,\ C_2=b \tag{3.11}$$

式中 b为任意常数．将式（3.4）及（3.9）代入式（2.1）得

$$(A_3+C_3)\frac{\partial^5\phi}{\partial x^2\partial y^3}+(A_4+C_2)\frac{\partial^5\phi}{\partial x^4\partial y}=0$$

要使得上式成立，我们令

$$A_3+C_3=0,\ A_4+C_2=0 \tag{3.12}$$

将式（3.1），(3.2）及（3.9）代入式（2.5）得

$$\frac{\partial^4\phi}{\partial x^4}\left[K_2-\frac{1}{E}\cdot 2(1+\mu)C_2\right]+\frac{\partial^4\phi}{\partial x^2\partial y^2}\left[K_3+D_3-\frac{1}{E}\cdot 2(1+\mu)C_3\right]+D_2\frac{\partial^4\phi}{\partial y^4}=0$$

要使得上式成为

$$\nabla^4\phi=0 \tag{3.13}$$

我们令

$$\left.\begin{aligned}&D_2=n,\ K_2-\frac{1}{E}\cdot 2(1+\mu)C_2=n\\&K_3+D_3-\frac{1}{E}\cdot 2(1+\mu)C_3=2n\end{aligned}\right\} \tag{3.14}$$

式中n为任意常数，由式（3.6），(3.7），(3.11），(3.12）及（3.14）等12个方程可决定A_3，A_4，B_3，B_4，C_2，C_3，D_2，D_3，K_2 及K_3等10个任意常数（把b及n作为参数），再代入式（3.1），(3.2），(3.4），(3.5)及（3.9），最后得

$$\left.\begin{aligned}&\sigma_x=(En+\mu b)\frac{\partial^4\phi}{\partial x\partial y^3}-b\frac{\partial^4\phi}{\partial x^3\partial y}\\&\sigma_y=b\frac{\partial^4\phi}{\partial x\partial y^3}+(2b+En+\mu b)\frac{\partial^4\phi}{\partial x^3\partial y}\\&\tau_{xy}=b\frac{\partial^4\phi}{\partial x^4}-(Fn+\mu b)\frac{\partial^4\phi}{\partial x^2\partial y^2}\\&u=n\frac{\partial^3\phi}{\partial y^3}-\left[\mu n+\frac{b}{E}(1+\mu)^2\right]\frac{\partial^3\phi}{\partial x^2\partial y}\\&v=\left[\frac{1}{E}\cdot 2(1+\mu)b+n\right]\frac{\partial^3\phi}{\partial x^3}+\frac{1}{E}[(1-\mu^2)b\\&\quad -E\mu n]\partial^3\phi/\partial x\partial y^2\\&\nabla^4\phi=0\end{aligned}\right\} \tag{3.15}$$

不难验证，式（3.15）满足全部基本方程（2.1）～(2.5），其中式（2.2）给出$\partial(\nabla^4\phi)/\partial x=0$，式（2.5）给出$\nabla^4\phi=0$。

对于多连体，还要考虑位移单值条件。

当$n=1$及$b=0$时，式（3.15）和文献[1]的结果完全一致。由于b是任意常数，故文献[1]的结果只是本文结果式（3.15）的一个特殊情况。

由于式（3.15）中含有b及n等任意常数，故和文献[1]相比较，本文公式（3.15）可以满足更多的边界条件，因而可以解决更多的实际问题。

在文献[2]中，如果令$\varphi=\varphi_1/x$，则文献[2]中的式（4.6）变为

$$\left.\begin{aligned}
&\sigma_x=G\left(-\frac{\partial^4\varphi}{\partial x^3\partial y}+\mu\frac{\partial^4\varphi_1}{\partial x\partial y^3}\right)\\
&\sigma_y=G\left[(2+\mu)\frac{\partial^4\varphi_1}{\partial x^3\partial y}+\frac{\partial^4\varphi_1}{\partial x\partial y^3}\right]\\
&\tau_{xy}=G\left[\frac{\partial^4\varphi_1}{\partial x^4}-\mu\frac{\partial^4\varphi_1}{\partial x^2\partial y^2}\right]\\
&u=-\frac{1}{2}(1+\mu)\frac{\partial^3\varphi_1}{\partial x^2\partial y}\\
&v=\frac{\partial^3\varphi_1}{\partial x^3}+\frac{1}{2}(1-\mu)\partial^3\varphi_1/\partial x\partial y^2\\
&\nabla^4\varphi_1=\beta_2(y)
\end{aligned}\right\}\tag{3.16}$$

式中$\beta_2(y)$为y的任意函数。

如果令$n=0$，$b=E/2(1+\mu)=G$及$\beta_2(y)=0$，则式（3.15）和（3.16）完全一致[式（3.16）中的φ_1相当于式（3.15）中的ϕ]。

参考文献

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The More General Displacement Solutions for the Plane Elasticity Problems

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Abstract

In this paper, the author obtains the more general displacement solutions for the isotropic plane elasticity problems. The general solution obtained in ref.[1]is merely the particular case of this paper. In camparison with ref.[1], the general solutions of this paper contain more arbitrary constants. Thus, they may satisfy more boundary conditions.

Key words plane elasticity problem, displacement solutions, biharmonic eguation