应用数学和力学，第13卷第8期（1992年8月）
Applied Mathematics and Mechanics

应用数学和力学编委会编
重庆出版社出版

可压缩流动的Fourier谱-有限元解法

郭本瑜　　曹卫明

（上海科技大学）

（刘人怀推荐，1991年4月15日收到）

摘　要

本文考虑n维（$n=2$，3）可压缩流动的带有单向周期边值条件问题的数值解。我们在周期方向采用Fourier谱方法，在非周期方向采用有限元方法，从而构造了一类谱-有限元格式。文中严格分析了计算误差，得到了收敛阶的估计。

关键词　可压缩流动　谱方法　有限元方法　误差估计

一、引　言

$n(n=2,3)$维可压缩流动满足下列方程组[1,2]：

$$\left.\begin{aligned}&\partial_t U^{(l)}+(U\cdot\nabla)U^{(l)}-\frac{1}{\rho}\partial_l(K\nabla\cdot U)-\frac{1}{\rho}\sum_{j=1}^{n}\partial_j[\nu(\partial_j U^{(l)}+\partial_l U^{(j)})]\\&\qquad+\frac{1}{\rho}\partial_l P=f^{(l)}\qquad(l=1,\cdots,n)\\&\partial_t T+(U\cdot\nabla)T-\frac{1}{\rho T S_T}(\nabla\cdot\mu\nabla)T-\frac{\nu}{2\rho T S_T}\sum_{l,j=1}^{n}(\partial_l U^{(j)}+\partial_j U^{(l)})^2\\&\qquad-\frac{K}{\rho T S_T}(\nabla\cdot U)^2-\frac{\rho S_\rho}{S_T}(\nabla\cdot U)=0\\&\partial_t\rho+(U\cdot\nabla)\rho+\rho(\nabla\cdot U)=0\end{aligned}\right\}\tag{1.1}$$

其中$\partial_t=\partial/\partial t$，$\partial_l=\partial/\partial x_l$，$\nabla=(\partial_1,\cdots,\partial_n)$，$U=(U^{(1)},\cdots,U^{(n)})$是速度向量，$T$是绝对温度，$\nu(T,\rho)$是粘性系数，$\nu'(T,\rho)$是第二粘性系数，$K(T,\rho)=\nu'(T,\rho)-(2/3)\nu(T,\rho)$，$\mu(T,\rho)$是热传导系数，$S(T,\rho)$是熵，$S_T=\partial S/\partial T$，$S_\rho=\partial S/\partial\rho$。

在一定条件下，Tani[2]证明了(1.1)的局部古典解的存在性．迄今已有许多文献致力于(1.1)的数值解，其中最经典的方法是差分法[1,3]．郭本瑜[4,5]曾应用这一方法求解(1.1)的周期边值问题和第一类边值条件问题，并且严格估计了误差．有限元方法也是一个被经常采用的解法[6]，它特别适合于求解区域不规则的情形．近年来，人们还应用Fourier谱方法求解(1.1)的周期问题[7]．但是，对于许多实际问题，其边界条件并非是完全周期或者完全非周期的．处理这类问题的一个适当的方法是把周期方向的Fourier谱方法与非周期方向

的差分方法或有限元方法结合起来．目前，已有一些文献分析了这种混合方法[8~10]．本文作者曾采用Fourier谱-有限元方法求解不可压缩流动的半周期问题，取得了较为满意的结果[11~13]．

本文推广[11～13]的工作，应用Fourier谱-有限元方法求解(1.1)的带有单向周期边值条件的问题．设$Q\subset\mathbf{R}^{n-1}$是一凸多角形区域（$n=2$时，即为一开区间），$I=(0,2\pi)$，$\Omega=Q\times I=\{x=(x_1,\cdots,x_{n-1},x_n)/(x_1,\cdots x_{n-1})\in Q,x_n\in I\}$，$t_0>0$．我们考虑在区域$(x,t)\in\Omega\times(0,t_0]$上求解(1.1)，并假定(1.1)中所有函数在$x_n$方向以$2\pi$为周期，$U$，$T$在其他空间方向上带有齐次的第一类边值条件，即：

$$\left.\begin{array}{ll}\eta|_{x_n=0}=\eta|_{x_n=2\pi}, & \forall(x_1,\cdots,x_{n-1},t)\in Q\times[0,t_0],\ \eta=U,T,\rho\\ \eta|_{(x_1,\cdots,x_{n-1})\in\partial Q}=0, & \forall(x_n,t)\in I\times[0,t_0],\ \eta=U,T\end{array}\right\}\tag{1.2}$$

另外，我们假定(1.1)的初始条件是

$$\eta|_{t=0}=\eta_0,\ \eta=U,T,\rho\tag{1.3}$$

为了避免在数值计算(1.1)～(1.3)的过程中因舍入误差出现“负密度”（即$\rho<0$），从而导致非物理解及计算不稳定性，我们采用郭本瑜[4,5]的思想，即不直接计算密度ρ，而是引入$\varphi=\ln\rho$．此外，我们还假定流体满足下述状态方程

$$P=R\rho T$$

其中R是一个正常数．这样，(1.1)可以改写为下列形式

$$\left.\begin{array}{l}\partial_tU^{(l)}+(U\cdot\nabla)U^{(l)}-e^{-\varphi}\partial_l(K\nabla\cdot U)-e^{-\varphi}\sum\limits_{j=1}^{n}\partial_j[\nu(\partial_jU^{(l)}+\partial_lU^{(j)})]+R\partial_lT\\ \qquad+RT\partial_l\varphi=f^{(l)}\qquad(l=1,\cdots,n)\\ \partial_tT+(U\cdot\nabla)T-e^{-\varphi}T^{-1}S_T^{-1}(\nabla\cdot\mu\nabla)T-\dfrac{1}{2}\nu e^{-\varphi}T^{-1}S_T^{-1}\sum\limits_{l,j=1}^{n}(\partial_lU^{(j)}+\partial_jU^{(l)})^2\\ \qquad-Ke^{-\varphi}T^{-1}S_T^{-1}(\nabla\cdot U)^2-S_\varphi S_T^{-1}(\nabla\cdot U)=0\\ \partial_t\varphi+(U\cdot\nabla)\varphi+\nabla\cdot U=0\end{array}\right\}\tag{1.4}$$

我们假定ν,μ,K和S对各变量充分光滑，并且存在正常数B_0，B_1，B_2，ν_0，ν_1，μ_0，μ_1,K_1，A_0，A_1，S_0，S_1，S_2，Φ_0和Φ_1，使得当$B_0<T<B_1$，$|\varphi|\leqslant B_2$时

$$\left.\begin{array}{l}\nu_0<\nu<\nu_1,\ \mu_0<\mu<\mu_1,\ |K|<K_1,\ \min(nK+(n+1)\nu,\nu)>A_0\\ S_0<S_T<S_1,\ |S_\varphi|<S_2,\ \Phi_0<e^{-\varphi}<\Phi_1\\ |\partial\eta/\partial z|\leqslant A_1,\ \text{其中}\eta=\nu,K,\mu,S_T,S_\varphi;\ z=T,\varphi\end{array}\right\}\tag{1.5}$$

二、一些记号

设$\mathscr{D}\subset\mathbf{R}^m$（$m=1,2$或3）是一个边界局部Lipschitz连续的有界开区域，$p\geqslant1$，$s\geqslant0$是实数．我们用$W^{s,p}(\mathscr{D})$表示通常的Sobolev空间，其范数和半范数分别记为$\|\cdot\|_{s,p,\mathscr{D}}$和$|\cdot|_{s,p}$，（参见[14]）。特别，记$H^s(\mathscr{D})=W^{s,2}(\mathscr{D})$，$\|\cdot\|_{s,\mathscr{D}}=\|\cdot\|_{s,2}$，$|\cdot|_{s}=|\cdot|_{s,2,\mathscr{D}}$，以及$L^p(\mathscr{D})=W^{0,p}(\mathscr{D})$，并且用$\|\cdot\|$和$(\cdot,\cdot)$分别表示$L^2(\mathscr{D})$上的范数与内积．此外，为简

单计，当$\mathscr{D}=\Omega$时，我们在上述范数、内积等记号中略去足标$\mathscr{D}$。

设$\mathscr{B}$是一个Banach空间，$\vartheta\subset\mathbf{R}^1$是一个开区间。我们定义$\vartheta$上的抽象可测函数空间如下：

$$L^2(\vartheta;\ \mathscr{B})=\{\eta/\eta:\vartheta\to\mathscr{B},\ \|\eta\|_{L^2(\vartheta;\ \mathscr{B})}=\Big(\int_\vartheta\|\eta(t')\|^2_{\mathscr{B}}dt'\Big)^{1/2}<\infty\}$$

$$H^1(\vartheta;\ \mathscr{B})=\{\eta/\eta:\vartheta\to\mathscr{B},\ \|\eta\|_{H^1(\vartheta;\ \mathscr{B})}=\Big(\|\eta\|^2_{L^2(\vartheta;\ \mathscr{B})}+\Big\|\frac{\partial\eta}{\partial t'}\Big\|^2_{L^2(\vartheta;\ \mathscr{B})}\Big)^{1/2}<\infty\}$$

$$C(\vartheta;\ \mathscr{B})=\{\eta/\eta:\vartheta\to\mathscr{B}\text{强连续},\ \|\eta\|_{C(\vartheta;\ \mathscr{B})}=\max_{t'\in\vartheta}\|\eta(t')\|_{\mathscr{B}}<\infty\}.\text{等等.}$$

又设α，$\beta\geqslant0$，我们定义Ω上的非各向同性Sobolev空间如下(参见[15])，

$$H^{\alpha,\beta}(\Omega)=L^2(I;\ H^\alpha(Q))\cap H^\beta(I;\ L^2(Q))$$

$$X^{\alpha,\beta}(\Omega)=H^{\beta+1}(I;\ H^\alpha(Q))\cap H^\beta(I,\ H^{\alpha+1}(Q))$$

其范数分别是，

$$\|\eta\|_{H^{\alpha,\beta}(\Omega)}=(\|\eta\|^2_{L^2(I,\ H^\alpha(Q))}+\|\eta\|^2_{H^\beta(I,L^2(Q))})^{1/2}$$

$$\|\eta\|_{X^{\alpha,\beta}(\Omega)}=(\|\eta\|^2_{H^{\beta+1}(I,H^\alpha(Q))}+\|\eta\|^2_{H^\beta(I,H^{\alpha+1}(Q))})^{1/2}$$

如果α，$\beta\geqslant1$，则还定义

$$Y^{\alpha,\beta}(\Omega)=H^{\alpha,\beta}(\Omega)\cap H^1(I;\ H^{\alpha-1}(Q))\cap H^{\beta-1}(I;\ H^1(Q))$$

其范数为

$$\|\eta\|_{Y^{\alpha,\beta}(\Omega)}=(\|\eta\|^2_{H^{\alpha,\beta}(\Omega)}+\|\eta\|^2_{H^1(I;\ H^{\alpha-1}(Q))}+\|\eta\|^2_{H^{\beta-1}(I;\ H^1(Q))})^{1/2}$$

用$C^\infty_p(\Omega)$表示$\bar{\Omega}$上无限次可微，且在x_n方向上以2π为周期的函数集合，$H^s_p(\Omega)$，$H^{\alpha,\beta}_p(\Omega)$，$X^{\alpha,\beta}_p(\Omega)$和$Y^{\alpha,\beta}_p(\Omega)$分别表示$C^\infty_p(\Omega)$在$H^s(\Omega)$，$H^{\alpha,\beta}(\Omega)$，$X^{\alpha,\beta}(\Omega)$和$Y^{\alpha,\beta}(\Omega)$中的完备化，此外，我们还记

$$H^s_{0,p}(\Omega)=H^s_p(\Omega)\cap L^2(I;\ H^1_0(\Omega)),\quad Y^{\alpha,\beta}_{0,p}(\Omega)=Y^{\alpha,\beta}_p(\Omega)\cap L^2(I,H^1_0(\Omega))$$

三、计算格式

利用分部积分可知，(1.2)～(1.4)的广义解$(U,T,\varphi)\in[C(0,t_0;\ H^1_{0,p}(\Omega)\cap C(\bar{\Omega}))]^n\times[C(0,t_0;\ H^1_{0,p}(\Omega)\cap C(\bar{\Omega}))]\times[C(0,t_0;\ H^1_p(\Omega)\cap C(\bar{\Omega}))]$满足下述方程

$$\left.\begin{aligned}&(\partial_tU,v)+([U\cdot\nabla]U,v)+R(\nabla T,v)+R(T\nabla\varphi,\ v)+\sum_{m=1}^3J_m(T,\varphi,U,v)\\&\quad=(f,v),\qquad\forall v\in(H^1_{0,p}(\Omega))^n\\&(\partial_tT,\omega)+([U\cdot\nabla]T,\omega)+J_4(T,\varphi,\omega)+J_5(T,\varphi,U,\omega)=0,\qquad\forall\omega\in H^1_{0,p}(\Omega)\\&(\partial_t\varphi,\chi)+([U\cdot\nabla]\varphi,\chi)+(\nabla\cdot U,\chi)=0,\qquad\forall\chi\in H^1_p(\Omega)\end{aligned}\right\}\tag{3.1}$$

其中

$$J_1(T,\varphi,U,v)=(K(T,\varphi)\nabla\cdot U,\nabla(e^{-\varphi}v))$$

$$J_2(T,\varphi,U,v)=\sum_{l,j=1}^n(\nu(T,\varphi)\partial_jU^{(l)},\partial_j(e^{-\varphi}v^{(l)}))$$

$$J_3(T,\varphi,U,v)=\sum_{l,j=1}^{n}(\nu(T,\varphi)\partial_l U^{(j)},\partial_j(e^{-\varphi}v^{(l)}))$$

$$J_4(T,\varphi,\omega)=(\mu(T,\varphi)\nabla T,\nabla(e^{-\varphi}T^{-1}S_T^{-1}\omega))$$

$$J_5(T,\varphi,U,\omega)=-\Big(\frac{1}{2}\nu e^{-\varphi}T^{-1}S_T^{-1}\sum_{l,j=1}^{n}(\partial_l U^{(j)}+\partial_j U^{(l)})^2$$

$$+Ke^{-\varphi}T^{-1}S_T^{-1}(\nabla\cdot U)^2+S_\varphi S_T^{-1}(\nabla\cdot U),\omega\Big)$$

显然，(1.1)～(1.3)的任何古典解都满足(3.1).下面我们构造逼近(3.1)的Fourier谱-有限元格式。设$\{C_h\}_h$是$\bar{Q}$的一个正规（三角）剖分族，$\bar{Q}=\bigcup_{m=1}^{M_h}K_m$（当$n=2$时，$K_m$为$\mathbf{R}^1$上的小区间，当$n=3$时，$K_m$为$\mathbf{R}^2$上的小三角形）.记

$$h_m=\mathrm{diam}(K_m),\quad h'_m=\begin{cases}h_m, & \text{当}n=2\text{时}\\ K_m\text{中内切圆之直径}, & \text{当}n=3\text{时}\end{cases}$$

$$h=\max_{1\leqslant m\leqslant M_h}h_m,\qquad h'=\min_{1\leqslant m\leqslant M_h}h'_m$$

我们还假定剖分族$\{C_h\}_h$满足“逆假设”，即存在一个正常数d，使得对任一剖分$C_h\in\{C_h\}_h$均有$h/h'\leqslant d$(参见[16]).

设$k\geqslant1$是一整数，我们用$\mathbf{P}_k$表示$\mathbf{R}^{n-1}$上所有次数$\leqslant k$的代数多项式之集合.定义非周期空间方向的有限元逼近子空间如下：

$$X_h^k(Q)=\{\eta/\eta|_{K_m}\in\mathbf{P}_k,\ 1\leqslant m\leqslant M_h\}\cap H^1(Q)$$

$$X_{0,h}^k(Q)=X_h^k(Q)\cap H_0^1(Q)$$

又设$N\geqslant1$为一正整数，我们定义周期空间方向的Fourier谱逼近子空间为

$$S_N(I)=\Big\{\eta(x_n)=\sum_{|j|\leqslant N}\eta_j\exp[ijx_n]/\eta_j=\bar{\eta}_{-j},|j|\leqslant N\Big\}$$

记$\delta=(h,N,k)$，综合上述两种空间方向的逼近方法，我们可以定义$H_p^1(\Omega)$和$H_{0,p}^1(\Omega)$的有限维逼近子空间如下：

$$V_\delta(\Omega)=X_h^k(Q)\otimes S_N(I),\quad V_{0,\delta}(\Omega)=X_{0,h}^k(Q)\otimes S_N(I)$$

用τ表示时间方向的步长，$\Theta_\tau=\{t=l\tau/0\leqslant l\leqslant[t_0/\tau]\}$.我们用下列一阶向前差商来逼近$\partial_t\eta(t)$：

$$\eta_t(t)=(\eta(t+\tau)-\eta(t))/\tau$$

(3.1)的全离散Fourier谱-有限元混合逼近格式是：对$t\in\Theta_\tau$，求$(u_\delta(t),T_\delta(t),\varphi_\delta(t))\in[V_{0,\delta}(\Omega)]^n\times V_{0,\delta}(\Omega)\times V_\delta(\Omega)$，使得满足

$$\left.\begin{aligned}&(u_{\delta t},v)+([u_\delta\cdot\nabla]u_\delta,v)+R(\nabla T_\delta,v)+R(T_\delta\nabla\varphi_\delta,v)+\sum_{m=1}^{3}J_m(T_\delta,\varphi_\delta,u_\delta,v)\\&\qquad=(f,v),\qquad\forall v\in[V_{0,\delta}(\Omega)]^n\\&(T_{\delta t},\omega)+([u_\delta\cdot\nabla]T_\delta,\omega)+J_4(T_\delta,\varphi_\delta,\omega)+J_5(T_\delta,\varphi_\delta,u_\delta,\omega)=0,\ \forall\omega\in V_{0,\delta}(\Omega)\\&(\varphi_{\delta t},\chi)+([u_\delta\cdot\nabla]\varphi_\delta,\chi)+(\nabla\cdot u_\delta,\chi)=0,\qquad\forall\chi\in V_\delta(\Omega)\end{aligned}\right\}\tag{3.2}$$

对初始条件(1.3)，我们采用周期方向x_n的L^2-投影与非周期方向的分片Lagrange插值来逼近. 具体地说，记P_N是从$L^2(I)$到$S_N(I)$上的正交投影算子，Π_h^k是从$C(\bar{Q})$到$X_h^k(Q)$上的分片k次Lagrange插值算子，即对任一$\xi\in C(\bar{Q})$，$\Pi_h^k\xi|_{K_m}(1\leqslant m\leqslant M_h)$是$\xi|_{K_m}$的$k$次Lagrange插值，并且$\Pi_h^k\xi$在$\bar{Q}$上连续. 复合算子$\mathscr{F}_\delta$: $L^2(I,C(\bar{Q}))\to V_\delta(\Omega)$定义为$\mathscr{F}_\delta=P_N\circ\Pi_h^k=\Pi_h^k\circ P_N$，即如果

$$\eta(x)=\sum_{|j|=0}^{\infty}\eta_j(x_1,\cdots,x_{n-1})\exp[ijx_n]\in L^2(I,C(\bar{Q}))$$

则
$$(\mathscr{F}_\delta\eta)(x)=\sum_{|j|\leqslant N}(\Pi_h^k\eta_j)(x_1,\cdots,x_{n-1})\exp[ijx_n]$$

根据上述逼近方法，我们选取下列近似初始条件

$$u_\delta(0)=\mathscr{F}_\delta U_0,\quad T_\delta(0)=\mathscr{F}_\delta T_0,\quad \varphi_\delta(0)=\mathscr{F}_\delta\varphi(0)=\mathscr{F}_\delta\ln\rho_0 \tag{3.3}$$

注记3.1 我们也可以采用其他方法逼近初始条件，例如在x_1，…，x_{n-1}和x_n方向上均采用插值逼近，$L^2(\Omega)$投影逼近，$H^1(\Omega)$投影逼近，等等.只要这些逼近方法具有与$\mathscr{F}_\delta$相同的逼近阶，那末本文中的所有结论仍然都成立.

四、一 些 引 理

引理1[8,12] 若$\alpha>(n-1)/2$，$\beta\geqslant0$，$\bar{\alpha}=\min(\alpha,k+1)$，则存在与$h,N$无关的正常数$C$，使得对所有$\eta\in H_p^{\alpha,\beta}(\Omega)$，都有

$$\|\eta-\mathscr{F}_\delta\eta\|\leqslant C(h^{\bar{\alpha}}+N^{-\beta})\|\eta\|_{H^{\bar{\alpha},\beta}(\Omega)}$$

引理2[8,12] 若$Nh\leqslant\text{const}$，$\alpha>(n-1)/2$，$\alpha\geqslant1$，$\beta\geqslant1$，$\bar{\alpha}=\min(\alpha,k+1)$，则存在与$h$，$N$无关的正常数$C$，使得对所有$\eta\in Y_p^{\alpha,\beta}(\Omega)$，都有

$$\|\eta-\mathscr{F}_\delta\eta\|_1\leqslant C(h^{\bar{\alpha}-1}+N^{1-\beta})\|\eta\|_{Y^{\bar{\alpha},\beta}(\Omega)}$$

引理3 若$\alpha>(n-1)/2$，$\beta>1/2$，则存在与h，N无关的正常数C，使得对所有$\eta\in X_p^{\alpha,\beta}(\Omega)$，都有

$$\|\mathscr{F}_\delta\eta\|_{1,\infty}\leqslant C\|\eta\|_{X^{\alpha,\beta}(\Omega)}$$

证明 假设

$$\eta(x)=\sum_{|j|=0}^{\infty}\eta_j(x_1,\cdots,x_{n-1})\exp[ijx_n]\in X_p^{\alpha,\beta}(\Omega)$$

由于$\alpha>(n-1)/2$，故根据Sobolev嵌入定理可知$X_p^{\alpha,\beta}(\Omega)\hookrightarrow L^2(I,C(\bar{Q}))$，从而

$$(\mathscr{F}_\delta\eta)(x)=\sum_{|j|\leqslant N}(\Pi_h^k\eta_j)(x_1,\cdots,x_{n-1})\exp[ijx_n]$$

又由于$\{C_h\}_h$是$\bar{Q}$的正规剖分族，并满足逆假设，根据有限元中的插值误差分析可知（参见[16]），

$$\|\Pi_h^k\eta_j\|_{0,\infty,Q}\leqslant C\|\eta_j\|_{\alpha,Q}$$

$$|\Pi_h^k\eta_j|_{1,\infty,Q}\leqslant C\|\eta_j\|_{1+\alpha,Q}$$

因此，

$$\begin{aligned}\|\mathscr{F}_\delta\eta\|_{1,\infty}&\leqslant\sum_{|j|\leqslant N}\{(1+|j|)\|\Pi_h^k\eta_j\|_{0,\infty,Q}+|\Pi_h^k\eta_j|_{1,\infty,Q}\}\\&\leqslant C\sum_{|j|\leqslant N}\{(1+|j|)\|\eta_j\|_{\alpha,Q}+\|\eta_j\|_{1+\alpha,Q}\}\\&\leqslant C\Big\{\sum_{|j|\leqslant N}[(1+|j|^{2+2\beta})\|\eta_j\|_{\alpha,Q}^2+(1+|j|^{2\beta})\|\eta_j\|_{1+\alpha,Q}^2]\Big\}^{1/2}\\&\quad\cdot\Big\{\sum_{|j|\leqslant N}(1+|j|)^{-2\beta}\Big\}^{1/2}\\&\leqslant C\|\eta\|_{X^{\alpha,\beta}(\Omega)}\end{aligned}$$

引理4 存在不依赖于h，N的正常数C_0，使得对所有$\eta\in V_\delta(\Omega)$，都有

(i) $|\eta|_1^2\leqslant(C_0h^{-2}+N^2)\|\eta\|^2$,

(ii) $\|\eta\|_{0,\infty}^2\leqslant C_0h^{1-n}N\|\eta\|^2$,

(iii) $\|\eta\|_{0,\infty}^2\leqslant C_0\bar{D}(h,N)\|\eta\|_1^2$,

其中

$$\bar{D}(h,N)=\begin{cases}\ln N, & \text{当}n=2\text{时}\\ N|\ln h|, & \text{当}n=3\text{时}\end{cases}$$

证明 结论(i)的证明可见[11，12]，下面证明结论(ii)．设

$$\eta(x)=\sum_{|j|\leqslant N}\eta_j(x_1,\cdots,x_{n-1})\exp[ijx_n]$$

则$\eta_j\in X_h^k(Q)\subset H^1(Q)$，$|j|\leqslant N$．由于$C_h$是$\bar{Q}$的正规剖分，并且满足“逆假设”，因此，我们根据有限元中的逆不等式可知[16]

$$\|\eta_j\|_{0,\infty,Q}\leqslant C_0h^{(1-n)/2}\|\eta_j\|_{0,2,Q}$$

从而

$$\begin{aligned}\|\eta\|_{0,\infty}&\leqslant\sum_{|j|\leqslant N}\|\eta_j\|_{0,\infty,Q}\leqslant C_0h^{(1-n)/2}\sum_{|j|\leqslant N}\|\eta_j\|_{0,2,Q}\\&\leqslant C_0h^{(1-n)/2}\Big(\sum_{|j|\leqslant N}\|\eta_j\|_{0,2,Q}^2)^{1/2}\Big(\sum_{|j|\leqslant N}1\Big)^{1/2}\leqslant C_0h^{(1-n)/2}N^{1/2}\|\eta\|\end{aligned}$$

因此结论(ii)得证．

下面我们分别$n=2$和$n=3$两种情形来证明结论(iii)．若$n=2$，则由 Sobolev 嵌入定理可知$H^1(Q)\hookrightarrow C(Q)$．并且

$$\|\eta_j\|_{0,\infty,Q}\leqslant C\|\eta_j\|_{0,2,Q}^{1/2}|\eta_j|_{1,2,Q}^{1/2}$$

因此

$$\|\eta\|_{0,\infty}\leqslant\sum_{|j|\leqslant N}\|\eta_j\|_{0,\infty,Q}\leqslant C\sum_{|j|\leqslant N}\|\eta_j\|_{0,2,Q}^{1/2}|\eta_j|_{1,2,Q}^{1/2}$$

$$\leqslant C\left\{\sum_{|j|\leqslant N}(1+j^2)\|\eta_j\|_{0,2,Q}^2\right\}^{1/4}\left\{\sum_{|j|\leqslant N}|\eta_j|_{1,2,Q}^2\right\}^{1/4}$$

$$\cdot\left\{\sum_{|j|\leqslant N}(1+j^2)^{-1/2}\right\}^{1/2}$$

$$\leqslant C_0(\ln N)^{1/2}\|\eta\|_1$$

若$n=3$，则由二维有限元子空间$X_h^1(Q)$上的"逆不等式"可知[17]

$$\|\eta_j\|_{0,\infty,Q}\leqslant C|\ln h|^{1/2}\|\eta_j\|_{1,Q}$$

从而

$$\|\eta\|_{0,\infty}\leqslant\sum_{|j|\leqslant N}\|\eta_j\|_{0,\infty,Q}\leqslant C|\ln h|^{1/2}\sum_{|j|\leqslant N}\|\eta_j\|_{1,Q}$$

$$\leqslant C|\ln h|^{1/2}\Big(\sum_{|j|\leqslant N}\|\eta_j\|_{1,Q}^2\Big)^{1/2}\Big(\sum_{|j|\leqslant N}1\Big)^{1/2}$$

$$\leqslant C_0N^{1/2}|\ln h|^{1/2}\|\eta\|_1$$

引理5 若$\psi\in H^1(\Omega)\cap C_p(\Omega)$，$\xi,\eta\in V_{0,\delta}(\Omega)$，则

$$(\psi\partial_l\xi,\partial_j\eta)=(\psi\partial_j\xi,\partial_l\eta)-(\partial_j\psi\partial_l\xi-\partial_l\psi\partial_j\xi,\eta),\qquad\forall\,1\leqslant l,j\leqslant n$$

证明 用$\boldsymbol{n}=(n_1,\cdots,n_n)$表示$\partial(K_m\times I)=(\partial K_m\times I)\cup(K_m\times\partial I)$的单位外法向矢量（见图1），于是由分部积分得到

$$(\psi\partial_l\xi,\partial_j\eta)=\sum_{m=1}^{M_h}\int_{K_m\times I}\psi(x)\partial_l\xi(x)\partial_j\eta(x)dx$$

$$=\sum_{m=1}^{M_h}\Big\{\psi\eta\partial_l\xi\cdot n_j\Big|_{\partial(K_m\times I)}-\int_{K_m\times I}\eta(x)(\psi(x)\partial_{jl}\xi(x)$$

$$+\partial_j\psi(x)\partial_l\xi(x))dx\Big\}$$

$$=\sum_{m=1}^{M_h}\Big\{\psi\eta(\partial_l\xi\cdot n_j-\partial_j\xi\cdot n_l)\Big|_{\partial(K_m\times I)}+\int_{K_m\times I}\psi(x)\partial_j\xi(x)\partial_l\eta(x)dx$$

$$-\int_{K_m\times I}\eta(x)(\partial_j\psi(x)\partial_l\xi(x)-\partial_l\psi(x)\partial_j\xi(x))dx\Big\}$$

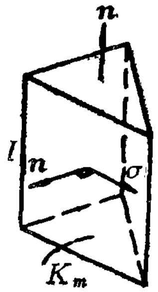

图 1

显然，若能证明

$$A_1=\sum_{m=1}^{M_h}\{\psi\eta(\partial_l\xi\cdot n_j-\partial_j\xi\cdot n_l)\}\Big|_{K_m\times\partial I}=0$$

$$A_2=\sum_{m=1}^{M_h}\{\psi\eta(\partial_l\xi\cdot n_j-\partial_j\xi\cdot n_l)\}\Big|_{\partial K_m\times I}=0$$

则就完成了引理的证明．根据ψ，ξ和η的周期性不难得到$A_1=0$．下面来证明$A_2=0$．首先若$l=j$，则显然有$A_2=0$；若$l\neq j$，并且l与j中之一等于n，不妨设$j=n$．那么在$\partial K_m\times I$上$n_j=0$，从而

$$\psi\eta\partial_l\xi\cdot n_j\Big|_{\partial K_m\times I}=0,\qquad 1\leqslant m\leqslant M_h$$

又由于$\xi\in V_{0,\delta}(\Omega)$，$j=n$，所以$\partial_j\xi\in V_{0,\delta}(\Omega)$，从而$\partial_j\xi\in C(\bar{Q})$，且$\partial_j\xi|_{\partial Q\times I}=0$.因此

$$\sum_{m=1}^{M_h}\psi\eta\partial_j\xi\cdot n_l\Big|_{\partial K_m\times I}=\psi\eta\partial_j\xi\cdot n_l\Big|_{\partial Q\times I}=0$$

从而亦有$A_2=0$. 若$l\neq j$，且l或j均不等于n，则必有$n=3$，此时$l=1$，$j=2$或者 $j=1$，$l=2$. 我们假定$l=1$，$j=2$，于是有

$$(\partial_1\xi\cdot n_2-\partial_2\xi\cdot n_1)|_{\partial K_m\times I}=\partial\xi/\partial\sigma|_{\partial K_m\times I}$$

其中$\sigma=(n_2,\ -n_1,\ 0)$是$\partial K_m\times I$上的单位切向矢量.由于ψ，ξ和η在$\bar{\Omega}$上连续，ξ在$\partial\Omega\times I$上为0，所以

$$\sum_{m=1}^{M_h}\psi\eta(\partial_1\xi\cdot n_2-\partial_2\xi\cdot n_1)|_{\partial K_m\times I}=\psi\eta\frac{\partial\xi}{\partial\sigma}|_{\partial Q\times I}==0$$

综合以上几种情况，即得到引理5.

引理6[3]　假设下列条件成立

(i)　η是定义于Θ_τ上的非负函数，ρ_0，B_0，$a_l(h,N)$和M_l，$0\leqslant l\leqslant m$，是非负常数；

(ii)　$\rho(t)=\rho(\eta(0),\eta(\tau),\cdots,\eta(t-\tau))$并且当

$$\eta(t')\leqslant M_0/a_0(h,N),\qquad \forall t'\leqslant t-\tau,\ t'\in\Theta_\tau$$

时，有$\rho(t)\leqslant\rho_0$；

(iii)　$$H_\eta(t)=\eta(t)[M(\eta(t))+a_0(h,N)B(\eta(t))\eta(t)]+\sum_{l=1}^{m}\xi_l(\eta(t))$$

其中当$\eta(t)\leqslant M_0/a_0(h,\ N)$时，$M(\eta(t))\leqslant M_0$，$B(\eta(t))\leqslant B_0$；而且当 $\eta(t)\leqslant M_l/a_l(h,N)$时，$\xi_l(\eta(t))\leqslant 0$，$1\leqslant l\leqslant m$；

(iv)　$G_\eta(t)=G(\eta(t),\eta(t-\tau))\geqslant\eta(t)$；

(v)　$\eta(0)\leqslant\rho(0)\leqslant\rho_0$，并且

$$G_\eta(t)\leqslant\rho(t)+\tau\sum_{t'=0}^{t-\tau}H_\eta(t'),\qquad \forall t\in\Theta_\tau;$$

(vi)　$\rho_0\exp[(1+B_0)M_0t_1]\leqslant\min\limits_{0\leqslant l\leqslant m}(M_l/a_l(h,N))$；

那么，对所有$t\in\Theta_\tau$，$t\leqslant t_1$，都有

$$\eta(t)\leqslant\rho_0\exp[(1+B_0)M_0t]$$

五、误 差 估 计

假设(U,T,φ)是(3.1)的解，记$u_*=\mathscr{F}_\delta U$，$T_*=\mathscr{F}_\delta T$，$\varphi_*=\mathscr{F}_\delta\varphi$.于是由(3.1)得到

$$\left.\begin{aligned}&(u_{*t},v)+([u_*\cdot\nabla]u_*,\ v)+R(\nabla T_*,\ v)+R(T_*\cdot\nabla\varphi_*,v)+\sum_{m=1}^{3}J_m(T_*,\varphi_*,u_*,v)\\&=(f_1,\ v)+(\tilde{f}_1,v)+\sum_{m=1}^{3}\tilde{H}_m(v),\qquad\forall v\in[V_{0,\delta}(\Omega)]^n\\&(T_{*t},\omega)+([u_*\cdot\nabla]T_*,\omega)+J_4(T_*,\varphi_*,\omega)+J_5(T_*,\varphi_*,u_*,\omega)\\&=(\tilde{f}_2,\omega)+\sum_{m=4}^{5}\tilde{H}_m(\omega),\qquad\forall\omega\in V_{0,\delta}(\Omega)\\&(\varphi_{*t},\chi)+([u_*\cdot\nabla]\varphi_*,\chi)+(\nabla\cdot u_*,\chi)=(\tilde{f}_3,\chi),\qquad\forall\chi\in V_\delta(\Omega)\end{aligned}\right\}\quad(5.1)$$

其中

$$\tilde{f}_1=u_{*t}-\partial_tU+[u_*\cdot\nabla]u_*-[U\cdot\nabla]U+R[\nabla(T_*-T)+T_*\nabla\varphi_*-T\nabla\varphi]$$

$$\tilde{f}_2=T_{*t}-\partial_tT+[u_*\cdot\nabla]T_*-[U\cdot\nabla]T$$

$$\tilde{f}_3=\varphi_{*t}-\partial_t\varphi+[u_*\cdot\nabla]\varphi_*-[U\cdot\nabla]\varphi+\nabla(u_*-U)$$

$$\tilde{H}_m(v)=J_m(T_*,\varphi_*,u_*,v)-J_m(T,\varphi,U,v)\qquad(m=1,2,3)$$

$$\tilde{H}_4(\omega)=J_4(T_*,\varphi_*,\omega)-J_4(T,\varphi,\omega)$$

$$\tilde{H}_5(\omega)=J_5(T_*,\varphi_*,u_*,\omega)-J_5(T,\varphi,U,\omega)$$

设$(u_\delta,T_\delta,\varphi_\delta)$是格式(3.2)～(3.3)的解，并记$\tilde{u}=u_\delta-u_*$，$\tilde{T}=T_\delta-T_*$，$\tilde{\varphi}=\varphi_\delta-\varphi_*$，则由(3.3)易知$\tilde{u}(0)=0$，$\tilde{T}(0)=\tilde{\varphi}(0)=0$。又把(5.1)与(3.2)各式对应相减，我们得到

$$\left.\begin{aligned}&(\tilde{u}_t,\ v)+\sum_{m=1}^{3}F_m(v)+\sum_{m=1}^{3}\tilde{J}_m(v)=-(\tilde{f}_1,v)-\sum_{m=1}^{3}\tilde{H}_m(v)\\&\qquad\forall v\in[V_{0,\delta}(\Omega)]^n\\&(\tilde{T}_t,\omega)+F_4(\omega)+\sum_{m=4}^{5}\tilde{J}_m(\omega)=-(\tilde{f}_2,\omega)-\sum_{m=4}^{5}\tilde{H}_m(\omega)\\&\qquad\forall\omega\in V_{0,\delta}(\Omega)\\&(\tilde{\varphi}_t,\ \chi)+\sum_{m=5}^{6}F_m(\chi)=-(\tilde{f}_3,\chi),\qquad\forall\chi\in V_\delta(\Omega)\end{aligned}\right\}\quad(5.2)$$

其中

$$F_1(v)=([\tilde{u}\cdot\nabla]u_*,v)+([(u_*+\tilde{u})\cdot\nabla]\tilde{u},v)$$

$$F_2(v)=R(\nabla\tilde{T},v)$$

$$F_3(v)=R([\tilde{T}\nabla]\varphi_*,v)+R([(T_*+\tilde{T})\nabla]\tilde{\varphi},v)$$

$$F_4(\omega)=([\tilde{u}\cdot\nabla]T_*,\omega)+([(u_*+\tilde{u})\cdot\nabla]\tilde{T},\omega)$$

$$F_5(\chi)=([\tilde{u}\cdot\nabla]\varphi_*,\chi)+([(u_*+\tilde{u})\cdot\nabla]\tilde{\varphi},\chi)$$

$$F_6(\chi)=(\nabla\cdot\tilde{u},\chi)$$

$$\tilde{J}_m(v)=J_m(T_*+\tilde{T},\varphi_*+\tilde{\varphi},u_*+\tilde{u},v)-J_m(T_*,\varphi_*,u_*,v)\qquad(m=1,2,3)$$

$$\tilde{J}_4(\omega)=J_4(T_*+\tilde{T},\varphi_*+\tilde{\varphi},\omega)-J_4(T_*,\varphi_*,\omega)$$

$$\tilde{J}_5(\omega)=J_5(T_*+\tilde{T},\varphi_*+\tilde{\varphi},u_*+\tilde{u},\omega)-J_5(T_*,\varphi_*,u_*,\omega)$$

在(5.2)中取$v=\tilde{u}+\tau\tilde{u}_t$，$\omega=\tilde{T}+\tau\tilde{T}_t$，$\chi=\tilde{\varphi}+\tau\tilde{\varphi}_t$。把所得三个等式相加，并注意到恒等式[8]：

$$2(\tilde{\eta}_t,\tilde{\eta})=(\|\tilde{\eta}\|^2)_t-\tau\|\tilde{\eta}_t\|^2$$

我们得到

$$(\|\tilde{u}\|^2+\|\tilde{T}\|^2+\|\tilde{\varphi}\|^2)_t+\tau(1-\varepsilon)(\|\tilde{u}_t\|^2+\|\tilde{T}_t\|^2+\|\tilde{\varphi}_t\|^2)+2\sum_{m=1}^{3}F_m(\tilde{u}+\tau\tilde{u}_t)$$

$$+2F_4(\tilde{T}+\tau\tilde{T}_t)+2\sum_{m=5}^{6}F_m(\tilde{\varphi}+\tau\tilde{\varphi}_t)+2\sum_{m=1}^{3}\tilde{J}_m(\tilde{u}+\tau\tilde{u}_t)+2\sum_{m=4}^{5}\tilde{J}_m(\tilde{T}+\tau\tilde{T}_t)$$

$$\leqslant\|\tilde{u}\|^2+\|\tilde{T}\|^2+\|\tilde{\varphi}\|^2+\left(1+\frac{\tau}{\varepsilon}\right)\sum_{m=1}^{3}\|\tilde{f}_m\|^2+2\sum_{m=1}^{3}\tilde{H}_m(\tilde{u}+\tau\tilde{u}_t)$$

$$+2\sum_{m=4}^{5}\tilde{H}_m(\tilde{T}+\tau\tilde{T}_t) \tag{5.3}$$

其中$\varepsilon>0$是一个适当小的常数.下面我们逐项估计(5.3)式中的F_m，$\tilde{J}_m$和$\tilde{H}_m$等.假定U，T和φ适当光滑，$B_0<T<B_1$，$|\varphi|<B_2$，于是若h^{-1}和N充分大，则有$B_0<T_*<B_1$，$|\varphi_*|<B_2$.又根据引理4中的结论(ii)，存在适当小的正常数$\tilde{B}>0$，使得当以下两式成立时

$$\|\tilde{T}\|\leqslant\tilde{B}h^{(n-1)/2}N^{-1/2},\quad \|\tilde{\varphi}\|\leqslant\tilde{B}h^{(n-1)/2}N^{-1/2} \tag{5.4}$$

有$B_0<T_*+\tilde{T}<B_1$和$|\varphi_*+\tilde{\varphi}|<B_2$成立.此时(1.5)中各项的有界性对$T_*+\tilde{T}$和$\varphi_*+\tilde{\varphi}$均成立.为方便计，记$D(h,N)=C_0\bar{D}(h,N)\cdot(C_0h^{-2}+N^2)$（参见引理4)，并用$M$表示与$h$，$N$，$\tau$无关的正常数，它可以依赖于$\varepsilon$，$R$，$\|\eta\|_{X^{\alpha',\beta'}(\Omega)}(\alpha'>(n-1)/2,\ \beta'>1/2,\ \eta=U,T,\varphi)$，$\|U\|_2$，$\|T\|_2$，以及$K_1$，$\nu_1$，$\Phi_1$，$S_1$等等，且在不同之处可以互不相等.首先，易由引理3和引理4得到

$$\begin{aligned}
|F_1(\tilde{u})|&\leqslant|u_*|_{1,\infty}\|\tilde{u}\|^2+\|u_*+\tilde{u}\|_{0,\infty}|\tilde{u}|_1\|\tilde{u}\|\\
&\leqslant\varepsilon|\tilde{u}|_1^2+M(1+\|\tilde{u}\|_{0,\infty}^2)\|\tilde{u}\|^2\\
&\leqslant\varepsilon|\tilde{u}|_1^2+M(1+D(h,N)\|\tilde{u}\|^2)\|\tilde{u}\|^2\\
|F_2(\tilde{u})|&\leqslant\varepsilon|\tilde{T}|_1^2+M\|\tilde{u}\|^2\\
|F_3(\tilde{u})|&\leqslant M\{(\|\tilde{T}\|+\|\tilde{\varphi}\|)\|\tilde{u}\|+\|\tilde{\varphi}\||\tilde{u}|_1+\|\tilde{\varphi}\|_{0,\infty}(\|\tilde{u}\||\tilde{T}|_1+\|\tilde{T}\||\tilde{u}|_1)\}\\
&\leqslant\varepsilon|\tilde{u}|_1^2+\varepsilon|\tilde{T}|_1^2+M(1+D(h,N)\|\tilde{\varphi}\|^2)(\|\tilde{u}\|^2+\|\tilde{T}\|^2)+M\|\tilde{\varphi}\|^2
\end{aligned}$$

类似地有

$$\begin{aligned}
|F_4(\tilde{T})|&\leqslant\varepsilon|\tilde{T}|_1^2+M(\|\tilde{u}\|^2+\|\tilde{T}\|^2+D(h,N)\|\tilde{u}\|^2\|\tilde{T}\|^2)\\
|F_5(\tilde{\varphi})|&\leqslant|([\tilde{u}\cdot\nabla]\varphi_*,\tilde{\varphi})|+|(\nabla(\tilde{\varphi}^2/2),u_*+\tilde{u})|\\
&=|([\tilde{u}\cdot\nabla]\varphi_*,\tilde{\varphi})|+|(\tilde{\varphi}^2/2,\nabla\cdot(u_*+\tilde{u}))|\\
&\leqslant M(\|\tilde{u}\|^2+\|\tilde{\varphi}\|^2)+MD(h,N)\|\tilde{\varphi}\|^2|\tilde{u}|_1^2\\
|F_6(\tilde{\varphi})|&\leqslant\varepsilon|\tilde{u}|_1^2+M\|\tilde{\varphi}\|^2
\end{aligned}$$

其次，我们来估计$\tilde{J}_m(\tilde{u})$，$1\leqslant m\leqslant3$和$\tilde{J}_4(\tilde{T})$.不难验证

$$\tilde{J}_1(\tilde{u})=(\exp[-\varphi_*-\tilde{\varphi}]K(T_*+\tilde{T},\varphi_*+\tilde{\varphi}),(\nabla\cdot\tilde{u})^2)+\sum_{q=1}^{5}\tilde{J}_{1,q}(\tilde{u}) \tag{5.5}$$

其中

$$\tilde{J}_{1,1}(\tilde{u})=([\exp[-\varphi_*-\tilde{\varphi}]K(T_*+\tilde{T},\varphi_*+\tilde{\varphi})-\exp[-\varphi_*]K(T_*,\varphi_*)]\cdot\nabla u_*,\nabla\cdot\tilde{u})$$

$$\tilde{J}_{1,2}(\tilde{u})=-([\exp[-\varphi_*-\tilde{\varphi}]K(T_*+\tilde{T},\varphi_*+\tilde{\varphi})-\exp[-\varphi_*]K(T_*,\varphi_*)]\cdot(\nabla\cdot u_*)(\nabla\varphi_*),\tilde{u})$$

$$\tilde{J}_{1,3}(\tilde{u})=-(\exp[-\varphi_*-\tilde{\varphi}]K(T_*+\tilde{T},\varphi_*+\tilde{\varphi})(\nabla\cdot\tilde{u})(\nabla\varphi_*),\tilde{u})$$

$$\tilde{J}_{1,4}(\tilde{u})=-(\exp[-\varphi_*-\tilde{\varphi}]K(T_*+\tilde{T},\varphi_*+\tilde{\varphi})(\nabla\cdot\tilde{u})(\nabla\tilde{\varphi}),\tilde{u})$$

$$\tilde{J}_{1,5}(\tilde{u})=-(\exp[-\varphi_*-\tilde{\varphi}]K(T_*+\tilde{T},\varphi_*+\tilde{\varphi})(\nabla\cdot u_*)(\nabla\tilde{\varphi}),\tilde{u})$$

由引理3，引理4以及(1.5)不难得到

$$|\tilde{J}_{1,1}(\tilde{u})|\leqslant\varepsilon|\tilde{u}|_1^2+M(\|\tilde{T}\|^2+\|\tilde{\varphi}\|^2)$$

$$|\tilde{J}_{1,2}(\tilde{u})|\leqslant M(\|\tilde{u}\|^2+\|\tilde{T}\|^2+\|\tilde{\varphi}\|^2)$$

$$|\tilde{J}_{1,3}(\tilde{u})|\leqslant\varepsilon|\tilde{u}|_1^2+M\|\tilde{u}\|^2$$

$$|\tilde{J}_{1,4}(\tilde{u})|\leqslant M\|\tilde{u}\|_{0,\infty}|\tilde{u}|_1|\tilde{\varphi}|_1\leqslant\varepsilon|\tilde{u}|_1^2+MD(h,N)\|\tilde{\varphi}\|^2|\tilde{u}|_1^2$$

又若$U\in[C(0,t_0;\ H_p^2(\Omega))]^n$，则由分部积分可以得到

$$\tilde{J}_{1,5}(\tilde{u})=\sum_{q=1}^{3}Z_q(\tilde{u})$$

其中

$$Z_1(\tilde{u})=-([\exp[-\varphi_*-\tilde{\varphi}]K(T_*+\tilde{T},\varphi_*+\tilde{\varphi})-\exp[-\varphi_*]K(T_*,\varphi_*)](\nabla\cdot u_*)(\nabla\tilde{\varphi}),\tilde{u})$$

$$Z_2(\tilde{u})=-(\exp[-\varphi_*]K(T_*,\varphi_*)[\nabla\cdot(u_*-U)]\nabla\tilde{\varphi},\tilde{u})$$

$$Z_3(\tilde{u})=-(\exp[-\varphi_*]K(T_*,\varphi_*)(\nabla\cdot U)(\nabla\tilde{\varphi},)\tilde{u})=(\tilde{\varphi},\nabla[\exp[-\varphi_*]K(T_*,\varphi_*)(\nabla\cdot U)\tilde{u}])$$

不难证明

$$|Z_1(\tilde{u})|\leqslant\varepsilon|\tilde{u}|_1^2+MD(h,N)(\|\tilde{T}\|^2+\|\tilde{\varphi}\|^2)\|\tilde{\varphi}\|^2$$

$$|Z_2(\tilde{u})|\leqslant M|u_*-U|_1^2+MD(h,N)\|\tilde{\varphi}\|^2|\tilde{u}|_1^2$$

$$|Z_3(\tilde{u})|\leqslant\varepsilon|\tilde{u}|_1^2+M(\|\tilde{u}\|^2+\|\tilde{\varphi}\|^2)$$

把上述各估计式代入(5.5)后得到

$$|\tilde{J}_1(\tilde{u})-(\exp[-\varphi_*-\tilde{\varphi}]K(T_*+\tilde{T},\varphi_*+\tilde{\varphi}),(\nabla\cdot\tilde{u})^2)|\leqslant\alpha(\tilde{u},\tilde{T},\tilde{\varphi},h,N)\tag{5.6}$$

其中

$$\alpha(\tilde{u},\tilde{T},\tilde{\varphi},h,N)=\varepsilon|\tilde{u}|_1^2+M(1+D(h,N)\|\tilde{\varphi}\|^2)(\|\tilde{u}\|^2+\|\tilde{T}\|^2+\|\tilde{\varphi}\|^2)+MD(h,N)(\|\tilde{T}\|^2+\|\tilde{\varphi}\|^2)|\tilde{u}|_1^2+M|u_*-U|_1^2$$

类似地有

$$|\tilde{J}_2(\tilde{u})-(\exp[-\varphi_*-\tilde{\varphi}]\nu(T_*+\tilde{T},\varphi_*+\tilde{\varphi}),\sum_{l,j=1}^{n}(\partial_j\tilde{u}^{(l)})^2)|\leqslant\alpha(\tilde{u},\tilde{T},\tilde{\varphi},h,N)$$

利用引理5，同样地可以得到

$$|\tilde{J}_3(\tilde{u})-(\exp[-\varphi_*-\tilde{\varphi}]\nu(T_*+\tilde{T},\varphi_*+\tilde{\varphi}),(\nabla\cdot\tilde{u})^2)|\leqslant\alpha(\tilde{u},\tilde{T},\tilde{\varphi},h,N)$$

另外，仿照(5.6)的证明，可以得到

$$|\tilde{J}_4(\tilde{T})-(\exp[-\varphi_*-\tilde{\varphi}]\mu(T_*+\tilde{T},\varphi_*+\tilde{\varphi})(T_*+\tilde{T})^{-1}S_T^{-1}(T_*+\tilde{T},\varphi_*+\tilde{\varphi}),\sum_{j=1}^{n}(\partial_j\tilde{T})^2)|$$

$$\leqslant\varepsilon|\tilde{T}|_1^2+M(1+D(h,N)\|\tilde{\varphi}\|^2)(\|\tilde{T}\|^2+\|\tilde{\varphi}\|^2)+MD(h,N)\|\tilde{\varphi}\|^2|\tilde{T}|_1^2$$
$$+M|T_*-T|_1^2$$

下面我们来估计 $\tilde{J}_m(2\tau\tilde{u}_t),1\leqslant m\leqslant 3$，和 $\tilde{J}_4(2\tau\tilde{T}_t)$．记$\lambda=\tau(C_0h^{-2}+N^2)$，并假设 $\lambda\leqslant\lambda^*=$ const，于是

$$\tau|\tilde{\eta}_t|_1^2\leqslant\lambda\|\tilde{\eta}_t\|^2,\qquad \eta=\tilde{u},\tilde{T},\tilde{\varphi}$$

由于

$$\|\nabla\cdot\eta\|^2\leqslant n|\eta|_1^2,\qquad \forall\eta\in[V_\delta(\Omega)]^n$$

所以利用 (1.5) 得到

$$|(\exp[-\varphi_*-\tilde{\varphi}]K(T_*+\tilde{T},\varphi_*+\tilde{\varphi})\nabla\cdot\tilde{u},2\tau\nabla\cdot\tilde{u}_t)|\leqslant 2\tau\Phi_1K_1\|\nabla\cdot\tilde{u}\|\|\nabla\cdot\tilde{u}_t\|$$
$$\leqslant\tau\|\tilde{u}_t\|^2/8+8\lambda n^2\Phi_1^2K_1^2|\tilde{u}|_1^2$$

并不难由此得到

$$|\tilde{J}_1(2\tau\tilde{u}_t)|\leqslant\tau\|\tilde{u}_t\|^2/8+8\lambda n^2\Phi_1^2K_1^2|\tilde{u}|_1^2+\beta(\tilde{u},\tilde{T},\tilde{\varphi},h,N)$$

其中

$$\beta(\tilde{u},\tilde{T},\tilde{\varphi},h,N)=\varepsilon\tau\|\tilde{u}_t\|^2+M(\|\tilde{u}\|^2+\|\tilde{T}\|^2+\|\tilde{\varphi}\|^2)+MD(h,N)\|\tilde{\varphi}\|^2|\tilde{u}|_1^2$$

完全类似地可以得到

$$|\tilde{J}_2(2\tau\tilde{u}_t)|\leqslant\tau\|\tilde{u}_t\|^2/8+8\lambda\Phi_1^2\nu_1^2|\tilde{u}|_1^2+\beta(\tilde{u},\tilde{T},\tilde{\varphi},h,N)$$
$$|\tilde{J}_3(2\tau\tilde{u}_t)|\leqslant\tau\|\tilde{u}_t\|^2/8+8\lambda n^2\Phi_1^2\nu_1^2|\tilde{u}|_1^2+\beta(\tilde{u},\tilde{T},\tilde{\varphi},h,N)$$
$$|\tilde{J}_4(2\tau\tilde{T}_t)|\leqslant\tau\|\tilde{T}_t\|^2/2+2\lambda\Phi_1^2\mu_1^2B_0^{-2}S_0^{-2}|\tilde{T}|_1^2+\varepsilon\tau\|\tilde{T}_t\|^2+M(\|\tilde{T}\|^2$$
$$+\|\tilde{\varphi}\|^2)+MD(h,N)\|\tilde{\varphi}\|^2|\tilde{T}|_1^2$$

此外，不难验证

$$|2\tilde{J}_5(\tilde{T}+\tau\tilde{T}_t)|\leqslant\varepsilon\tau\|\tilde{T}_t\|^2+\varepsilon|\tilde{u}|_1^2+M(\|\tilde{u}\|^2+\|\tilde{T}\|^2+\|\tilde{\varphi}\|^2)$$

最后，应用前面估计 $\tilde{J}_m$ 的同样方法，我们可以估计 $\tilde{H}_m(\tilde{u}+\tau\tilde{u}_t)$，$1\leqslant m\leqslant 3$，和 $\tilde{H}_m(\tilde{T}+\tau\tilde{T}_t)$，$4\leqslant m\leqslant 5$，如下：

$$2\sum_{m=1}^{3}|\tilde{H}_m(\tilde{u}+\tau\tilde{u}_t)|+2\sum_{m=4}^{5}|\tilde{H}_m(\tilde{T}+\tau\tilde{T}_t)|\leqslant\varepsilon\tau(\|\tilde{u}_t\|^2+\|\tilde{T}_t\|^2)+\varepsilon|\tilde{u}|_1^2+\varepsilon|\tilde{T}|_1^2$$
$$+M(\|\tilde{u}\|^2+\|\tilde{T}\|^2+\|\tilde{\varphi}\|^2)+M(\|u_*-U\|_1^2+\|T_*-T\|_1^2+\|\varphi_*-\varphi\|_1^2)$$

记

$$\tilde{A}(T_*+\tilde{T},\varphi_*+\tilde{\varphi})=\min(nK(T_*+\tilde{T},\varphi_*+\tilde{\varphi})+(n+1)\nu(T_*+\tilde{T},\varphi_*+\tilde{\varphi}),$$
$$\nu(T_*+\tilde{T},\varphi_*+\tilde{\varphi}))$$
$$\tilde{F}(\mathscr{D})=(\exp[-\varphi_*-\tilde{\varphi}][K(T_*+\tilde{T},\varphi_*+\tilde{\varphi})+\nu(T_*+\tilde{T},\varphi_*+\tilde{\varphi})],(\nabla\tilde{u})^2)_{\mathscr{D}}$$
$$+(\exp[-\varphi_*-\tilde{\varphi}]\nu(T_*+\tilde{T},\varphi_*+\tilde{\varphi}),\sum_{l,j=1}^{n}(\partial\tilde{u}_j^{(l)})^2)$$

$$\Omega^+=\{x\in\Omega/K(T_*+\tilde{T},\varphi_*+\tilde{\varphi})+\nu(T_*+\tilde{T},\varphi_*+\tilde{\varphi})>0\},\qquad \Omega^-=\Omega\backslash\Omega^+$$

则根据(5.4)和(1.5)可知$\tilde{A}(T_*+\tilde{T},\varphi_*+\tilde{\varphi})>0$．又显然可见

$$\tilde{F}(\Omega^+)\geqslant(\exp[-\varphi_*-\tilde{\varphi}]\tilde{A}(T_*+\tilde{T},\ \varphi_*+\tilde{\varphi}),\ \sum_{l,j=1}^{n}(\partial_j\tilde{u}^{(l)})^2)_{\Omega^+}\qquad(5.7)$$

根据不等式

$$(\nabla\cdot\tilde{u})^2\leqslant n\sum_{j=1}^{n}(\partial_j\tilde{u}^{(j)})^2$$

我们不难验证(5.7)式对Ω^-亦成立.因此

$$\begin{aligned}&(\exp[-\varphi_*-\tilde{\varphi}][K(T_*+\tilde{T},\varphi_*+\tilde{\varphi})+\nu(T_*+\tilde{T},\varphi_*+\tilde{\varphi})],(\nabla\cdot\tilde{u})^2)\\&\quad+(\exp[-\varphi_*-\tilde{\varphi}]\nu(T_*+\tilde{T},\ \varphi_*+\tilde{\varphi}),\ \sum_{l,j=1}^{n}(\partial_j\tilde{u}^{(l)})^2)\\&=\tilde{F}(\Omega^+)+\tilde{F}(\Omega^-)\\&\geqslant(\exp[-\varphi_*-\tilde{\varphi}]\tilde{A}(T_*+\tilde{T},\ \varphi_*+\tilde{\varphi}),\ \sum_{l,j=1}^{n}(\partial_j\tilde{u}^{(l)})^2)\end{aligned}$$

综合上式以及前面关于F_m，$\tilde{J}_m$和$\tilde{H}_m$的估计式，我们从(5.3)得到

$$\begin{aligned}&(\|\tilde{u}\|^2+\|\tilde{T}\|^2+\|\tilde{\varphi}\|^2)_t+\tau(5/8-3\varepsilon)(\|\tilde{u}_t\|^2+\|\tilde{T}_t\|^2+\|\tilde{\varphi}_t\|^2)\\&\quad+(2\exp[-\varphi_*-\tilde{\varphi}]\tilde{A}(T_*+\tilde{T},\varphi_*+\tilde{\varphi})-12\varepsilon-MD(h,N)(\|\tilde{T}\|^2+\|\tilde{\varphi}\|^2),\\&\quad\sum_{l,j=1}^{n}(\partial_j\tilde{u}^{(l)})^2)+(2\exp[-\varphi_*-\tilde{\varphi}](T_*+\tilde{T})^{-1}S_T^{-1}(T_*+\tilde{T},\\&\quad\varphi_*+\tilde{\varphi})\mu(T_*+\tilde{T},\ \varphi_*+\tilde{\varphi})-6\varepsilon-MD(h,\ N)\|\tilde{\varphi}\|^2,\ \sum_{j=1}^{n}(\partial_j\tilde{T})^2)\\&\quad-8\lambda\Phi_1^2(n^2K_1^2+(n^2+1)\nu_1^2)|\tilde{u}|_1^2-2\lambda\Phi_1^2\mu_1^2B_0^{-2}S_0^{-2}|\tilde{T}|_1^2\leqslant\tilde{R}(\tilde{u},\tilde{T},\tilde{\varphi})+\tilde{Z}\end{aligned}\tag{5.8}$$

其中

$$\tilde{R}(\tilde{u},\tilde{T},\tilde{\varphi})=M(1+D(h,N)\cdot(\|\tilde{u}\|^2+\|\tilde{T}\|^2+\|\tilde{\varphi}\|^2))\cdot(\|\tilde{u}\|^2+\|\tilde{T}\|^2+\|\tilde{\varphi}\|^2)$$

$$\tilde{Z}=M\cdot(\|u_*-U\|_1^2+\|T_*-T\|_1^2+\|\varphi_*-\varphi\|_1^2)+\left(1+\frac{\tau}{\varepsilon}\right)\sum_{m=1}^{3}\|\tilde{f}_m\|^2$$

设a是一个正数，

$$(\lambda<\min\left(\frac{A_0\Phi_0}{8\Phi_1^2[n^2K_1^2+(n^2+1)\nu_1^2]},\ \frac{\mu_0\Phi_0B_0^2S_0^2}{2\mu_1^2\Phi_1^2B_1S_1}\right)\tag{5.9}$$

定义

$$\tilde{E}_a(\tilde{\eta},\ t)=\|\tilde{\eta}(t)\|^2+\tau\sum_{t'=0}^{t-\tau}\left(a,\ \sum_{j=1}^{n}(\partial_j\tilde{\eta}(t'))^2\right)+\frac{\tau^2}{2}\sum_{t'=0}^{t-\tau}\|\tilde{\eta}_t(t')\|^2$$

$$\tilde{G}(t)=\tilde{E}_{\Phi_0A_0/2}(\tilde{u},t)+\tilde{E}_{\Phi_0A_0B_1^{-1}S_1^{-1}/2}(\tilde{T},t)+\tilde{E}_0(\tilde{\varphi},t)$$

把(5.8)式对所有$t'\in\Theta_\tau$，$t'\leqslant t-\tau$求和后得到

$$\tilde{G}(t)\leqslant\rho(t)+\tau\sum_{t'=0}^{t-\tau}\{\tilde{R}(\tilde{u}(t'),\tilde{T}(t'),\ \tilde{\varphi}(t'))+\sum_{m=1}^{2}\xi_m(\tilde{u}(t'),\tilde{T}(t'),\tilde{\varphi}(t'))\}$$

其中

$$\rho(t)=\tau\sum_{t'=0}^{t-\tau}\tilde{Z}(t')$$

$$\xi_1(\tilde{u},\tilde{T},\tilde{\varphi})=-(2\exp[-\varphi_*-\tilde{\varphi}]\tilde{A}(T_*+\tilde{T},\varphi_*+\tilde{\varphi})-\Phi_0A_0/2$$
$$-8\lambda\Phi_1^2(n^2K_1^2+(n^2+1)\nu_1^2)-12\varepsilon-MD(h,N)(\|\tilde{T}\|^2+\|\tilde{\varphi}\|^2),$$

$$\sum_{l,j=1}^{n}(\partial_j\tilde{u}^{(l)})^2)$$

$$\xi_2(\tilde{u},\tilde{T},\tilde{\varphi})=-(2\exp[-\varphi_*-\tilde{\varphi}](T_*+\tilde{T})^{-1}S_T^{-1}(T_*+\tilde{T},\ \varphi_*+\tilde{\varphi})$$
$$\cdot\mu(T_*+\tilde{T},\ \varphi_*+\tilde{\varphi})-\Phi_0\mu_0B_1^{-1}S_1^{-1}/2-2\lambda\Phi_1^2\mu_1^2B_0^{-2}S_0^{-2}-6\varepsilon$$

$$-MD(h,\ N)\|\tilde{\varphi}\|^2,\ \sum_{j=1}^{n}(\partial_j\tilde{T})^2)$$

现在我们应用引理6，其中$a_0(h,N)=C_0h^{-2}+N^2$，$a_1(h,N)=a_2(h,N)=D(h,N)$，于是，如果(5.4)成立，并存在$t_1\in\Theta_\tau$，使得

$$\rho(t_1)\leqslant M_1\min(1/D(h,N),\ 1/(C_0h^{-2}+N^2))\tag{5.10}$$

其中$M_1>0$是一个不依赖于h，N，τ的适当小的正常数，那么，对所有$t\leqslant t_1$，$t\in\Theta_\tau$，都有

$$\tilde{G}(t)\leqslant M_2\rho(t)\exp[M_3t]\tag{5.11}$$

其中M_2，M_3是不依赖于h，N，τ的正常数.又因为(5.10)蕴含了(5.4)，所以，我们要获得收敛阶的估计，只须要估计出$\rho(t)$的阶，并验证(5.10)成立.

首先，类似于$|F_m|$，$1\leqslant m\leqslant 6$，的估计，可以得到

$$\sum_{m=1}^{3}\|\tilde{f}_m(t)\|^2\leqslant\|u_{*t}(t)-\partial_tU(t)\|^2+\|T_{*t}(t)-\partial_tT(t)\|^2+\|\varphi_{*t}(t)-\partial_t\varphi(t)\|^2$$
$$+M(\|u_*(t)-U(t)\|_1^2+\|T_*(t)-T(t)\|_1^2+\|\varphi_*(t)-\varphi(t)\|_1^2)$$

利用Taylor公式和引理1，得到

$$\|\eta_{*t}(t)-\partial_t\eta(t)\|\leqslant\|\eta_{*t}(t)-\eta_t(t)\|+\|\eta_t(t)-\partial_t\eta(t)\|$$

$$=\frac{1}{\tau}\left\|\int_t^{t+\tau}\left[\mathscr{F}_\delta\left(\frac{\partial\eta}{\partial t}(t')\right)-\frac{\partial\eta}{\partial t}(t')\right]dt'\right\|+\frac{1}{\tau}\left\|\int_t^{t+\tau}(t+\tau-t')\frac{\partial^2\eta}{\partial t^2}(t')dt'\right\|$$

$$\leqslant\frac{1}{\tau}\int_t^{t+\tau}\left\|\mathscr{F}_\delta\left(\frac{\partial\eta}{\partial t}(t')\right)-\frac{\partial\eta}{\partial t}(t')\right\|dt'+\int_t^{t+\tau}\left\|\frac{\partial^2\eta}{\partial t^2}(t')\right\|dt'$$

$$\leqslant M\tau^{-1/2}(h^{\bar{\alpha}-1}+N^{1-\beta})\left[\int_t^{t+\tau}\left\|\frac{\partial\eta}{\partial t}(t')\right\|^2_{H^{\alpha-1,\beta-1}(\Omega)}dt'\right]^{1/2}$$

$$+M\tau^{1/2}\left[\int_t^{t+\tau}\left\|\frac{\partial\eta}{\partial t^2}(t)'\right\|^2dt'\right]^{1/2}$$

其中$\alpha>(n+1)/2$，$\beta\geqslant 1$，$\bar{\alpha}=\min(\alpha,k+1)$，$\eta=U,T,\varphi$.又由引理2得到

$$\|\eta_*(t)-\eta(t)\|_1\leqslant M(h^{\bar{\alpha}-1}+N^{1-\beta})\|\eta(t)\|_{Y^{\alpha,\beta}(\Omega)},\qquad \eta=U,T,\varphi$$

因此，

$$\rho(t)\leqslant M(t)(\tau^2+h^{2(\bar{\alpha}-1)}+N^{2(1-\beta)})$$

其中$M(t)>0$是依赖于

$$\|\eta\|_{C(0,t;\ Y^{\bar{\alpha},\beta}(\Omega)\cap X^{\alpha',\beta'}(\Omega))},\quad \left\|\frac{\partial\eta}{\partial t}\right\|_{L^2(0,t;\ H^{\alpha-1,\beta-1}(\Omega))},\quad \left\|\frac{\partial^2\eta}{\partial t^2}\right\|_{L^2(0,t;\ L^2(\Omega))}$$

$\eta=U,\ T,\ \varphi$，以及$\|U\|_{C(0,t;\ H^2(\Omega))}$，$\|T\|_{C(0,t;\ H^2(\Omega))}$，$R$，$\varepsilon$等等的正数。假设$h=N^{-a}$，$a\geqslant 1$。并且满足下列条件：

$$\left.\begin{aligned}&\bar{\alpha}>2-1/a+n/2a\\&\beta>a+n/2\end{aligned}\right\}\tag{5.12}$$

那么$\rho(t)=o(1/D(h,N))$，因此，当h^{-1}和N充分大时，(5.10)对$t_1=t_0$成立。

综上所述，我们根据(5.11)，并利用三角不等式，

$$\|\eta_\delta-\eta\|\leqslant\|\eta_*-\eta\|+\|\tilde{\eta}\|,\qquad \eta=U,T,\varphi$$

便可以得到如下结论：

定理1 假设(3.1)的解(U,T,φ)满足下列光滑性条件：

$$U\in C(0,t_0;\ [Y^{\alpha,\beta}_{0,p}(\Omega)\cap X^{\alpha',\beta'}_p(\Omega)\cap H^2(\Omega)]^n)\cap H^1(0,t_0;\ [H^{\alpha-1,\beta-1}_p(\Omega)]^n)\cap H^2(0,t_0;[L^2(\Omega)]^n)$$

$$T\in C(0,t_0;\ Y^{\alpha,\beta}_{0,p}(\Omega)\cap X^{\alpha',\beta'}_p(\Omega)\cap H^2(\Omega))\cap H^1(0,t_0;\ H^{\alpha-1,\beta-1}_p(\Omega))\cap H^2(0,t_0;\ L^2(\Omega))$$

$$\varphi\in C(0,t_0;\ Y^{\alpha,\beta}_{0,p}(\Omega)\cap X^{\alpha',\beta'}_p(\Omega))\cap H^1(0,t_0;H^{\alpha-1,\beta-1}_p(\Omega))\cap H^2(0,t_0;\ L^2(\Omega))$$

其中$\alpha>(n+1)/2,\beta\geqslant 1,\ \alpha'>(n-1)/2,\ \beta'>1/2$。又设$(u_\delta,T_\delta,\varphi_\delta)$是格式(3.2),(3.3)的解，并记$\bar{\alpha}=\min(\alpha,k+1)$，那么，如果以下两条件成立

(i) $B_0<T<B_1$，$|\varphi|<B_2$，并且条件(1.5)成立；

(ii) $h=N^{-a}$，$a\geqslant 1$，$\lambda=\tau(C_0h^{-2}+N^2)$，并满足(5.9)和(5.12)；

则存在不依赖于h，N，τ的正常数M_4和M_5，使得当h，N^{-1}和τ充分小时，对所有$t\in\Theta_\tau$，都有

$$\begin{aligned}&\|u_\delta(t)-U(t)\|^2+\|T_\delta(t)-T(t)\|^2+\|\varphi_\delta(t)-\varphi(t)\|^2\\&\leqslant M_4\exp[M_5t](\tau^2+h^{2(\bar{\alpha}-1)}+N^{2(1-\beta)})\end{aligned}$$

注记5.1 一般说来，(3.1)的解在周期方向具有较高的光滑性，因而此方向的谱方法具有较高的分辨率，所以，x_n方向上的步长N^{-1}一般可大于其他非周期方向上的步长，以节省计算工作量，也就是说条件$a\geqslant 1$是合理的。一些具体计算实例也证实了这一点（参见[8,11]）。

参 考 文 献

[1] Roache, P.J., *Computational Fluid Dynamics*, 2nd Ed., Hermosa Publishers, Albuqueque (1976).

[2] Atusi, Tani, The existence and uniqueness of the solution of equations describing compressible viscous fluid flow in a domain, *Proc. Japan Acad.*, 52 (1976), 334—337.

[3] 郭本瑜，《偏微分方程的差分方法》，科学出版社（1988）.

[4] Kuo Pen-yu, Résolution numérique de fluide compressible, *C.R. Acad. Sc. Paris*, 291A (1980), 167—171.

[5] Guo Ben-yu, Strict error estimation of numerical solution of compressible flow in two-dimensional space, *Scientia Sinica*, **26A** (1983), 482—498.

[6] Chung, T.J., *Finite Element Analysis in Fluid Dynamics*, McGraw-Hill International Book Company (1978).

[7] Guo Ben-yu and Ma He-ping, Strict error estimation for a spectral method of compressible fluid flow, *CalColo*, **24** (1987), 263—282.

[8] Canuto, C., Y. Maday and A. Quarteroni, Analysis of the combined finite element and Fourier interpolation, *Numer. Math.*, **39** (1982), 205—220.

[9] Canuto, C., Y. Maday and A. Quarteroni, Combined finite element and spectral approximation of the Navier-Stokes equations, *Numer. Math.*, **44** (1984), 201—217.

[10] Guo Ben-yu, Spectral-difference method for solving two-dimensional vorticity equations, *J. Comput. Math.*, **6** (1988), 238—257.

[11] Guo Ben-yu and Cao Wei-ming, Spectral-finite element method for solving two-dimensional vorticity equations, *Acta Mathematicae Applicatae Sinica*, 7 (1991), 257—271.

[12] Guo Ben-yu and Cao Wei-ming, Spectral-finite element method for solving three-dimensional vorticity equations, *J. Greek Math. Soc.*, **32** (1990), 258—280.

[13] Guo Ben-yu and Cao Wei-ming, Spectral-finite element method for solving two-dimensional Navier-Stokes equations (1988). (unpublished)

[14] Adams, R.A., *Sobolev Spaces*, Academic Press, New York (1975).

[15] Grisvard, P., Equations differentielles abstraites, *Ann. Sci. Ecole Norm. Sup.*, **4** (1969), 311—395.

[16] Ciarlet, P.G., *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam (1978).

[17] Schatz, A., V. Thomee and L.B. Wahlbin, Maximum norm stability and error estimates in parabolic finite element equations, *Commun. Pure Appl. Math.*, **33** (1980), 265—304.

Spectral-Finite Element Method for Compressible Fluid Flow

Guo Ben-yu Cao Wei-ming

(Shanghai University of Science and Technology, Shanghai)

Abstract

In this paper, a combined Fourier spectral-finite element method is proposed for solving n-dimensional ($n=2, 3$), semi-periodic compressible fluid flow problems. The strit error estimation, as well as the convergence rate, is presented.

Key words compressible fluid flow, spectral method, finite element method, error estimation