·应用数学和力学，第14卷第1期（1993年1月）
Applied Mathematics and Mechanics

应用数学和力学编委会编
重庆出版社出版

高速扩展平面应力裂纹尖端的各向异性塑性场*

林 拜 松

（中南工业大学，1991年7月1日收到）

摘 要

在裂纹尖端的应力分量都只是 θ 的函数的条件下，利用定常运动方程，Hill 各向异性屈服条件及应力应变关系，我们得到高速扩展平面应力裂纹尖端的各向异性塑性场的一般解。将这个一般解用于四种各向异性特殊情形，我们就导出这四种特殊情形的一般解。最后，本文给出 $X=Y=Z$ 情形的高速扩展平面应力Ⅰ型裂纹尖端的各向异性塑性场。

关键词 各向异性塑性场 裂纹尖端 平面应力

一、引 言

关于高速扩展平面应力裂纹尖端的理想塑性场问题，我们研究过各向同性塑性应力场[1]和各向同性塑性场[2]。至今没有人研究过高速扩展平面应力裂纹尖端的各向异性塑性场。为此，我们用文献[1]、[2]和[3]的方法来解决上述问题。

在理想弹塑性材料中，高速扩展裂纹尖端的应力分量都只是 θ 的函数，利用这个条件以及定常运动方程，Hill 各向异性屈服条件和应力应变关系，我们得到高速扩展平面应力裂纹尖端的各向异性塑性场的一般解。将这个一般解用于四种各向异性特殊情形，我们就导出这四种特殊情形的一般解。最后，本文给出 $X=Y=Z$ 情形的高速扩展平面应力Ⅰ型裂纹尖端的各向异性塑性场。

图1表示一沿裂纹线高速扩展的平面应力裂纹。(x_1, y_1, z_1) 和 (x, y, z) 分别是静止坐标系和运动坐标系。这些坐标轴亦是各向异性主轴。裂纹尖点是运动坐标系的原点。裂纹尖端的速度为 $c=\text{const}$. 设裂纹作定常运动，则有如下关系：

$$\frac{\partial t}{\partial}\frac{\partial}{\partial t}=-c\frac{\partial}{\partial x},\quad \frac{\partial^2}{\partial t^2}=c^2\frac{c^2}{\partial x^2} \tag{1.1}$$

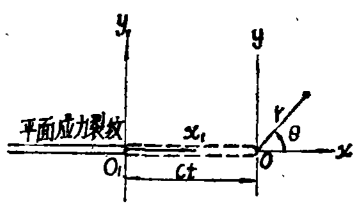

图 1

今后取

$$a=c/\sqrt{\mu/\rho}\leqslant 1 \tag{1.2}$$

其中，$c_s=\sqrt{\mu/\rho}$ 为剪切波波速，μ为剪切模量，ρ为材料密度。

* 叶开沅推荐。

二、基本方程

在运动坐标系(x, y, z)中，定常运动平面应力裂纹的基本方程为：

1. 定常运动方程

$$\left.\begin{aligned}\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}-\rho c^2\frac{\partial u_x}{\partial x}=0\\ \frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{xy}}{\partial x}-\rho c^2\frac{\partial v_x}{\partial x}=0\end{aligned}\right\}\tag{2.1}$$

其中 $u_x=\partial u/\partial x$，$v_x=\partial v/\partial x$；$u$和$v$分别是$x$方向和$y$方向的位移分量；$\sigma_x$，$\sigma_y$，$\tau_{xy}$为应力分量。

2. Hill各向异性屈服条件[3]

$$(2-\beta)\sigma_+^2+(2+\beta)\sigma_-^2+\sigma_{xy}^2=1\tag{2.2}$$

其中

$$\left.\begin{aligned}&\sigma_+=\frac{1}{2}\left(\frac{\sigma_x}{X}+\frac{\sigma_y}{Y}\right),\ \sigma_-=\frac{1}{2}\left(\frac{\sigma_x}{X}-\frac{\sigma_y}{Y}\right),\sigma_{xy}=\frac{\tau_{xy}}{T}\\ &\beta=\frac{X}{Y}+\frac{Y}{X}-\frac{XY}{Z^2}\end{aligned}\right\}\tag{2.3}$$

这里X，Y，Z分别是在各向异性主轴x，y，z方向上的拉伸屈服应力，而T是相对于各向异性主轴x和y的剪切屈服应力。

若取

$$\sigma_+=\frac{1}{\sqrt{2-\beta}}\cdot\cos\omega,\quad \sigma_-=-\frac{1}{\sqrt{2+\beta}}\cdot\sin\omega\cos\varphi,\quad \sigma_{xy}=\sin\omega\sin\varphi\tag{2.4}$$

则式(2.2)恒被满足。

3. 应力应变关系

对于正交各向异性塑性材料，应力应变关系为：

$$\left.\begin{aligned}X\frac{\partial u_x}{\partial x}+Y\frac{\partial v_x}{\partial y}&=-\frac{2\dot\lambda}{c}(2-\beta)\sigma_++\left(D_1\frac{\partial\sigma_+}{\partial x}+D_2\frac{\partial\sigma_-}{\partial x}\right)\\ X\frac{\partial u_x}{\partial x}-Y\frac{\partial v_x}{\partial y}&=-\frac{2\dot\lambda}{c}(2+\beta)\sigma_-+\left(D_2\frac{\partial\sigma_+}{\partial x}+D_3\frac{\partial\sigma_-}{\partial x}\right)\\ T\left(\frac{\partial u_x}{\partial y}+\frac{\partial v_x}{\partial x}\right)&=-\frac{2\dot\lambda}{c}\sigma_{xy}+D_4\frac{\partial\sigma_{xy}}{\partial x}\end{aligned}\right\}\tag{2.5}$$

其中

$$\left.\begin{aligned}D_1&=\frac{X^2+Y^2-2\nu XY}{E},\quad D_2=\frac{X^2-Y^2}{E}\\ D_3&=\frac{X^2+Y^2+2\nu XY}{E},\quad D_4=\frac{T^2}{\mu}\end{aligned}\right\}\tag{2.6}$$

而E是材料的弹性模量，ν是泊松比．对于各向同性塑性材料，$X=Y=Z=\sqrt{3}T$；若取$\dot{\lambda}_1=\dot{\lambda}/X^2$，则式(2.5)就变成各向同性塑性材料的应力应变关系．

三、一 般 解

利用(2.1)，(2.3)和(2.5)，我们得到如下的偏微分方程组：

$$\left.\begin{aligned}&X\frac{\partial\sigma_+}{\partial x}+X\frac{\partial\sigma_-}{\partial x}+T\frac{\partial\sigma_{xy}}{\partial y}-\rho c^2\frac{\partial u_x}{\partial x}=0\\ &Y\frac{\partial\sigma_+}{\partial y}-Y\frac{\partial\sigma_-}{\partial y}+T\frac{\partial\sigma_{xy}}{\partial x}-\rho c^2\frac{\partial v_x}{\partial x}=0\\ &X\sigma_{xy}\frac{\partial u_x}{\partial x}-T(2-\beta)\sigma_+\frac{\partial u_x}{\partial y}+Y\sigma_{xy}\frac{\partial v_x}{\partial y}-T(2-\beta)\sigma_+\frac{\partial v_x}{\partial x}\\ &\quad-\sigma_{xy}\left(D_1\frac{\partial\sigma_+}{\partial x}+D_2\frac{\partial\sigma_-}{\partial x}\right)+D_4(2-\beta)\sigma_+\frac{\partial\sigma_{xy}}{\partial x}=0\\ &X\sigma_{xy}\frac{\partial u_x}{\partial x}-T(2+\beta)\sigma_-\frac{\partial u_x}{\partial y}-Y\sigma_{xy}\frac{\partial v_x}{\partial y}-T(2+\beta)\sigma_-\frac{\partial v_x}{\partial x}\\ &\quad-\sigma_{xy}\left(D_2\frac{\partial\sigma_+}{\partial x}+D_3\frac{\partial\sigma_-}{\partial x}\right)+D_4(2+\beta)\sigma_-\frac{\partial\sigma_{xy}}{\partial x}=0\end{aligned}\right\}\tag{3.1}$$

在裂纹尖端的应力分量都只是θ的函数的条件下，σ_+，σ_-，τ_{xy}，u_x和v_x亦只是θ的函数．这样，式(2.4)中的ω和φ亦是θ的函数．将式(2.4)代入(3.1)，并采用如下变换：

$$\frac{\partial}{\partial x}=-\frac{\sin\theta}{r}\frac{d}{d\theta},\quad\frac{\partial}{\partial y}=\frac{\cos\theta}{r}\frac{d}{d\theta}\tag{3.2}$$

(3.1)就变成关于新变量$d\omega/d\theta$，$d\varphi/d\theta$，$du_x/d\theta$和$dv_x/d\theta$的方程组：

$$\begin{aligned}&\left[\frac{X}{\sqrt{2-\beta}}\sin\theta\sin\omega+\left(\frac{X}{\sqrt{2+\beta}}\sin\theta\cos\varphi+T\cos\theta\sin\varphi\right)\cos\omega\right]\frac{d\omega}{d\theta}\\ &\quad+\left(T\cos\theta\cos\varphi-\frac{X}{\sqrt{2-\beta}}\sin\theta\sin\varphi\right)\sin\omega\frac{d\varphi}{d\theta}+\alpha^2\mu\sin\theta\frac{du_x}{d\theta}=0\\ &\left[\frac{Y}{\sqrt{2-\beta}}\cos\theta\sin\omega-\left(\frac{Y}{\sqrt{2+\beta}}\cos\theta\cos\varphi-T\sin\theta\sin\varphi\right)\cos\omega\right]\frac{d\omega}{d\theta}\\ &\quad+\left(T\sin\theta\cos\varphi+\frac{Y}{\sqrt{2+\beta}}\cos\theta\sin\varphi\right)\sin\omega\frac{d\varphi}{d\theta}-\alpha^2\mu\sin\theta\frac{dv_x}{d\theta}=0\\ &\left[\frac{D_2}{\sqrt{2-\beta}}\sin\omega-\left(D_4\sqrt{2+\beta}-\frac{D_3}{\sqrt{2+\beta}}\right)\cos\varphi\cos\omega\right]\sin\theta\sin\varphi\frac{d\omega}{d\theta}\\ &\quad-\left[\frac{D_3}{\sqrt{2+\beta}}\sin^2\varphi+D_4\sqrt{2+\beta}\cos^2\varphi\right]\sin\theta\sin\omega\frac{d\varphi}{d\theta}\end{aligned}$$

$$
\left.\begin{aligned}
&+(X\sin\theta\sin\varphi-T\sqrt{2+\beta}\cos\theta\cos\varphi)\frac{du_z}{d\theta}+(Y\cos\theta\sin\varphi\\
&+T\sqrt{2+\beta}\sin\theta\sin\varphi)\frac{dv_z}{d\theta}=0\\
&\left(\frac{D_1}{\sqrt{2-\beta}}\sin^2\omega+D_4\sqrt{2-\beta}\cos^2\omega+\frac{D_2}{\sqrt{2+\beta}}\cos\varphi\sin\omega\cos\omega\right)\sin\theta\sin\varphi\frac{d\omega}{d\theta}\\
&-\left(\frac{D_2}{\sqrt{2+\beta}}\sin\theta\sin^2\varphi\sin^2\omega-D_4\sqrt{2-\beta}\sin\theta\cos\varphi\sin\omega\cos\omega\right)\frac{d\varphi}{d\theta}\\
&+\left(X\sin\theta\sin\varphi\sin\omega+T\sqrt{2-\beta}\cos\theta\cos\omega\right)\frac{du_z}{d\theta}\\
&-\left(Y\cos\theta\sin\varphi\sin\omega+T\sqrt{2-\beta}\sin\theta\cos\omega\right)\frac{dv_z}{d\theta}=0
\end{aligned}\right\}\tag{3.3}
$$

由 (3.3) 得到下面两种塑性区：

1. **均匀塑性区** ($d\omega/d\theta=0$, $d\varphi/d\theta=0$, $du_z/d\theta=0$, $dv_z/d\theta=0$)

在均匀塑性区内，我们有：

$$
\left.\begin{aligned}
&\omega=\text{const},\ \varphi=\text{const}\\
&u_z=\text{const},\ v_z=\text{const}
\end{aligned}\right\}\tag{3.4}
$$

所以，均匀塑性区是均匀应力区。

2. **非均匀塑性区**($d\omega/d\theta\neq0$, $d\varphi/d\theta\neq0$, $du_z/d\theta\neq0$, $dv_z/d\theta\neq0$)

非均匀塑性区的存在条件是(3.3)的系数行列式为零，即

$$A\sin^2\omega+B\cos^2\omega+C\sin\omega\cos\omega=0\tag{3.5}$$

其中

$$
\left.\begin{aligned}
A=&\frac{1}{2-\beta}[\sqrt{2+\beta}\,T(X\sin^2\theta-Y\cos^2\theta)\cos\varphi+XY\sin2\theta\sin\varphi]^2\\
&-\frac{\alpha^2\mu\sin^2\theta}{2-\beta}\left\{\frac{4X^2Y^2}{E}\sin^2\varphi+\frac{\sqrt{2+\beta}\,TXY}{2\mu}(X-Y)\sin2\theta\sin2\varphi\right.\\
&\left.+(2+\beta)[T^2D_1+D_4(X^2\sin^2\theta+Y^2\cos^2\theta]\cos^2\varphi\right\}\\
&+\frac{\alpha^4\sin^4\theta}{2-\beta}\left[\frac{\mu}{\mu_1}X^2Y^2\sin^2\varphi+(2+\beta)T^2D_1\mu\cos^2\varphi\right]\\
B=&T^2(X\sin^2\theta+Y\cos^2\theta)^2-\alpha^2\mu\sin^2\theta[T^2D_3+D_4(X^2\sin^2\theta+Y^2\cos^2\theta)]\\
&+\alpha^4\mu^2\sin^4\theta\cdot D_3D_4\\
C=&\frac{2T}{\sqrt{2-\beta}}(X\sin^2\theta+Y\cos^2\theta)[\sqrt{2+\beta}\,T(X\sin^2\theta-Y\cos^2\theta)\cos\varphi\\
&+XY\sin2\theta\sin\varphi]-\sqrt{\frac{2+\beta}{2-\beta}}\alpha^2\mu\sin^2\theta\left\{2D_4(X^2\sin^2\theta\right.\\
&-Y^2\cos^2\theta)\cos\varphi+\frac{TD_3}{\sqrt{2+\beta}}(X+Y)\sin2\theta\sin\varphi\\
&\left.+TD_2\left[2T\cos\varphi-\frac{X-Y}{2\sqrt{2+\beta}}(1+\cos^2\theta+\sin^2\varphi)\sin2\theta\sin\varphi\right]\right\}\\
&+2\sqrt{\frac{2+\beta}{2-\beta}}\alpha^4\mu^2\sin^4\theta D_2D_4\cos\varphi
\end{aligned}\right\}\tag{3.6}
$$

由（3.5）得到：

$$\left.\begin{aligned}
\sin^2\omega&=\frac{C^2-2B(A-B)\mp\sqrt{C^2-4AB}}{2[(A-B)^2+C^2]}\\
\cos^2\omega&=\frac{C^2+2A(A-B)\pm\sqrt{C^2-4AB}}{2[(A-B)^2+C^2]}\\
\frac{d\omega}{d\theta}&=\frac{(A+B)\frac{d}{d\theta}[(A-B)^2-C^2]-2\frac{d}{d\theta}(A+B)}{4(C^2-4AB)^{1/2}}\\
&\quad+\frac{(A-B)^2}{2[(A-B)^2-C^2]}\cdot\frac{d}{d\theta}\left(\frac{C}{A-B}\right)
\end{aligned}\right\}\tag{3.7}$$

由（3.3）的前三式得到$d\varphi/d\theta$，$du_x/d\theta$和$dv_x/d\theta$，然后积分得到：

$$\left.\begin{aligned}
\varphi&=\varphi_0-\int_{\omega_0}^{\omega}\left[\frac{j_1(\theta,\ \varphi)}{f(\theta,\ \varphi)}+\frac{j_2(\theta,\ \varphi)}{f(\theta,\ \varphi)}\cdot\cot\omega\right]d\omega\\
u_x&=u_{x0}+\int_{\omega_0}^{\omega}\left[\frac{k_1(\theta,\ \varphi)}{\alpha^2\mu\sin\theta\cdot f(\theta,\varphi)}\sin\omega+\frac{k_2(\theta,\ \varphi)}{\alpha^2\mu\sin\theta\cdot f(\theta,\varphi)}\cos\omega\right]d\omega\\
v_x&=v_{x0}+\int_{\omega_0}^{\omega}\left[\frac{l_1(\theta,\ \varphi)}{\alpha^2\mu\sin\theta\cdot f(\theta,\varphi)}\sin\omega+\frac{l_2(\theta,\ \varphi)}{\alpha^2\mu\sin\theta\cdot f(\theta,\varphi)}\cos\omega\right]d\omega
\end{aligned}\right\}\tag{3.8}$$

其中

$$\left.\begin{aligned}
f(\theta,\varphi)&=T^2\sqrt{2+\beta}\cos^2\varphi-\frac{T}{2}(X-Y)\sin2\theta\sin2\varphi+\frac{X^2\sin^2\theta+Y^2\cos^2\theta}{\sqrt{2+\beta}}\sin^2\varphi\\
&\quad-\alpha^2\mu\sin^2\theta\cdot\left(\frac{D_3}{\sqrt{2+\beta}}\sin^2\varphi+\sqrt{2+\beta}\,D_4\cos^2\varphi\right)\\
j_1(\theta,\varphi)&=\sqrt{\frac{2+\beta}{2-\beta}}\frac{T}{2}(X+Y)\sin2\theta\cos\varphi+\frac{Y^2\cos^2\theta-X^2\sin^2\theta}{\sqrt{2-\beta}}\sin\varphi\\
&\quad+\frac{D_2\mu}{\sqrt{2-\beta}}\sin^2\theta\sin\varphi\\
j_2(\theta,\varphi)&=\frac{T}{2}(X-Y)\sin2\theta\cos2\varphi+\frac{\sin2\varphi}{2\sqrt{2+\beta}}[(2+\beta)T^2-(X^2\sin^2\theta+Y^2\cos^2\theta)]\\
&\quad-\left(D_4\sqrt{2+\beta}-\frac{D_3}{\sqrt{2+\beta}}\right)\cdot\alpha^2\mu\sin^2\theta\cos\theta\sin\varphi\\
k_1(\theta,\varphi)&=\frac{1}{\sqrt{2-\beta}}\left\{\alpha^2\mu\sin^2\theta\cdot\left[\frac{TD_2}{2}\cos\theta\sin2\varphi+\left(\frac{X(D_3-D_2)}{\sqrt{2+\beta}}\sin^2\varphi\right.\right.\right.\\
&\quad\left.\left.\left.+\sqrt{2+\beta}\,XD_4\cos^2\varphi\right)\sin\theta\right]+(Yg_1\cos\theta-Xg_2\sin\theta)\right\}\\
k_2(\theta,\varphi)&=\alpha^2\mu\sin\theta\left[\left(D_4-\frac{D_3}{2+\beta}\right)\left(\frac{X}{2}\sin2\theta\sin^2\varphi+\frac{T\sqrt{2+\beta}}{2}\sin2\varphi\cos^2\theta\right)\right.\\
&\quad\left.+\left(D_4\cos^2\varphi+\frac{D_3}{2+\beta}\sin^2\varphi\right)(X\sin\theta\cos\varphi+T\sqrt{2+\beta}\cos\theta\sin\varphi)\right]\\
&\quad+T\sin\varphi(g_1\sin\theta-g_2\cos\theta)-\frac{\cos\varphi}{\sqrt{2+\beta}}(Yg_1\cos\theta+Xg_2\sin\theta)
\end{aligned}\right\}\tag{3.9}$$

$$
\left.\begin{aligned}
l_1(\theta,\varphi)=&\frac{1}{\sqrt{2-\beta}}\Big\{-\alpha^2\mu\sin^2\theta\cdot\Big[\frac{TD_2}{2}\sin\theta\sin2\varphi+\Big(\frac{Y(D_2+D_3)}{\sqrt{2+\beta}}\sin^2\varphi\\
&+\sqrt{2+\beta}\ YD_4\cos^2\varphi\Big)\cos\theta\Big]-(Xg_1\sin\theta-Yg_2\cos\theta)\Big\}\\
l_2(\theta,\varphi)=&\alpha^2\mu\sin\theta\Big[\Big(D_4-\frac{D_3}{2+\beta}\Big)\Big(\frac{Y}{2}\cos^2\theta\sin2\varphi+\frac{T\sqrt{2+\beta}}{2}\sin2\theta\cos^2\varphi\Big)\\
&+\Big(D_4\cos^2\varphi+\frac{D_3}{2+\beta}\sin^2\varphi\Big)(Y\cos\theta\cos\varphi-T\sqrt{2+\beta}\sin\theta\sin\varphi)\Big]\\
&+T\sin\varphi(g_3\sin\theta-g_1\cos\theta)-\frac{\cos\varphi}{\sqrt{2+\beta}}(Yg_3\mathrm{cs}\theta+Xg_1\sin\theta)
\end{aligned}\right\}
$$

其中

$$
\left.\begin{aligned}
g_1(\theta,\varphi)=&\frac{T}{2}(Y\cos^2\theta-X\sin^2\theta)\sin2\varphi+\Big(\frac{\sqrt{2+\beta}T^2}{2}\cos^2\varphi-\frac{XY}{2\sqrt{2+\beta}}\sin^2\varphi\Big)\sin2\theta\\
g_2(\theta,\varphi)=&\Big(\sqrt[4]{2+\beta}\,T\sin\theta\cos\varphi+\frac{Y}{\sqrt[4]{2+\beta}}\cos\theta\sin\varphi\Big)^2\\
g_3(\theta,\varphi)=&\Big(\sqrt[4]{2+\beta}\,T\cos\theta\cos\varphi-\frac{X}{\sqrt[4]{2+\beta}}\sin\theta\sin\varphi\Big)^2
\end{aligned}\right\}\quad(3.10)
$$

如果高速扩展平面应力裂纹尖端的各向异性塑性场存在着径向的应力间断线，则有[8]

$$
\sigma_\theta^+=\sigma_\theta^-,\quad \tau_{r\theta}^+=\tau_{r\theta}^-,\quad \sigma_r^+-\sigma_r^-=\frac{1}{A_1}\cdot\sqrt{B_1^2-4A_1C_1}\qquad(3.11)
$$

其中

$$
\left.\begin{aligned}
A_1(\theta)=&1+\alpha_1\cos^22\theta+\alpha_2\sin^22\theta+2\alpha_3\cos2\theta\\
B_1(\theta)=&2\{(1-\alpha_1\cos^22\theta-\alpha_2\sin^22\theta)\sigma_\theta\\
&-[(\alpha_1-\alpha_2)\sin4\theta+2\alpha_3\sin2\theta]\tau_{r\theta}\}\\
C_1(\theta)=&(1+\alpha_1\cos^22\theta+\alpha_2\sin^22\theta-2\alpha_3\mathrm{co}2\theta)\sigma_\theta^2\\
&+4(\alpha_1\sin^22\theta+\alpha_2\cos^22\theta)\tau_{r\theta}^2\\
&-2[(\alpha_1-\alpha_2)\sin4\theta+2\alpha_3\sin2\theta]\sigma_\theta\tau_{r\theta}-4Z^2
\end{aligned}\right\}\quad(3.12)
$$

这里

$$
\alpha_1=2\Big(\frac{Z^2}{X^2}+\frac{Z^2}{Y^2}\Big)-1,\quad \alpha_2=\frac{Z^2}{T^2},\quad \alpha_3=\frac{Z^2}{X^2}-\frac{Z^2}{Y^2}\qquad(3.13)
$$

四、特殊情形的一般解

现在我们来研究四种特殊情形的一般解．

1．$X=Y=Z$ **的情形** $(\beta_1=\sqrt{3}T/X)$

对于这种情形，$\beta=1$，于是，(3.6)，(3.9)，(3.10)和(3.12)就分别变为：

$$
\left.\begin{aligned}
A=&3\Big\{(\sin2\theta\sin\varphi-\beta_1\cos2\theta\cos\varphi)^2-\alpha^2\sin^2\theta(\sin^2\varphi+\beta_1\cos^2\varphi)\\
&\cdot\Big[1+\frac{\mu}{\mu_1}(1-\alpha^2\sin^2\theta)\Big]\Big\}\\
B=&\beta_1^2(1-\alpha^2\sin^2\theta)^2\\
C=&-2\sqrt{3}(1-\alpha^2\sin^2\theta)(\beta_1^2\cos2\theta\cos\varphi-\beta_1\sin2\theta\sin\varphi)
\end{aligned}\right\}\quad(4.1)
$$

和

$$
\left.\begin{aligned}
&f(\theta,\varphi)=\frac{1}{\sqrt{3}}\cdot(1-\alpha^2\sin^2\theta)(\sin^2\varphi+\beta_1^2\cos^2\varphi)\\
&j_1(\theta,\varphi)=\beta_1\sin2\theta\cos\varphi+\cos2\theta\sin\varphi\\
&j_2(\theta,\varphi)=\frac{\beta_1^2-1}{2\sqrt{3}}(\sin2\varphi-2\alpha^2\sin^2\theta\cos\theta\sin\varphi)\\
&k_1(\theta,\varphi)=X\left[\frac{\alpha^2\sin^3\theta}{\sqrt{3}}(\sin^2\varphi+\beta_1^2\cos^2\varphi)+(g_1\cos\theta-g_2\sin\theta)\right]\\
&k_2(\theta,\varphi)=\frac{X}{\sqrt{3}}\left\{\frac{\alpha^2\sin\theta}{\sqrt{3}}\left[\frac{(\beta_1^2-1)}{2}(\sin2\theta\sin^2\varphi+\beta_1\sin2\varphi\cos^2\theta)\right.\right.\\
&\qquad\left.+(\beta_1^2\cos^2\varphi+\sin^2\varphi)(\sin\theta\cos\varphi+\beta_1\cos\theta\sin\varphi)\right]\\
&\qquad\left.+\beta_1\sin\varphi(g_1\sin\theta-g_2\cos\theta)-\mathrm{oos}\varphi(g_1\cos\theta+g_2\sin\theta)\right\}\\
&l_1(\theta,\varphi)=X\left[-\frac{\alpha^2\cos^2\theta\cos\theta}{\sqrt{3}}(\sin^2\varphi+\beta_1^2\cos^2\varphi)-(g_1\sin\theta-g_2\cos\theta)\right]\\
&l_2(\theta,\varphi)=\frac{X}{\sqrt{3}}\left\{\frac{\alpha^2\sin\theta}{\sqrt{3}}\left[\frac{\beta_1^2-1}{2}(\cos^2\theta\sin2\varphi+\beta_1\sin2\theta\cos^2\varphi)\right.\right.\\
&\qquad\left.+(\beta_1^2\cos^2\varphi+\sin^2\varphi)(\cos\theta\cos\varphi-\beta_1\sin\theta\sin\varphi)\right]\\
&\qquad\left.+\beta_1\sin\varphi(g_3\sin\theta-g_1\cos\theta)-\cos\varphi(g_3\cos\theta+g_1\sin\theta)\right\}
\end{aligned}\right\}\tag{4.2}
$$

和

$$
\left.\begin{aligned}
&g_1=\frac{1}{2\sqrt{3}}[\beta_1\cos2\theta\sin2\varphi+(\beta_1^2\cos^2\varphi-\sin^2\varphi)\sin2\theta\\
&g_2=\frac{1}{\sqrt{3}}(\beta_1\sin\theta\cos\varphi+\cos\theta\sin\varphi)^2\\
&g_3=\frac{1}{\sqrt{3}}(\beta_1\cos\theta\cos\varphi-\sin\theta\sin\varphi)^2
\end{aligned}\right\}\tag{4.3}
$$

和

$$
\left.\begin{aligned}
&A_1(\theta)=4-3\left(1-\frac{1}{\gamma_1^2}\right)\sin^22\theta\\
&B_1(\theta)=-2\left\{\left[2-3\left(1-\frac{1}{\gamma_1^2}\right)\sin^22\theta\right]\sigma_\theta+3\left(1-\frac{1}{\gamma_1^2}\right)\sin4\theta\cdot\tau_{r\theta}\right\}\\
&C_1(\theta)=\left[4-3\left(1-\frac{1}{\gamma_1^2}\right)\sin^22\theta\right]\sigma_\theta^2\\
&\qquad+12\left[\frac{1}{\gamma_1^2}+\left(1-\frac{1}{\gamma^2}\right)\sin^22\theta\right]\tau_{r\theta}^2\\
&\qquad-2\left[3\left(1-\frac{1}{\gamma_1^2}\right)\sin4\theta\right]\sigma_\theta\tau_{r\theta}-4X^2
\end{aligned}\right\}\tag{4.4}
$$

2. $X=Y=\sqrt{3}T$ 的情形($\sqrt{3}T/Z=\beta_2$)

对于这种情形，$\beta=2-\beta_2^2$，于是(3.6)，(3.9)，(3.10)和(3.12)就分别变为：

$$\left.\begin{aligned}
A&=\frac{1}{\beta_2^2}\Big\{(\sqrt{4-\beta_2^2}\cos2\theta\cos\varphi-\sqrt{3}\sin2\theta\sin\varphi)^2\\
&\quad-\alpha^2\sin^2\theta\Big[\Big(1+\frac{\mu}{\mu_1}(1-\alpha^2\sin^2\theta)\Big)((4-\beta_2^2)\cos^2\varphi+3\sin^2\varphi)\Big]\Big\}\\
B&=(1-\alpha^2\sin^2\theta)^2\\
C&=-\frac{2}{\beta^2}(1-\alpha^2\sin^2\theta)(\sqrt{4-\beta_2^2}\cos2\theta\cos\varphi-\sin2\theta\sin\varphi)
\end{aligned}\right\}\tag{4.5}$$

和

$$\left.\begin{aligned}
f(\theta,\varphi)&=\frac{1}{\sqrt{4-\beta_2^2}}\{(1-\alpha^2\sin^2\theta)[3\sin^2\varphi+(4-\beta_2^2)\cos^2\varphi]\}\\
j_1(\theta,\varphi)&=\frac{1}{\beta_2}[\sqrt{3(4-\beta_2^2)}\sin2\theta\cos\varphi+3\cos2\theta\sin\varphi]\\
j_2(\theta,\varphi)&=\frac{1-\beta_2^2}{2\sqrt{4-\beta_2^2}}\cdot(\sin2\varphi-2\alpha^2\sin^2\theta\cos\theta\sin\varphi)\\
k_1(\theta,\varphi)&=\frac{\sqrt{3}T}{\beta_2}\Big\{\frac{\alpha^2\sin^3\theta}{\sqrt{4-\beta_2^2}}[3\sin^2\varphi+(4-\beta_2^2)\cos^2\varphi]+g_1\cos\theta-g_2\sin\theta\Big\}\\
k_2(\theta,\varphi)&=\sqrt{3}T\Big\{\alpha^2\sin\theta\Big[\frac{1-\beta_2^2}{2(4-\beta_2^2)}\Big(\sin2\theta\sin^2\varphi+\sqrt{\frac{4-\beta_2^2}{3}}\sin2\varphi\cos^2\theta\Big)\\
&\quad+\Big(\cos^2\varphi+\frac{3}{4-\beta_2^2}\sin^2\varphi\Big)\Big(\sin\theta\cos\varphi+\sqrt{\frac{4-\beta_2^2}{3}}\cos\theta\sin\varphi\Big)\Big]\\
&\quad+\frac{\sin\varphi}{\sqrt{3}}(g_1\sin\theta-g_2\cos\theta)-\frac{\cos\varphi}{\sqrt{4-\beta_2^2}}(g_1\cos\theta+g_2\sin\theta)\Big\}\\
l_1(\theta,\varphi)&=\frac{\sqrt{3}T}{\beta_2}\Big\{\frac{-\alpha^2\sin^2\theta\cos\theta}{\sqrt{4-\beta_2^2}}[3\sin^2\varphi+(4-\beta_2^2)\cos^2\varphi]\\
&\quad-(g_1\sin\theta-g_2\cos\theta)\Big\}\\
l_2(\theta,\varphi)&=\sqrt{3}T\Big\{\alpha^2\sin\theta\Big[\frac{1-\beta_2^2}{2(4-\beta_2^2)}\Big(\sin2\varphi\cos^2\theta+\sqrt{\frac{4-\beta_2^2}{3}}\sin2\theta\cos^2\varphi\Big)\\
&\quad+\Big(\cos^2\varphi+\frac{3}{4-\beta_2^2}\sin^2\varphi\Big)\Big(\cos\theta\cos\varphi-\sqrt{\frac{4-\beta_2^2}{3}}\sin\theta\sin\varphi\Big)\Big]\\
&\quad+\frac{\sin\varphi}{\sqrt{3}}(g_3\sin\theta-g_1\cos\theta)-\frac{\cos\varphi}{\sqrt{4-\beta_2^2}}(g_3\cos\theta+g_1\sin\theta)\Big\}
\end{aligned}\right\}\tag{4.6}$$

和

$$\left.\begin{aligned}
g_1(\theta,\varphi)&=\frac{1}{2\sqrt{4-\beta_2^2}}\Big\{\sqrt{3(4-\beta_2^2)}\cos2\theta\sin2\varphi+[(4-\beta_2^2)\cos^2\varphi-3\sin^2\varphi]\sin2\theta\Big\}\\
g_2(\theta,\varphi)&=\frac{1}{\sqrt{4-\beta_2^2}}(\sqrt{4-\beta_2^2}\sin\theta\cos\varphi+\sqrt{3}\cos\theta\sin\varphi)^2\\
g_3(\theta,\varphi)&=\frac{1}{\sqrt{4-\beta_2^2}}(\sqrt{4-\beta_2^2}\cos\theta\cos\varphi-\sqrt{3}\sin\theta\sin\varphi)^2
\end{aligned}\right\}\tag{4.7}$$

和

$$
\left.\begin{aligned}
&A_1(\theta)=4-\left(1-\frac{1}{\beta_2^2}\right)\sin^2 2\theta\\
&B_1(\theta)=2\{\left[\frac{2}{\beta_2}-4+\left(1-\frac{1}{\beta_2^2}\right)\sin^2 2\theta\right]\sigma_\theta-\left(1-\frac{1}{\beta_2^2}\right)\sin 4\theta\cdot\tau_{r\theta}\\
&C_1(\theta)=\left[4-\left(1-\frac{1}{\beta_2^2}\right)\sin^2 2\theta\right]\sigma_\theta^2+4\left[3+\left(1-\frac{1}{\beta_2^2}\right)\sin^2 2\theta\right]\tau_{r\theta}^2\\
&\qquad-2\left(1-\frac{1}{\beta_2^2}\right)\sin 4\theta\cdot\sigma_\theta\tau_{r\theta}-4X^2
\end{aligned}\right\}\tag{4.8}
$$

3. $X=Z=\sqrt{3}T$的情形

对于这种情形，$\beta=\sqrt{3}T/Y$，于是，（3.6），（3.9），（3.10）和（3.12）分别变为：

$$
\left.\begin{aligned}
A=&\frac{1}{2-\beta}\left[\sqrt{\frac{2+\beta}{3}}(\beta\sin^2\theta-\cos^2\theta)\cos\varphi+\sin 2\theta\sin\varphi\right]^2\\
&-\frac{\alpha^2\mu\sin^2\theta}{2-\beta}\left\{\frac{4}{E}\sin^2\varphi+\sqrt{\frac{2+\beta}{3}}\,\frac{\beta-1}{2\mu}\sin 2\theta\sin 2\varphi\right.\\
&\left.+\frac{2+\beta}{3}\left[D_1+\frac{1}{\mu}(\beta^2\sin^2\theta+\cos^2\theta)\right]\cos^2\varphi\right\}\\
&+\frac{\alpha^4\mu\sin^4\theta}{2-\beta}\left(\frac{\sin^2\varphi}{\mu_1}+\frac{2+\beta}{3}D_1\cos^2\varphi\right)\\
B=&\frac{1}{3}\left\{(\beta\sin^2\theta+\cos^2\theta)^2-\alpha^2\mu\sin^2\theta\left[D_3(1-\alpha^2\sin^2\theta)+\frac{\beta\sin^2\theta+\cos^2\theta}{\mu}\right]\right\}\\
C=&\frac{2}{\sqrt{3(2-\beta)}}(\beta\sin^2\theta+\cos^2\theta)\left[\sqrt{\frac{2+\beta}{3}}(\beta\sin^2\theta-\cos^2\theta)\cos\varphi\right.\\
&\left.+\sin 2\theta\sin\varphi\right]-\sqrt{\frac{2+\beta}{2-\beta}}\alpha^2\mu\sin^2\theta\left\{\frac{2}{3}(\beta^2\sin^2\theta-\cos^2\theta)\cos\varphi\right.\\
&+\frac{1+\frac{1}{\beta}}{\sqrt{3(2+\beta)}}\sin 2\theta\sin\varphi+\frac{D_2}{\sqrt{3}}\left[\frac{2}{\sqrt{3}}\cos\varphi\right.\\
&\left.\left.-\frac{1-\frac{1}{\beta}}{2\sqrt{2+\beta}}(1+\cos^2\theta+\sin^2\varphi)\sin 2\theta\sin\varphi\right]\right\}\\
&+\frac{2D_2}{3}\sqrt{\frac{2+\beta}{2-\beta}}\alpha^4\mu\sin^4\theta\cos\varphi
\end{aligned}\right\}\tag{4.9}
$$

和

$$
\left.\begin{aligned}
f(\theta,\varphi)=&\frac{\sqrt{2+\beta}}{3}\beta^2\cos^2\varphi-\frac{B}{2\sqrt{3}}(\beta-1)\sin 2\theta\sin 2\varphi+\frac{\beta^2\sin^2\theta+\cos^2\theta}{\sqrt{2+\beta}}\sin^2\varphi\\
&-\alpha^2\mu\sin^2\theta\left(\frac{D_3}{\sqrt{2+\beta}}\sin^2\varphi+\frac{\sqrt{2+\beta}}{3\mu}\beta^2\cos^2\varphi\right)\\
j_1(\theta,\varphi)=&\sqrt{\frac{2+\beta}{2-\beta}}\cdot\frac{\beta}{2\sqrt{3}}(\beta+1)\sin 2\theta\cos\varphi+\frac{\cos^2\theta-\beta^2\sin^2\theta}{\sqrt{2-\beta}}\sin\varphi
\end{aligned}\right.
$$

$$
\left.
\begin{aligned}
&+\frac{D_2\mu}{\sqrt{2-\beta}}\sin^2\theta\sin\varphi\\
&j_2(\theta,\varphi)=\frac{\beta}{2\sqrt{3}}(\beta-1)\sin2\theta\cos2\varphi+\frac{\sin2\varphi}{2\sqrt{2+\beta}}\left[\frac{\beta^2(2+\beta)}{3}-(\beta^2\sin^2\theta+\cos^2\theta)\right]\\
&\quad-\left(\frac{\sqrt{2+\beta}\,\beta^2}{3\mu}-\frac{D_3}{\sqrt{2+\beta}}\right)\alpha^2\mu\sin^2\theta\cos\theta\sin\varphi\\
&k_1(\theta,\varphi)=\frac{\sqrt{3}\,T}{\sqrt{2-\beta}}\left\{\alpha^2\mu\sin^2\theta\left[\frac{D_2}{2\sqrt{3}}\cos\theta\sin2\varphi+\left(\frac{2(1+\nu\beta)}{\sqrt{2+\beta}}\sin^2\varphi\right.\right.\right.\\
&\quad\left.\left.\left.+\frac{\sqrt{2+\beta}}{3\mu}\beta^2\cos^2\varphi\right)\sin\theta\right]+(g_1\cos\theta-\beta g_2\sin\theta)\right\}\\
&k_2(\theta,\varphi)=\sqrt{3}\,T\left\{\alpha^2\mu\sin\theta\left[\left(\frac{\beta^2}{3\mu}-\frac{D_3}{2+\beta}\right)\left(\frac{1}{2}\sin2\theta\sin^2\varphi\right.\right.\right.\\
&\quad\left.+\frac{1}{2}\sqrt{\frac{2+\beta}{3}}\sin2\varphi\cos^2\theta\right)+\left(\frac{\beta^2}{3\mu}\cos^2\varphi+\frac{D_3}{2+\beta}\sin^2\varphi\right)\left(\sin\theta\cos\varphi\right.\\
&\quad\left.\left.+\sqrt{\frac{2+\beta}{3}}\cos\theta\sin\varphi\right)\right]+\frac{\beta}{\sqrt{3}}\sin\varphi(g_1\sin\theta-g_2\cos\theta)\\
&\quad\left.-\frac{\cos\varphi}{\sqrt{2+\beta}}(g_1\cos\theta+\beta g_2\sin\theta)\right\}\\
&l_1(\theta,\varphi)=\frac{\sqrt{3}\,T}{\sqrt{2-\beta}}\left\{-\alpha^2\mu\sin^2\theta\left[\frac{D_2}{2\sqrt{3}}\sin\theta\sin2\varphi+\left(\frac{2\left(\frac{1}{\beta}+\nu\right)}{\sqrt{2+\beta}}\sin^2\varphi\right.\right.\right.\\
&\quad\left.\left.\left.+\frac{\sqrt{2+\beta}}{3\mu}\beta\cos^2\varphi\right)\cos\theta\right]-(\beta g_1\sin\theta-g_2\cos\theta)\right\}\\
&l_2(\theta,\varphi)=\sqrt{3}\,T\left\{\alpha^2\mu\sin\theta\left[\left(\frac{\beta^2}{3\mu}-\frac{D_3}{2+\beta}\right)\left(\frac{1}{2\beta}\cos^2\theta\sin2\varphi\right.\right.\right.\\
&\quad\left.+\frac{1}{2}\sqrt{\frac{2+\beta}{3}}\sin2\theta\cos^2\varphi\right)+\left(\frac{\beta^2}{3\mu}\cos^2\varphi+\frac{D_3}{2+\beta}\sin^2\varphi\right)\left(\frac{1}{\beta}\cos\theta\cos\varphi\right.\\
&\quad\left.\left.-\sqrt{\frac{2+\beta}{3}}\sin\theta\sin\varphi\right)\right]+\frac{\beta}{\sqrt{3}}\sin\varphi(g_3\sin\theta-g_1\cos\theta)\\
&\quad\left.-\frac{\cos\varphi}{\sqrt{2+\beta}}(g_3\cos\theta+\beta g_1\sin\theta)\right\}\\
&\qquad(D_1=1+\beta(\beta-2\nu),\ D_2=\beta^2-1,\ D_3=1+\beta(\beta+2\nu))
\end{aligned}
\right\}\quad(4.10)
$$

和

$$
\left.
\begin{aligned}
&g_1(\theta,\varphi)=\frac{1}{2\sqrt{3}}(\cos^2\theta-\beta\sin^2\theta)\sin2\varphi+\frac{1}{2}\left(\frac{\sqrt{2+\beta}}{3}\beta\cos^2\varphi\right.\\
&\quad\left.-\frac{1}{\sqrt{2+\beta}}\sin^2\varphi\right)\sin2\theta\\
&g_2(\theta,\varphi)=\frac{1}{\beta\sqrt{2+\beta}}\left(\sqrt{\frac{2+\beta}{3}}\beta\sin\theta\cos\varphi+\cos\theta\sin\varphi\right)^2\\
&g_3(\theta,\varphi)=\frac{1}{\beta\sqrt{2+\beta}}\left(\sqrt{\frac{2+\beta}{3}}\cos\theta\cos\varphi-\sin\theta\sin\varphi\right)^2
\end{aligned}
\right\}\quad(4.11)
$$

和

$$\left.\begin{aligned}
&A_1(\theta)=3+\beta^2+(1-\beta^2)(\cos 4\theta+2\cos 2\theta)\\
&B_1(\theta)=2\{[(1-\beta^2)\cos 4\theta-1-\beta^2]\sigma_\theta+2[(1-\beta^2)(\sin 4\theta-\sin 2\theta)]\tau_{r\theta}\}\\
&C_1(\theta)=[3+\beta^2-(1-\beta^2)(\cos 4\theta+2\cos 2\theta)]\sigma_\theta^2\\
&\qquad+4[2+\beta^2+(1-\beta^2)\cos 4\theta]\tau_{r\theta}^2\\
&\qquad+4[(1-\beta^2)(\sin 4\theta-\sin 2\theta)]\sigma_\theta\tau_{r\theta}-4X^2
\end{aligned}\right\}\tag{4.12}$$

4. $Y=Z=\sqrt{3}\,T$ 的情形

对于这种情形，我们有 $\beta=\sqrt{3}\,T/X$，于是，(3.6)，(3.9)，(3.10)和(3.12)就分别变为：

$$\left.\begin{aligned}
A=&\frac{1}{2-\beta}\left[\sqrt{\frac{2+\beta}{3}}\sin^2\theta-\beta\cos^2\theta)\cos\varphi+\sin 2\theta\sin\varphi\right]^2\\
&-\frac{\alpha^2\mu\sin^2\theta}{2-\beta}\left\{\frac{4}{E}\sin^2\varphi+\sqrt{\frac{2+\beta}{3}}(1-\beta)\sin 2\theta\sin 2\varphi\right.\\
&\left.+\frac{2+\beta}{3}\left[D_1+\frac{1}{\mu}(\sin^2\theta+\beta^2\cos^2\theta)\right]\cos^2\varphi\right\}\\
&+\frac{\alpha^4\mu\sin^4\theta}{2-\beta}\left(\frac{\sin^2\varphi}{\mu_1}+\frac{2+\beta}{3}D_1\cos^2\varphi\right)\\
B=&\frac{1}{3}\left\{(\sin^2\theta+\beta\cos^2\theta)^2-\alpha^2\mu\sin^2\theta\left[D_3(1-\alpha^2\sin^2\theta)+\frac{\sin^2\theta+\beta^2\cos^2\theta}{\mu}\right]\right\}\\
C=&\frac{2}{\sqrt{3(2-\beta)}}(\sin^2\theta+\beta\cos^2\theta)\left[\sqrt{\frac{2+\beta}{3}}(\sin^2\theta-\beta\cos^2\theta)\cos\varphi+\sin 2\theta\sin\varphi\right]\\
&-\sqrt{\frac{2+\beta}{2-\beta}}\alpha^2\mu\sin^2\theta\left\{\frac{2}{3}(\sin^2\theta-\beta\cos^2\theta)\cos\varphi+\frac{1+\frac{1}{\beta}}{\sqrt{3(2+\beta)}}\sin 2\theta\sin\varphi\right.\\
&\left.+\frac{D_2}{\sqrt{3}}\left[\frac{2}{\sqrt{3}}\cos\varphi-\frac{\frac{1}{\beta}-1}{2\sqrt{2+\beta}}(1+\cos^2\theta+\sin^2\varphi)\sin 2\theta\sin\varphi\right]\right\}\\
&+\frac{2D_2}{3}\sqrt{\frac{2+\beta}{2-\beta}}\alpha^4\mu^2\sin^4\theta\cos\varphi\quad.
\end{aligned}\right\}\tag{4.13}$$

和

$$\begin{aligned}
f(\theta,\varphi)=&\frac{\sqrt{3+\beta}}{3}\beta^2\cos^2\varphi-\frac{\beta}{2\sqrt{3}}(1-\beta)\sin 2\theta\sin 2\varphi+\frac{\sin^2\theta+\beta^2\cos^2\theta}{\sqrt{2+\beta}}\sin^2\varphi\\
&-\alpha^2\mu\sin^2\theta\left(\frac{D_3}{\sqrt{2+\beta}}\sin^2\varphi+\frac{\sqrt{2+\beta}}{3\mu}\beta^2\cos^2\varphi\right)\\
j_1(\theta,\varphi)=&\sqrt{\frac{2+\beta}{2-\beta}}\cdot\frac{\beta}{2\sqrt{3}}(1+\beta)\sin 2\theta\cos\varphi+\frac{\beta^2\cos^2\theta-\sin^2\theta}{\sqrt{2-\beta}}\sin\varphi\\
&+\frac{D_2\mu}{\sqrt{2-\beta}}\sin^2\theta\sin\varphi\\
j_2(\theta,\varphi)=&\frac{\beta}{2\sqrt{3}}(1-\beta)\sin 2\theta\cos 2\varphi+\frac{\sin 2\varphi}{2\sqrt{2+\beta}}\left[\frac{\beta^2(2+\beta)}{3}-(\sin^2\theta+\beta^2\cos^2\theta)\right]
\end{aligned}$$

$$
\left.\begin{aligned}
&-\left(\frac{\sqrt{2+\beta}\cdot\beta^2}{3\mu}-\frac{D_3}{\sqrt{2+\beta}}\right)\alpha^2\mu\sin^2\theta\cos\theta\sin\varphi\\
&k_1(\theta,\varphi)=\frac{\sqrt{3}\,T}{\sqrt{2-\beta}}\left\{\alpha^2\mu\sin^2\theta\left[\frac{D_2}{2\sqrt{3}}\cos\theta\sin2\varphi+\left(\frac{2(\beta+\nu)}{\sqrt{2+\beta}}\sin^2\varphi\right.\right.\right.\\
&\qquad\left.\left.\left.+\frac{\sqrt{2+\beta}}{3\mu}\beta\cos^2\varphi\right)\sin\theta\right]+(\beta g_1\cos\theta-g_2\sin\theta)\right\}\\
&k_2(\theta,\varphi)=\sqrt{3}\,T\left\{\alpha^2\mu\sin\theta\left[\left(\frac{\beta^2}{3\mu}-\frac{D_3}{2+\beta}\right)\left(\frac{1}{2\beta}\sin2\theta\sin^2\varphi\right.\right.\right.\\
&\qquad\left.+\frac{1}{2}\sqrt{\frac{2+\beta}{3}}\sin2\varphi\cos^2\theta\right)+\left(\frac{\beta^2}{3\mu}\cos^2\varphi+\frac{D_3}{2+\beta}\sin^2\varphi\right)\left(\frac{1}{\beta}\sin\theta\cos\varphi\right.\\
&\qquad\left.\left.+\sqrt{\frac{2+\beta}{3}}\cos\theta\sin\varphi\right)\right]+\frac{\beta}{\sqrt{3}}\sin\varphi(g_1\sin\theta-g_2\cos\theta)\\
&\qquad\left.-\frac{\cos\varphi}{\sqrt{2+\beta}}(\beta g_1\cos\theta+g_2\sin\theta)\right\}\\
&l_1(\theta,\varphi)=\frac{\sqrt{3}\,T}{\sqrt{2-\beta}}\left\{-\alpha^2\mu\sin^2\theta\left[\frac{D_2}{2\sqrt{3}}\sin\theta\sin2\varphi+\left(\frac{2(1+\nu\beta)}{\sqrt{2+\beta}}\sin^2\varphi\right.\right.\right.\\
&\qquad\left.\left.\left.+\frac{\sqrt{2+\beta}}{3\mu}\beta^2\cos^2\varphi\right)\cos\theta\right]-(g_1\sin\theta-\beta g_2\cos\theta)\right\}\\
&l_2(\theta,\varphi)=\sqrt{3}\,T\left\{\alpha^2\mu\sin\theta\left[\left(\frac{\beta^2}{3\mu}-\frac{D_3}{2+\beta}\right)\left(\frac{1}{2}\cos^2\theta\sin2\varphi\right.\right.\right.\\
&\qquad\left.+\frac{1}{2}\sqrt{\frac{2+\beta}{3}}\sin2\theta\cos^2\varphi\right)\\
&\qquad\left.+\left(\frac{\beta^2}{3\mu}\cos^2\varphi+\frac{D_3}{2+\beta}\sin^2\varphi\right)\left(\cos\theta\cos\varphi-\sqrt{\frac{2+\beta}{3}}\sin\theta\sin\varphi\right)\right]\\
&\qquad\left.+\frac{\beta}{\sqrt{3}}\sin\varphi(g_3\sin\theta-g_1\cos\theta)-\frac{\cos\varphi}{\sqrt{2+\beta}}(\beta g_3\cos\theta+g_1\sin\theta)\right\}\\
&\qquad(D_1=1+\beta(\beta-2\nu),\ D_2=1-\beta^2,\ D_3=1+\beta(\beta+2\nu))
\end{aligned}\right\}\tag{4.14}
$$

和

$$
\left.\begin{aligned}
g_1(\theta,\varphi)&=\frac{1}{2\sqrt{3}}(\beta\cos^2\theta-\sin^2\theta)\sin2\varphi\\
&\qquad-\frac{1}{2}\left(\frac{\sqrt{2+\beta}}{3}\beta\cos^2\varphi-\frac{1}{\sqrt{2+\beta}}\sin^2\varphi\right)\sin2\theta\\
g_2(\theta,\varphi)&=\frac{1}{\beta\sqrt{2+\beta}}\left(\sqrt{\frac{2+\beta}{3}}\sin\theta\cos\varphi+\cos\theta\sin\varphi\right)^2\\
g_3(\theta,\varphi)&=\frac{1}{\beta\sqrt{2+\beta}}\left(\sqrt{\frac{2+\beta}{3}}\beta\cos\theta\cos\varphi-\sin\theta\sin\varphi\right)^2
\end{aligned}\right\}\tag{4.15}
$$

$$
\left.\begin{aligned}
&A_1(\theta)=3+\beta^2+(\beta^2-1)(\cos4\theta+2\cos2\theta)\\
&B_1(\theta)=2\{[(1-\beta^2)\cos4\theta-1-\beta^2]\sigma_\theta+2[(1-\beta^2)(\sin4\theta+\sin2\theta)]\tau_{r\theta}\}\\
&C_1(\theta)=[3+\beta^2-(1-\beta^2)(\cos4\theta-2\cos2\theta)]\sigma_\theta^2\\
&\qquad+4[2+\beta^2+(1-\beta^2)\cos4\theta]\tau_{r\theta}^2\\
&\qquad+4[(1-\beta^2)(\sin4\theta+\sin2\theta)]\sigma_\theta\tau_{r\theta}-4Z^2
\end{aligned}\right\}\tag{4.16}
$$

五、各向异性塑性场

作为实例，我们给出 $X=Y=Z$ 的情形的高速扩展平面应力 I 型裂纹尖端的各向异性塑性场，即

1. **非均匀塑性区**($0\leqslant\theta\leqslant\theta_1$)

$$\left.\begin{aligned}
&\sigma_x=\frac{X}{\sqrt{3}}(\sqrt{3}\cos\omega-\sin\omega\cos\varphi)\\
&\sigma_y=\frac{X}{\sqrt{3}}(\sqrt{3}\cos\omega+\sin\omega\cos\varphi)\\
&\tau_{xy}=T\sin\omega\sin\varphi\\
&u_x=u_{x0}+\int_{\pi/6}^{\omega}\left[\frac{k_1(\theta,\varphi)\cdot\sin\omega+k_2(\theta,\varphi)\cos\omega}{a^2\mu\sin\theta\cdot f(\theta,\varphi)}\right]d\omega\\
&v_x=\int_{\pi/6}^{\omega}\left[\frac{l_1(\theta,\varphi)\sin\omega+l_2(\theta,\varphi)\cos\omega}{a^2\mu\sin\theta\cdot f(\theta,\varphi)}\right]d\omega\\
&\varphi=-\int_{\pi/6}^{\omega}\left[\frac{j_1(\theta,\varphi)+j_2(\theta,\varphi)\cot\omega}{f(\theta,\varphi)}\right]d\omega
\end{aligned}\right\}\tag{5.1}$$

这里，$\sin\omega$和$\cos\omega$由 (3.7) 和 (4.1) 给出；$k_1(\theta,\ \varphi)$，$k_2(\theta,\ \varphi)$，$l_1(\theta,\ \varphi)$，$l_2(\theta,\ \varphi)$，$j_1(\theta,\ \varphi)$，$j_2(\theta,\ \varphi)$和$f(\theta,\ \varphi)$由 (4.2) 和 (4.3) 给出。

2. **均匀塑性区**($\theta_1\leqslant\theta\leqslant\theta_2$)

$$\left.\begin{aligned}
&\sigma_x=(1+\cos2\theta_2)[\sigma_r]_{\theta=\theta_2}-X\\
&\sigma_y=(1-\cos2\theta_2)[\sigma_r]_{\theta=\theta_2}\\
&\tau_{xy}=[\sigma_r]_{\theta=\theta_2}\sin2\theta_2\\
&\varphi=-\int_{\pi/6}^{\omega(\theta_1)}\left[\frac{j_1(\theta,\varphi)+j_2(\theta,\varphi)\cot\omega}{f(\theta,\varphi)}\right]d\omega\\
&u_x=u_{x0}+\int_{\pi/6}^{\omega(\theta_1)}\left[\frac{k_1(\theta,\varphi)\sin\omega+k_2(\theta,\varphi)\cos\omega}{a^2\mu\sin\theta\cdot f(\theta,\varphi)}\right]d\omega\\
&v_x=\int_{\pi/6}^{\omega(\theta_1)}\left[\frac{l_1(\theta,\varphi)\sin\omega+k(\theta,\varphi)\cos\omega}{a^2\mu\sin\theta\cdot f(\theta,\varphi)}\right]d\omega
\end{aligned}\right\}\tag{5.2}$$

其中

$$[\sigma_r]_{\theta=\theta_2}=\frac{\sqrt{B_1^2-4A_1C_1}}{A_1}\tag{5.3}$$

而A_1，B_1和C_1则由式(3.12)来确定。

3. **均匀塑性区**($\theta_2\leqslant\theta\leqslant\pi$)

$$\left.\begin{aligned}
&\sigma_x=-X,\quad \sigma_y=\tau_{xy}=0\\
&u_x=u_{x0}+\int_{\pi/6}^{\omega(\theta_1)}\left[\frac{k_1(\theta,\varphi)\sin\omega+k_2(\theta,\varphi)\cos\omega}{a^2\mu\sin\theta\cdot f(\theta,\varphi)}\right]d\omega\\
&v_x=\int_{\pi/6}^{\omega(\theta_1)}\left[\frac{l_1(\theta,\varphi)\sin\omega+l_2(\theta,\varphi)\cos\omega}{a^2\mu\sin\theta\cdot f(\theta,\varphi)}\right]d\omega
\end{aligned}\right\}\tag{5.4}$$

这里$\theta=\theta_2$是应力间断线。

$\theta=\theta_1$上的应力连续条件给出确定θ_1和θ_2的两个方程为：

$$\left.\begin{aligned}&2[\sigma_r]_{\theta=\theta_2}-X=2X\cos\omega(\theta_1)\\&[\sigma_r]_{\theta=\theta_2}\sin2\theta_2=\frac{\beta_1X}{\sqrt{3}}\sin\omega(\theta_1)\cdot\sin\varphi(\theta_1)\end{aligned}\right\}\tag{5.5}$$

参 考 文 献

[1] 林拜松，高速扩展平面应力裂纹尖端的理想塑性应力场，应用数学和力学，7(8)(1986)，751—758.

[2] 林拜松，高速扩展平面应力裂纹尖端的理想塑性场，应用数学和力学，12(7)1991)，613—620.

[3] 林拜松，静止平面应力裂纹尖端的各向异性塑性应力场，应用数学和力学（待发表）.

Anisotropic Plastic Fields at a Rapidly Propagating Plane-Stress Crack-Tip

Lin Bai-song

(Central-South University of Technology, Changsha)

Abstract

Under the condition that all the stress components at a crack-tip are the functions of θ only, making use of the equations of steady-state motion, Hill anisotropic yield condition and stress-strain relations, we obtain the general solution of anisotropic plastic field at a rapidly propagating plane-stress crack-tip. Applying this general solution to four particular cases of anisotropy the general solutions of these four particular cases are derived. Finally, We give the anisotropic plastic field at the rapidly propagating plane-stress mode I crack-tip in the case of $Z=Y=Z$.

Key words anisotropic plastic field, crack-tip, plane-stress