

正交各向异性体梁弯曲的弹性理论*

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摘 要

本文由文献[1]横观各向同性板的弯曲弹性理论关于二维问题的特例, 通过比拟, 得到了正交各向异性梁弯曲的弹性理论, 文中给出了求解正交各向异性梁弯曲问题的一种方法。提出了一种新的深梁理论, 并指出了考虑横向剪切变形影响的Reissner理论对于应力分量的近似程度较差。

关键词 弹性理论 正交各向异性体梁 Reissner理论

一、引 言

在梁的弯曲和板的板条的柱状弯曲问题中, 为了修正经典梁的理论, 人们提出了各种考虑剪切变形的深梁理论, 由于在推导中采用了各种假设, 使得这些理论的解不能满足弹性理论的全部方程。

文献[1]采用算子符号法, 分析了横观各向同性板的三维弹性弯曲问题及相应的厚板理论。本文利用横观各向同性体平面应变状态和正交各向异性体平面应力状态基本方程的比较, 由文献[1]结论的比拟, 导出正交各向异性梁弯曲的弹性理论。

二、正交各向异性体平面问题的弯曲弹性理论

(一) 正交各向异性体平面应力状态的基本方程

1. 平面应力状态:

$$\sigma_y = 0, \tau_{xy} = 0, \tau_{zy} = 0 \quad (2.1)$$

2. 选取材料主方向1, 2, 3分别与直角坐标系的 x, y, z 轴重合, 则物理方程

$$\left. \begin{aligned} \sigma_x &= A_{11} \frac{\partial u}{\partial x} + A_{13} \frac{\partial w}{\partial z} \\ \sigma_z &= A_{13} \frac{\partial u}{\partial x} + A_{33} \frac{\partial w}{\partial z} \\ \tau_{xz} &= A_{44} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \end{aligned} \right\} \quad (2.2)$$

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$$\text{式中 } \left. \begin{aligned} A_{11} &= \frac{E_1}{1-\nu_{13}\nu_{31}}, & A_{33} &= \frac{E_3}{1-\nu_{13}\nu_{31}} \\ A_{13} &= \frac{E_1\nu_{31}}{1-\nu_{13}\nu_{31}} = \frac{E_3\nu_{13}}{1-\nu_{13}\nu_{31}}, & A_{44} &= G_{31} \end{aligned} \right\} \quad (2.3)$$

为了简明起见, 对平面应变状态不另讨论. 平面应力状态和平面应变状态的区别只是在(2.3)式不同.

3. 以位移表示平衡方程 (无体力):

$$\left(B_{11} \frac{\partial^2}{\partial x^2} + B_{44} \frac{\partial^2}{\partial z^2} \right) u + B_{13} \frac{\partial^2 w}{\partial x \partial y} = 0 \quad (2.4)$$

$$B_{13} \frac{\partial^2 u}{\partial x \partial z} + \left(B_{44} \frac{\partial^2}{\partial x^2} + B_{33} \frac{\partial^2}{\partial z^2} \right) w = 0$$

$$\text{式中: } B_{11} = A_{11}, \quad B_{33} = A_{33}, \quad B_{44} = A_{44}, \quad B_{13} = A_{13} + A_{44} \quad (2.5)$$

(二) 横观各向同性体平面应变问题^[1]的基本方程

1. 平面应变状态:

$$v = 0, \quad u = u(x, y), \quad w = w(x, x) \quad (2.6)$$

2. 物理方程:

$$\left. \begin{aligned} \sigma_x &= A_{11} \frac{\partial u}{\partial x} + A_{13} \frac{\partial w}{\partial z} \\ \sigma_z &= A_{13} \frac{\partial u}{\partial x} + A_{33} \frac{\partial w}{\partial z} \\ \tau_{xz} &= A_{44} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \end{aligned} \right\} \quad (2.7)$$

此式中 A_{ij} 与(2.3)中 A_{ij} 表达式不同, 参见文献[1].

3. 以位移表示的平衡方程 (无体力)

$$\left. \begin{aligned} \left(B_{11} \frac{\partial^2}{\partial x^2} + B_{44} \frac{\partial^2}{\partial z^2} \right) u + B_{13} \frac{\partial^2 w}{\partial x \partial z} = 0 \\ B_{13} \frac{\partial^2 u}{\partial x \partial z} + \left(B_{44} \frac{\partial^2}{\partial x^2} + B_{33} \frac{\partial^2}{\partial z^2} \right) w = 0 \end{aligned} \right\} \quad (2.8)$$

式中 $B_{11}, B_{13}, B_{33}, B_{44}$ 与(2.5)形式上相同.

(三) 两种平面问题的比较

将横观各向同性体平面应变问题的基本方程(2.7)、(2.8)与正交各向异性体平面应力问题的基本方程(2.2)、(2.4)比较发现, 二者在形式上是一样的, 只是 $A_{11}, A_{33}, A_{13}, A_{44}$ 表达式不同, 所以横观各向同性体平面应变弯曲问题的解形式上也能作为正交各向异性体平面应力问题的解. 从物理意义上讲, 就是这两种材料在 Oxz 平面上的性质是一样的.

为便于上述两种状态的转换, 通过一些运算, 利用中间变量 A_{ij} 将文献[1]中的弹性常数转换为用正交各向异性弹性常数表示:

$$\frac{E}{1-\nu^2} = E_1, \quad S = \frac{E_1}{2G_{31}}, \quad S' = \frac{E_1}{G_{31}} - \nu_{13} \quad (2.9)$$

(四) 弯曲弹性理论基本方程

考虑高为 h 的梁承受反对称横向荷载的弯曲问题,如图1所示.选取图示坐标,上下表面应力边界条件为:

$$\tau_{xz}|_{z=\pm h/2}=0$$

$$\sigma_z|_{z=\pm h/2}=\pm q/2$$

$q(x, y)$ 为梁所承受的反对称横向荷载.

根据(三)的讨论和转换公式,由文献[1]的平面应变问题转换得到正交各向异性体梁弯曲弹性理论的基本方程为:

$$\nabla^4 \bar{\varphi}_1 = 0 \quad (2.10a)$$

$$L^* \bar{\varphi} = 0 \quad (2.10b)$$

$$L^*(\nabla^4 \varphi^*) = q/JE_1 \quad (2.10c)$$

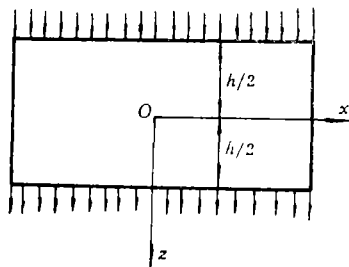


图 1

$$\text{式中: } L^* = \frac{2}{\nabla^2(S_1^2 - S_2^2)} J \left(\frac{\sin S_1 \nabla h/2}{S_1 \nabla} \cos \frac{S_2 \nabla h}{2} - \frac{\sin \frac{S_2 \nabla h}{2}}{S_2 \nabla} \cos \frac{S_1 \nabla h}{2} \right) \quad (2.11a)$$

$$S_1^2 = (B_{44}^2 + B_{11}B_{33} - B_{13}^2 + \sqrt{(B_{44}^2 + B_{11}B_{33} - B_{13}^2) - 4B_{11}B_{33}B_{44}^2}) / 2B_{33}B_{44} \quad (2.11b)$$

$$S_2^2 = (B_{44}^2 + B_{11}B_{33} - B_{13}^2 - \sqrt{(B_{44}^2 + B_{11}B_{33} - B_{13}^2) - 4B_{11}B_{33}B_{44}^2}) / 2B_{33}B_{44} \quad (2.11c)$$

$$\nabla^2 = \frac{d^2}{dx^2}, \quad J = h^3/12$$

(2.10a)为退化的双调和方程,(2.10b)不产生横向剪力,故讨论横向剪力对变形的影响时,可不予考虑,(2.10c)为特解方程.它们构成了平面弯曲弹性理论的基本方程,这个理论能满足梁端侧面上精确给出的任何数目的应力、位移边界条件.

三、一种新的深梁理论

对于特解方程(2.10c)总可以写成进行了量级处理的形式:

$$\nabla^4 \varphi^* = q/JE_1 + O(h^2 \nabla^2) \nabla^4 \varphi^*$$

考虑到梁的特征尺寸,将 $O(h^2 \nabla^2) \nabla^4 \varphi^*$ 与 q/JE_1 比较略去不计,就得到一种新的深梁理论,基本方程和基本表达式的形式也可由文献[1]中的厚板理论关于平面应变的特例而转换得到,以轴线挠度 w_0 为基本变量来表示.

$$1. \text{ 基本方程: } \frac{d^4 w_0}{dx^4} = q(x)/JE_1 \quad (3.1)$$

2. 位移分量和应力分量及剪力、弯矩.

$$\begin{aligned} u = & -z \left[1 + \left(\frac{h^2}{4} S - \frac{z^2}{6} S' \right) \frac{d^2}{dx^2} \right] \frac{dw_0}{dx} + \frac{h^2 z^3}{24} \left(S_1^2 + S_2^2 - \frac{A_{44}}{A_{33}} \right) S \frac{d^5 w_0}{dx^5} \\ & + \frac{zh^4}{32} S \left[\frac{1}{6} (S_1^2 + S_2^2) - S' \right] \frac{d^6 w_0}{dx^6} \\ & - \frac{z^5}{120} \left[S_1^4 + S_2^4 + S_1^2 S_2^2 + \frac{A_{13}}{A_{33}} (S_1^2 + S_2^2) \right] \frac{d^6 w_0}{dx^6} \end{aligned} \quad (3.2)$$

$$w = w_0 + \frac{1}{2} \frac{A_{13}}{A_{33}} z^2 \frac{d^2 w_0}{dx^2} + \frac{1}{8} \frac{B_{13}}{A_{33}} h^2 z^2 S \frac{d^4 w_0}{dx^4}$$

$$-\frac{z^4}{24} \left[\frac{A_{13}}{A_{33}} (S_1^2 + S_2^2) + S_1^2 S_2^2 \right] \frac{d^4 w_0}{dx^4} \quad (3.3)$$

$$\sigma_x = -E_1 z \left\{ \frac{d^2 w_0}{dx^2} + \frac{q(x)}{2E_1 J} \left[\frac{h^2}{4} S' + \frac{2}{3} z^2 \left(\frac{A_{13}}{A_{33}} - S \right) \right] \right\} \quad (3.4)$$

$$\sigma_z = \frac{q(x)z}{2J} \left(\frac{h^2}{4} - z^2 \right) \quad (3.5)$$

$$\begin{aligned} \tau_{xz} = & -\frac{E_1}{2} \left(\frac{h^2}{4} - z^2 \right) \frac{d^3 w_0}{dx^3} - \frac{E_1}{2} \left(\frac{h^2}{4} - z^2 \right) \frac{h^2}{8} S' \frac{d^5 w_0}{dx^5} \\ & + \frac{E_1}{24} (S_1^2 + S_2^2) \left(\frac{h^4}{16} - z^4 \right) \frac{d^5 w_0}{dx^5} \end{aligned} \quad (3.6)$$

$$M_x = -JE_1 \frac{d^2 w_0}{dx^2} - \frac{1}{8} q h^2 \left[S' + \frac{2}{5} \left(\frac{A_{13}}{A_{33}} - S \right) \right] \quad (3.7)$$

$$Q_x = -JE_1 \frac{d^3 w_0}{dx^3} - \frac{1}{8} h^2 \left[S' + \frac{2}{5} \left(\frac{A_{13}}{A_{33}} - S \right) \right] \frac{dq}{dx} \quad (3.8)$$

可以验证, 当 q 为线性荷载时, 新的深梁理论的解就是弹性理论的解, 当 q 为非线性荷载时, 新理论只是在弹性理论基础上, 对特解方程进行了量级简化。考虑到梁的尺寸特征, 这个简化所带来的误差是很小的, 所以, 与常见的深梁理论比较, 本文理论最靠近弹性理论的结果。新理论不但考虑了剪切变形的影响, 而且考虑了横向正应力和横向正应变的影响, 与两个广义位移的深梁理论相比, 新的深梁理论只有一个基本变量, 在解题时更加简明。

四、受线性分布荷载作用梁弯曲问题的理论解

在梁弯曲问题的求解中, 如果梁端以应力合力 (或中面位移) 给出两个边界, 则这两个边界条件由(2.10a)来满足, (2.10b)的解取为零, 这时所求得解是边界条件在圣维南原理意义下得到满足的弹性理论解。对于受线性分布荷载作用的梁, 采用 w_0 为基本未知量, 则基本方程(2.10a)和(2.10c)等价于基本方程(3.1), 相应的各力学量如(3.2)~(3.8)所示。求解 w_0 的过程与材料力学解梁问题过程很相似。用应力函数法解下列问题的结果, 请见文献[2]。

1. 受均布荷载的两端简支梁 (见图2)

梁端边界条件: $M_x|_{x=\pm l/2} = 0, w_0|_{x=\pm l/2} = 0$

$$\begin{aligned} w_0 = & \frac{qx^4}{24JE_1} - \frac{ql^2}{2JB_1} \left[\frac{1}{8} + \frac{h^2}{40l^2} \left(\frac{4E_1}{G_{31}} - 3\nu_{13} \right) \right] x^2 \\ & + \frac{5ql^4}{384JE_1} \left[1 + \frac{6h^2}{25l^2} \left(\frac{4E_1}{G_{31}} - 3\nu_{13} \right) \right] \end{aligned}$$

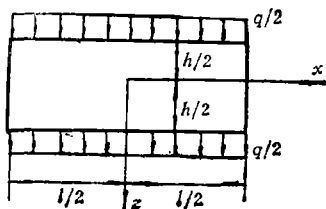


图 2

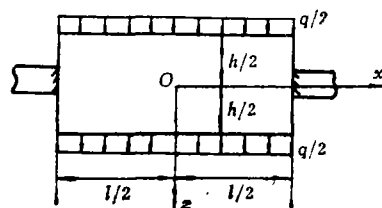


图 3

$$\sigma_z \Big|_{\substack{x=0 \\ z=h/2}} = \frac{ql^2h}{16J} + \frac{q}{10} \left(\frac{E_1}{G_{31}} - 2\nu_{13} \right)$$

2. 受均布荷载的两端固支梁 (见图3).

边界条件: $\frac{\partial u}{\partial z} \Big|_{x=\pm l/2} = 0, w_0 \Big|_{z=0} = 0$

$$w_0 = \frac{qx^4}{24JE_1} - \left(\frac{ql^2}{48JE_1} + \frac{qh^2}{16JG_{31}} \right) x^2 + \frac{ql^4}{384JE_1} + \frac{ql^2h^2}{64JG_{31}}$$

$$\sigma_x = \frac{qz}{2J} \left(\frac{l^2}{12} - x^2 \right) + \frac{qz^3}{3J} \left(\frac{E_1}{2G_{31}} - \nu_{13} \right) + \frac{qh^2}{8J} z\nu_{13}$$

$$\sigma_z = \frac{qz}{2J} \left(\frac{h^2}{4} - \frac{z^2}{3} \right)$$

$$\tau_{xz} = \frac{qx}{2J} \left(z^2 - \frac{h^2}{4} \right)$$

$$\sigma_x \Big|_{\substack{x=0 \\ z=h/2}} = \frac{qh}{48J} \left(l^2 + \frac{E_1}{G_{31}} h^2 + 5\nu_{13} h^2 \right)$$

3. 受均布荷载的悬臂梁的弯曲 (见图4).

边界条件: $M_x \Big|_{z=0} = 0, Q_x \Big|_{z=0} = 0$

$$w_0 \Big|_{z=l} = 0, \frac{\partial u}{\partial z} \Big|_{z=0} = 0$$

$$w_0 = \frac{qx^4}{24JE_1} - \frac{qh^2}{80JE_1} \left(\frac{4E_1}{G_{31}} - 3\nu_{13} \right) x^2 - \frac{qh}{2J} \left(\frac{l^2}{3E_1} + \frac{h^2}{20G_{31}} + \frac{3h^2}{20E_1} \nu_{13} \right) x + \frac{ql^4}{8JE_1} \left(\frac{3E_1h^2}{5G_{31}l^2} + \frac{3h^2}{10l^2} \nu_{13} + 1 \right)$$

$$\sigma_x = -\frac{qzx^2}{2J} + \frac{q}{2h} \left(\frac{E_1}{G_{31}} - 2\nu_{13} \right) \left(\frac{4z^3}{h^2} - \frac{3}{5}z \right)$$

$$\sigma_z = \frac{qz}{2J} \left(\frac{h^2}{4} - \frac{z^2}{3} \right)$$

$$\tau_{xz} = \frac{qx}{2J} \left(z^2 - \frac{h^2}{4} \right)$$

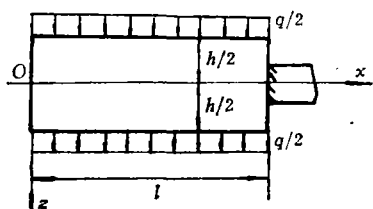


图 4

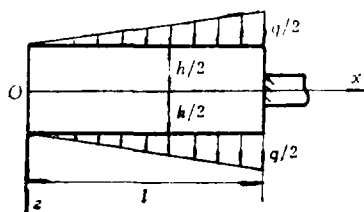


图 5

4. 承受按线性规律分布的荷载的悬臂梁的弯曲(见图5).

边界条件: $M_x|_{x=0}=0, Q_x|_{x=0}=0$

$$w_0|_{x=l}=0, \left. \frac{\partial w}{\partial z} \right|_{z=0} = 0$$

$$\begin{aligned} w_0 = & \frac{qx^5}{120JE_1l} - \frac{qh^2}{240JE_1l} \left(\frac{4E_1}{G_{31}} - 3\nu_{13} \right) x^3 + \left[-\frac{ql^3}{24JE_1} + \frac{qh^4}{960JG_{31}l} \left(\nu_{13} - \frac{E_1}{2G_{31}} \right) \right. \\ & \left. - \frac{qlh^2}{80JG_{31}} \left(\nu_{13} + \frac{E_1}{G_{31}} \right) \right] x + \frac{ql^4}{30JE_1} + \frac{ql^2h^2}{240JE_1} \left(\frac{4E_1}{G_{31}} - 3\nu_{13} \right) \\ & + \frac{qh^4}{960JG_{31}} \left(\frac{E_1}{G_{31}} - \nu_{13} \right) + \frac{qlh^3}{80JG_{31}} \left(\nu_{13} + \frac{E_1}{G_{31}} \right) \end{aligned}$$

五、本文理论与Reissner理论比较

以简支梁受均布荷载为例进行比较, 如图6所示, 计算 w_0, σ_z 的最大值为:

1. 本文

$$\begin{aligned} w_0|_{x=0} &= \frac{5ql^4}{384JE_1} \left[1 + \frac{6h^2}{25l^2} \left(\frac{4E_1}{G_{31}} - 3\nu_{13} \right) \right] \\ \sigma_z \Big|_{\substack{x=0 \\ z=h/2}} &= \frac{ql^2h}{16J} + \frac{q}{10} \left(\frac{E_1}{G_{31}} - 2\nu_{13} \right) \end{aligned}$$

2. R氏理论

$$\begin{aligned} w_0|_{x=0} &= \frac{5ql^4}{384JE_1} \left[1 + \frac{24h^2}{25l^2} \left(\frac{E_1}{G_{31}} - \nu_{13} \right) \right] \\ \sigma_z \Big|_{\substack{x=0 \\ z=h/2}} &= \frac{ql^2h}{16J} \end{aligned}$$

3. 经典理论

$$\begin{aligned} w_{j0}|_{x=0} &= \frac{5ql^4}{384JE_1} \\ \sigma_{jz} \Big|_{\substack{x=0 \\ z=h/2}} &= \frac{ql^2h}{16J} \end{aligned}$$

从图中可以看到, 关于 w_0 , Reissner理论与本文的理论吻合较好, Reissner理论关于 w_0 是较好的近似理论, 关于 σ_z , Reissner理论与经典理论一致, 与本文理论相差较大, 原因是Reissner理论采用了 σ_z 沿 z 方向线性分布的假设. 在 q 为线性函数时, 本文理论与弹性理论是一致的.

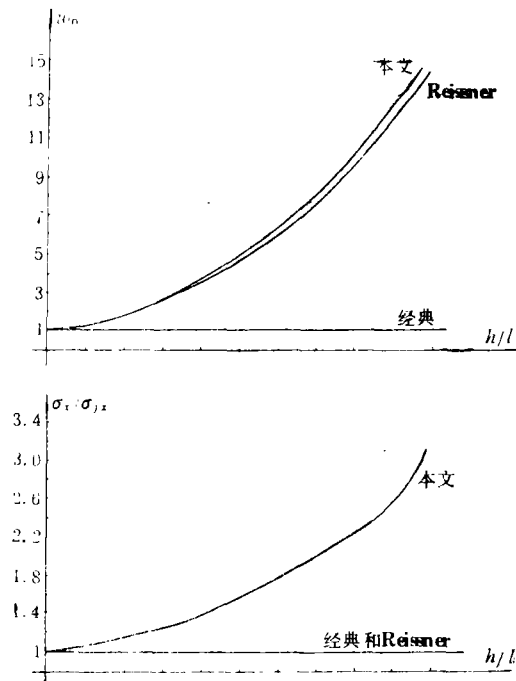


图 6

图中所用材料为胶合板, 其材料常数为: $E_1=12\text{GPa}$, $E_2=6\text{GPa}$, $\nu_{13}=0.071$, $G_{31}=0.7\text{GPa}$.

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Theory of Elasticity of an Anisotropic Body for the Bending of Beams

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Abstract

A theory of elasticity for the bending of orthogonal anisotropic beams has been developed by analogy with the special case, which can be obtained by applying the theory of elasticity for bending of transversely isotropic plates to the problems of two deminsions. In this paper, we present a method to solve the problems of bending of orthogonal anisotropic beams and a new theory of the deep-beam whose ratio of depth to length is larger. It is pointed out that Reissner's theory to account for the effect of transverse shear deformation is not very approximate in the components of stress.

Key words: elasticity, orthogonal anisotropic beams, Reissner's theory