关于非线性初边值问题的奇摄动(Ⅱ)*

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摘　要

本文研究一类拟线性双曲一抛物型方程，具有变动边界的初边值问题的奇摄动：

$$\varepsilon\frac{\partial^2 u_\varepsilon}{\partial t^2}+\frac{\partial u_\varepsilon}{\partial t}=A\left(x,t,u_\varepsilon,\frac{\partial u_\varepsilon}{\partial x}\right)\frac{\partial^2 u_\varepsilon}{\partial x^2}+F\left(x,t,u_\varepsilon,\frac{\partial u_\varepsilon}{\partial x}\right)$$

$$u_\varepsilon(x,0)=\varphi_0(x),\quad \frac{\partial u_\varepsilon}{\partial t}(x,0)=\varphi_1(x);\quad \frac{\partial u_\varepsilon}{\partial x}(l_i(t),t)=\psi_i(t),\qquad (i=0,1)$$

在某些条件成立，且ε充分小时，此问题的解具有以退化问题充分光滑解为首项的广义渐近展开式（Van der Corput意义），它在充分光滑解存在的区域$Q=\{(x,\ t)|l_0(t)\leqslant x\leqslant l_1(t),\ 0\leqslant t\leqslant T\}$上一致有效．其边界层存在于$t=0$附近．本文是工作[3]～[5]的进一步发展．

关键词　奇摄动　变动边界　渐近展开式

一、引　言

在部分高阶导数带有小参数的偏微分方程奇摄动问题的研究中，椭圆一抛物型方程研究得较多，而双曲一抛物型方程的研究则比较少．自1959年起有这样一些成果：M.Zlamal和江福汝分别研究了线性双曲一抛物型方程混合问题[1],[2]．至1983年，高汝熹研究了拟线性双曲一抛物型方程的混合问题[3]．作者则对具有非线性初边值条件的拟线性双曲一抛物型方程的奇摄动进行过研究[4],[5]．在本文中，我们将对初边值问题的更一般情况，即具有变动边界的情况进行讨论．

当某些假定条件满足而ε充分小时，这样的问题的解具有一个广义渐近展开式(Van der Corput意义)．通过能量不等式的建立，得到渐近展开式在一个有界区域上一致有效的证明．这个区域则是退化问题的充分光滑解存在的区域．

二、形式渐近解

我们考察下列拟线性双曲一抛物型方程，具有变动边界的初边值问题的奇摄动：

$$\varepsilon\frac{\partial^2 u_\varepsilon}{\partial t^2}+\frac{\partial u_\varepsilon}{\partial t}=A\left(x,t,u_\varepsilon,\frac{\partial u_\varepsilon}{\partial x}\right)\frac{\partial^2 u_\varepsilon}{\partial x^2}+F\left(x,t,u_\varepsilon,\frac{\partial u_\varepsilon}{\partial x}\right)\tag{2.1}$$

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$$(l_0(t)<x<l_1(t),0<t<T)$$

$$u_\varepsilon(x,0)=\varphi_0(x) \qquad (l_0(0)\leqslant x\leqslant l_1(0)) \tag{2.2}$$

$$\frac{\partial u_\varepsilon}{\partial t}(x,0)=\varphi_1(x) \qquad (l_0(0)\leqslant x\leqslant l_1(0)) \tag{2.3}$$

$$\frac{\partial u_\varepsilon}{\partial x}(l_i(t),t)=\psi_i(t) \qquad (0\leqslant t\leqslant T),(i=0,1) \tag{2.4}$$

其中$0<\varepsilon\ll 1$，假设

(H1) $A(x,t,u,p)\geqslant\alpha_0>0$，$F'_u(x,t,u,p)\leqslant c_0<0$

(H2) 系数$A(x,t,u,p)$及$F(x,t,u,p)$在$\{l_0(t)\leqslant x\leqslant l_1(t)$，$0\leqslant t\leqslant T$，$|u|\leqslant M_0$，$|p|<+\infty\}$上关于所有变元为充分光滑.

(H3) 当$0\leqslant t\leqslant T$，$l_0(t)$，$l_1(t)$连续可微，且$l_0(t)<l_1(t)$，$l_0'(t)\geqslant 0$，$l_1'(t)\leqslant 0$.

(H4) $\varphi_0(x)$ 连续可微，$\varphi_1(x)$ 与 $\psi_i(t)$ $(i=0,1)$ 连续，且 $\varphi_0'(l_0(0))=\psi_0(0)$，$\varphi_0'(l_1(0))=\psi_1(0)$.

迄今为止，这样的问题，其整体光滑解的存在性还没有一般性的结论. 因此我们限定在充分光滑解存在的局部区域上讨论. 为简单起见，假定此区域即为$Q=\{(x,t)\,|\,l_0(t)\leqslant x\leqslant l_1(t)$，$0\leqslant t\leqslant T\}$.

当$\varepsilon=0$时，得到退化问题：

$$\frac{\partial u_0}{\partial t}=A\left(x,t,u_0,\frac{\partial u_0}{\partial x}\right)\frac{\partial^2 u_0}{\partial x^2}+F\left(x,t,u_0,\frac{\partial u_0}{\partial x}\right) \tag{2.5}$$

$$(l_0(t)<x<l_1(t),\ 0<t<T)$$

$$u_0(x,\ 0)=\varphi_0(x) \qquad (l_0(0)\leqslant x\leqslant l_1(0)) \tag{2.6}$$

$$\frac{\partial u_0}{\partial x}(l_i(t),t)=\psi_i(t) \qquad (0\leqslant t\leqslant T),(i=0,1) \tag{2.7}$$

显然，这是一个拟线性抛物型方程，具有变动边界的初边值问题. 许多文献[8][9]都作过研究，其光滑解是存在唯一的，我们将以此解作为原问题(2.1)～(2.4)解的渐近展开式的首项.

首先，我们构造问题(2.1)～(2.4)的形式渐近解. 设其外部解的二阶展开式为

$$u_\varepsilon=u_0+\varepsilon u_1+\varepsilon^2u_2+O(\varepsilon^3) \tag{2.8}$$

将(2.8)式代入方程(2.1). 为简便起见，记

$$A\left(x,t,u_\varepsilon,\frac{\partial u_\varepsilon}{\partial x}\right)=A(u_\varepsilon),\ F\left(x,t,u_\varepsilon,\frac{\partial u_\varepsilon}{\partial x}\right)=F(u_\varepsilon),$$

$$A_u\left(x,t,u_\varepsilon,\frac{\partial u_\varepsilon}{\partial x}\right)=A_u(u_\varepsilon),\ F_u\left(x,t,u_\varepsilon,\frac{\partial u_\varepsilon}{\partial x}\right)=F_u(u_\varepsilon)$$

等等，余此类推.

由于$A(u_\varepsilon)$,$F(u_\varepsilon)$的光滑性，我们有

$$A(u_\varepsilon)=A(u_0)+\varepsilon\left[u_1A_u(u_0)+\frac{\partial u_1}{\partial x}A_{u'}(u_0)\right]+\varepsilon^2\left[u_2A_u(u_0)\right.$$

$$\left.+\frac{\partial u_2}{\partial x}A_{u'}(u_0)+\frac{1}{2}u_1^2A_{uu}(u_0)+\frac{1}{2}2u_1\frac{\partial u_1}{\partial x}A_{uu'}(u_0)\right.$$

$$+\frac{1}{2!}\left(\frac{\partial u_1}{\partial x}\right)^2 A_{u'u'}(u_0)\Big]+O(\varepsilon^3)$$

$$F(u_\varepsilon)=F(u_0)+\varepsilon\left[u_1F_u(u_0)+\frac{\partial u_1}{\partial x}F_{u'}(u_0)\right]+\varepsilon^2\Big[u_2F_u(u_0)$$

$$+\frac{\partial u_2}{\partial x}F_{u'}(u_0)+\frac{1}{2!}u_1^2F_{uu}(u_0)+\frac{1}{2!}2u_1\frac{\partial u_1}{\partial x}F_{uu'}(u_0)$$

$$+\frac{1}{2!}\left(\frac{\partial u_1}{\partial x}\right)^2F_{u'u'}(u_0)\Big]+O(\varepsilon^3)$$

将上述展开式及(2.8)代入(2.1)，并使$\varepsilon^i(i=0,1)$的系数求和，分别令其为零，我们得到关于u_0，u_1，u_2的递推方程：

$$\frac{\partial u_0}{\partial t}=A(u_0)\frac{\partial^2u_0}{\partial x^2}+F(u_0) \tag{2.9}$$

$$\frac{\partial u_1}{\partial t}=A(u_0)\frac{\partial^2u_1}{\partial x^2}+\frac{\partial u_1}{\partial x}\left[\frac{\partial^2u_0}{\partial x^2}A_{u'}(u_0)+F_{u'}(u_0)\right]+u_1\Big[\frac{\partial^2u_0}{\partial x^2}A_u(u_0)$$

$$+F_u(u_0)\Big]-\frac{\partial^2u_0}{\partial t^2} \tag{2.10}$$

$$\frac{\partial u_2}{\partial t}=A(u_0)\frac{\partial^2u_2}{\partial x^2}+\frac{\partial u_2}{\partial x}\left[\frac{\partial^2u_0}{\partial x^2}A_{u'}(u_0)+F_{u'}(u_0)\right]+u_2\Big[\frac{\partial^2u_0}{\partial x^2}A_u(u_0)$$

$$+F_u(u_0)\Big]-\frac{\partial^2u_1}{\partial t^2}+\frac{\partial^2u_1}{\partial x^2}\left[u_1A_u(u_0)+\frac{\partial u_1}{\partial x}A_{u'}(u_0)\right]+\frac{\partial^2u_0}{\partial x^2}$$

$$\cdot\left[\frac{1}{2}u_1^2A_{uu}(u_0)+u_1\frac{\partial u_1}{\partial x}A_{uu'}(u_0)+\frac{1}{2}\left(\frac{\partial u_1}{\partial x}\right)^2+A_{uu'}(u_0)\right]\frac{1}{2}u_1^2F_{uu}(u_0)$$

$$+u_1\frac{\partial u_1}{\partial x}F_{uu'}(u_0)+\frac{1}{2}\left(\frac{\partial u_1}{\partial x}\right)^2F_{u'u'}(u_0) \tag{2.11}$$

它们的定解条件将在后面依次给出。

由于方程(2.9)～(2.11)分别为拟线性方程(2.5)及线性方程，加上各自的定解条件(2.29a)～(2.29c)，(2.31a)～(2.31c)，显然都存在光滑解（在Q上），依次递推，我们得到u_0，u_1，u_2，也得到了(2.8)。一般地讲，解(2.8)不可能再满足初值条件(2.3)。为此我们必须在$t=0$附近（即所谓内部区域）构造边界层项。

设边界层项为

$$V=\varepsilon v_1+\varepsilon^2v_2+O(\varepsilon^3) \tag{2.12}$$

记$U=u_0+\varepsilon u_1+\varepsilon^2u_2$，将$U+V$代入方程(2.1)，由于(2.9)～(2.11)，得

$$\varepsilon\frac{\partial^2V}{\partial t^2}+\frac{\partial V}{\partial t}=A(U)\frac{\partial^2V}{\partial x^2}+\left[A(U+V)-A(U)\right]\left[\frac{\partial^2U}{\partial x^2}+\frac{\partial^2V}{\partial x^2}\right]$$

$$+F(U+V)-F(U)+O(\varepsilon^3) \tag{2.13}$$

将$A(U)$，$A(U+V)-A(U)$，$F(U+V)-F(U)$展开为ε的幂级数形式，

$$A(U)=A(u_0)+\varepsilon\left[u_1A_u(u_0)+\frac{\partial u_1}{\partial x}A_{u'}(u_0)\right]+\varepsilon^2\Big[u_2A_u(u_0)+\frac{\partial u_2}{\partial x}A_{u'}(u_0)$$

$$+\frac{1}{2!}u_1^2A_{uu}(u_0)+\frac{1}{2!}2u_1\frac{\partial u_1}{\partial x}A_{uu'}(u_0)+\frac{1}{2!}\left(\frac{\partial u_1}{\partial x}\right)^2A_{u'u'}(u_0)\Big]+O(\varepsilon^3) \tag{2.14}$$

$$A(U+V)-A(U)=\varepsilon\Big[v_1A_u(u_0)+\frac{\partial v_1}{\partial x}A_{u'}(u_0)\Big]+\varepsilon^2\Big[u_1v_1A_{uu}(u_0)$$

$$+\frac{\partial u_1}{\partial x}v_1A_{uu'}(u_0)+u_1\frac{\partial v_1}{\partial x}A_{u'u}(u_0)+\frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x}A_{u'u'}(u_0)+v_2A_u(u_0)$$

$$+\frac{\partial v_2}{\partial x}A_{u'}(u_0)+\frac{1}{2}v_1^2A_{uu}(u_0)+v_1\frac{\partial v_1}{\partial x}A_{uu'}(u_0)+\frac{1}{2}\Big(\frac{\partial v_1}{\partial x}\Big)^2A_{u'u'}(u_0)\Big]$$

$$+O(\varepsilon^3) \tag{2.15}$$

$$F(U+V)-F(U)=\varepsilon\Big[v_1F_u(u_0)+\frac{\partial v_1}{\partial x}F_{u'}(u_0)\Big]+\varepsilon^2[u_1v_1F_{uu}(u_0)$$

$$+\frac{\partial u_1}{\partial x}v_1F_{uu'}(u_0)+u_1\frac{\partial v_1}{\partial x}F_{u'u}(u_0)+\frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x}F_{u'u'}(u_0)+v_2F_u(u_0)$$

$$+\frac{\partial v_2}{\partial x}F_{u'}(u_0)+\frac{1}{2}v_1^2F_{uu}(u_0)+v_1\frac{\partial v_1}{\partial x}F_{uu'}(u_0)+\frac{1}{2}\Big(\frac{\partial v_1}{\partial x}\Big)^2$$

$$\cdot F_{u'u'}(u_0)]+O(\varepsilon^3) \tag{2.16}$$

引进伸展变量并应用两变量展开直接构造边界层[6]，

$$\xi=\frac{t}{\varepsilon},\ \eta=t,\ x=x \tag{2.17}$$

$$\therefore\ \frac{\partial}{\partial t}=\frac{\partial}{\partial\eta}+\frac{1}{\varepsilon}\frac{\partial}{\partial\xi};\quad \frac{\partial^2}{\partial t^2}=\frac{\partial^2}{\partial\eta^2}+2\frac{1}{\varepsilon}\frac{\partial^2}{\partial\xi\partial\eta}+\frac{1}{\varepsilon^2}\frac{\partial^2}{\partial\xi^2} \tag{2.18}$$

将(2.14)～(2.18)代入(2.13)，得到

$$\Big(\frac{\partial^2v_1}{\partial\xi^2}+\frac{\partial v_1}{\partial\xi}\Big)+\varepsilon\Big(\frac{\partial^2v_2}{\partial\xi^2}+\frac{\partial v_2}{\partial\xi}+2\frac{\partial^2v_1}{\partial\xi\partial\eta}+\frac{\partial v_1}{\partial\eta}\Big)+\varepsilon^2\Big(\frac{\partial v_2}{\partial\eta}+\frac{\partial^2v_1}{\partial\eta^2}+2\frac{\partial^2v_2}{\partial\xi\partial\eta}\Big)+\varepsilon^3\frac{\partial^2v_2}{\partial\eta^2}$$

$$=\varepsilon\Big\{\frac{\partial^2v_1}{\partial x^2}A(u_0)+\frac{\partial v_1}{\partial x}\Big[\frac{\partial^2u_0}{\partial x^2}A_{u'}(u_0)+F_{u'}(u_0)\Big]+v_1\Big[\frac{\partial^2u_0}{\partial x^2}$$

$$\cdot A_u(u_0)+F_u(u_0)\Big]\Big\}+O(\varepsilon^2) \tag{2.19}$$

由此我们得到了关于v_1，v_2的递推方程

$$\frac{\partial^2v_1}{\partial\xi^2}+\frac{\partial v_1}{\partial\xi}=0 \tag{2.20}$$

$$\frac{\partial^2v_2}{\partial\xi^2}+\frac{\partial v_2}{\partial\xi}=-2\frac{\partial^2v_1}{\partial\xi\partial\eta}-\frac{\partial v_1}{\partial\eta}+\frac{\partial^2v_1}{\partial x^2}A(u_0)+\frac{\partial v_1}{\partial x}\Big[\frac{\partial^2u_0}{\partial x^2}A_{u'}(u_0)$$

$$+F_{u'}(u_0)\Big]+v_1\Big[\frac{\partial^2u_0}{\partial x^2}A_u(u_0)+F_u(u_0)\Big] \tag{2.21}$$

显然，方程(2.20)有指数型衰减的解

$$v_1=P_1(x,\eta)e^{-\xi} \tag{2.22}$$

其中P_1为满足下列方程的解。

$$\frac{\partial P_1}{\partial\eta}=A(u_0)\frac{\partial^2P_1}{\partial x^2}+\frac{\partial P_1}{\partial x}\Big[\frac{\partial^2u_1}{\partial x^2}A_{u'}(u_0)+F_{u'}(u_0)\Big]+P_1\Big[\frac{\partial^2u_0}{\partial x^2}A_u(u_0)+F_u(u_0)\Big] \tag{2.23}$$

其定解条件为(2.30a)、(2.31a)。由于(2.22)、(2.23)，方程(2.21)化简为

$$\frac{\partial^2v_2}{\partial\xi^2}+\frac{\partial v_2}{\partial\xi}=2\frac{\partial P_1}{\partial\eta}e^{-\xi} \tag{2.21$'$}$$

显然，方程(2.21)′也有指数型衰减的解，

$$v_2=P_2(x,\eta)e^{-\xi}-2\frac{\partial P_1}{\partial\eta}\xi e^{-\xi} \tag{2.24}$$

其中P_2为只要满足条件(2.30b)，(2.31c)的，对x，η为适当光滑的任意函数．这样我们得到了在Q中解u_ε的形式渐近展开式，

$$u_\varepsilon=u_0+\varepsilon u_1+\varepsilon^2u_2+\varepsilon v_1+\varepsilon^2v_2+O(\varepsilon^3) \tag{2.25}$$

用Z记为其余项，

$$\therefore\quad u_\varepsilon=U+V+Z \tag{2.26}$$

下面我们来确定递推方程(2.9)、(2.10)、(2.11)和(2.20)、(2.21)、(2.23)的初边值条件．

初始条件

$$\because\quad u_\varepsilon(x,0)=[u_0+\varepsilon(u_1+v_1)+\varepsilon^2(u_2+v_2)+O(\varepsilon^3)]_{t=0}=\varphi_0(x)$$

$$\frac{\partial u_\varepsilon}{\partial t}(x,0)=\left[\frac{\partial u_0}{\partial t}+\frac{\partial v_1}{\partial\xi}+\varepsilon\left(\frac{\partial u_1}{\partial t}+\frac{\partial v_2}{\partial\xi}+\frac{\partial v_1}{\partial\eta}\right)+\varepsilon^2\left(\frac{\partial u_2}{\partial t}+\frac{\partial v_2}{\partial\eta}\right)+O(\varepsilon^3)\right]_{t=0}=\varphi_1(x)$$

$$\therefore\quad u_0(x,0)=\varphi_0(x) \tag{2.27a}$$

$$u_1(x,0)=-v_1(x,0,0) \tag{2.27b}$$

$$u_2(x,0)=-v_2(x,0,0) \tag{2.27c}$$

$$\frac{\partial v_1}{\partial\xi}(x,0,0)=\varphi_1(x)-\frac{\partial u_0}{\partial t}(x,0) \tag{2.28a}$$

$$\frac{\partial v_2}{\partial\xi}(x,0,0)=-\frac{\partial u_1}{\partial t}(x,0)-\frac{\partial v_2}{\partial\eta}(x,0,0) \tag{2.28b}$$

再根据v_1，v_2的具体表达式，上述条件化为

$$u_0(x,0)=\varphi_0(x) \tag{2.29a}$$

$$u_1(x,0)=\varphi_1(x)-\frac{\partial u_0}{\partial t}(x,0) \tag{2.29b}$$

$$u_2(x,0)=\frac{\partial P_1}{\partial\eta}(x,0)-\frac{\partial u_1}{\partial t}(x,0) \tag{2.29c}$$

$$P_1(x,0)=\frac{\partial u_0}{\partial\eta}(x,0)-\varphi_1(x) \tag{2.30a}$$

$$P_2(x,0)=\frac{\partial u_1}{\partial\eta}(x,0)-\frac{\partial P_1}{\partial\eta}(x,0) \tag{2.30b}$$

边界条件

在边界上可以导出下列条件，

$$\frac{\partial u_0}{\partial x}(l_i(t),t)=\psi_i(t)\qquad(i=0,1) \tag{3.31a}$$

$$\frac{\partial u_1}{\partial x}(l_i(t),t)=\frac{\partial P_1}{\partial x}(l_i(t),t)=0\qquad(i=0,1) \tag{3.31b}$$

$$\frac{\partial u_2}{\partial x}(l_i(t),t)=\frac{\partial P_2}{\partial x}(l_i(t),t)=0\qquad(i=0,1) \tag{3.31c}$$

根据上述初边值条件的特点，关于u_0，u_1，u_2与v_1，v_2的递推求解应当按u_0，u_1，v_1，u_2，v_2的次序交替进行．

三、余项的估计

容易发现余项Z满足下面的方程与条件，

$$\varepsilon\frac{\partial^2 Z}{\partial t^2}+\frac{\partial Z}{\partial t}-A(U+V+Z)\frac{\partial^2 Z}{\partial x^2}+\frac{\partial^2(U+V)}{\partial x^2}\Big[ZA_u(U+V+\theta_1 Z)$$

$$+\frac{\partial Z}{\partial x}A_{u'}(U+V+\theta_2 Z)\Big]+\Big[ZF_u(U+V+\theta_3 Z)+\frac{\partial Z}{\partial x}F_{u'}(U+V+\theta_4 Z)\Big]$$

$$=g(x,t,\varepsilon) \tag{3.1}$$

$$Z(x,0)=0 \tag{3.2}$$

$$\frac{\partial Z}{\partial t}(x,0)=O(\varepsilon^2) \tag{3.3}$$

$$\frac{\partial Z}{\partial x}(l_i(t),t)=0 \qquad (i=0,1) \tag{3.4}$$

其中，$0<\theta_i<1\,(i=1,2,3,4)$，而$g(x,t,\varepsilon)=O(\varepsilon^2)$. 为方便计，如文[5]将(3.3)改为$\partial Z(x,0)/\partial t=0$，这种改变并不影响在$\varepsilon^2$量级上对$Z$的估计.

记$Q_{\bar t}=\{(x,t)\,|\,l_0(t)\leqslant x\leqslant l_1(t),0\leqslant t\leqslant \bar t<T\}$. 方程(3.1)的两边同乘以$2\partial Z/\partial t$并在$Q_{\bar t}$上积分之.

$$2\int_{Q_{\bar t}}\Big(\frac{\partial Z}{\partial t}\Big)^2 dQ+\varepsilon\int_{Q_{\bar t}}2\frac{\partial Z}{\partial t}\Big(\frac{\partial^2 Z}{\partial t^2}\Big)dQ-2\int_{Q_{\bar t}}A(U+V+Z)\frac{\partial^2 Z}{\partial x^2}\frac{\partial Z}{\partial t}dQ$$

$$-2\int_{Q_{\bar t}}Z\frac{\partial Z}{\partial t}\Big[A_u(U+V+\theta_1 Z)\frac{\partial^2(U+V)}{\partial x^2}+F_u(U+V+\theta_3 Z)\Big]dQ$$

$$-2\int_{Q_{\bar t}}\frac{\partial Z}{\partial t}\frac{\partial Z}{\partial x}\Big[A_{u'}(U+V+\theta_2 Z)\frac{\partial^2(U+V)}{\partial x^2}+F_{u'}(U+V+\theta_4 Z)\Big]dQ$$

$$=2\int_{Q_{\bar t}}\frac{\partial Z}{\partial t}g(x,t,\varepsilon)dQ \tag{3.5}$$

其中

(I) $$\varepsilon\int_{Q_{\bar t}}2\frac{\partial Z}{\partial t}\Big(\frac{\partial^2 Z}{\partial t^2}\Big)dQ=\varepsilon\int_{Q_{\bar t}}\frac{\partial}{\partial t}\Big(\frac{\partial Z}{\partial t}\Big)^2 dxdt=\varepsilon\int_{l_0(t)}^{l_0(\bar t)}\Big(\frac{\partial Z}{\partial t}\Big)^2\Big|_{t=l_0^{-1}(x)}dx$$

$$+\varepsilon\int_{l_0(\bar t)}^{l_1(\bar t)}\Big(\frac{\partial Z}{\partial t}\Big)^2\Big|_{t=\bar t}dx+\varepsilon\int_{l_1(\bar t)}^{l_1(0)}\Big(\frac{\partial Z}{\partial t}\Big)^2\Big|_{t=l_1^{-1}(x)}dx\geqslant 0$$

(II) $$2\int_{Q_{\bar t}}A(U+V+Z)\frac{\partial^2 Z}{\partial x^2}\frac{\partial Z}{\partial t}dQ=2\int_{Q_{\bar t}}\Big[\frac{\partial}{\partial x}\Big(A\frac{\partial Z}{\partial x}\frac{\partial Z}{\partial t}\Big)$$

$$-\frac{\partial A}{\partial x}\frac{\partial Z}{\partial x}\frac{\partial Z}{\partial t}-\frac{1}{2}A\frac{\partial}{\partial t}\Big(\frac{\partial Z}{\partial x}\Big)^2\Big]dQ$$

$$=-2\int_{Q_{\bar t}}\frac{\partial A}{\partial x}\frac{\partial Z}{\partial x}\frac{\partial Z}{\partial t}dQ-\int_{l_0(\bar t)}^{l_1(\bar t)}A\Big(\frac{\partial Z}{\partial x}\Big)^2\Big|_{t=\bar t}dx+\int_{Q_{\bar t}}\frac{\partial A}{\partial t}\Big(\frac{\partial Z}{\partial x}\Big)^2 dQ$$

(Ⅲ) $$2\int_{Q_t} Z\frac{\partial Z}{\partial t}\Big[A_u(U+V+\theta_1 Z)\frac{\partial^2(U+V)}{\partial x^2}+F_u(U+V+\theta_3 Z)\Big]dQ$$

$$=\int_{Q_t}\frac{\partial Z^2}{\partial t}\Big[A_u(U+V+\theta_1 Z)\frac{\partial^2(U+V)}{\partial x^2}+F_u(U+V+\theta_3 Z)\Big]dQ$$

$$=\int_{l_0(0)}^{l_0(\bar t)} Z^2\Big[A_u(\cdot)\frac{\partial^2(U+V)}{\partial x^2}+F_u(\cdot)\Big]_{t=l_0^{-1}(x)}dx+\int_{l_0(\bar t)}^{l_1(\bar t)} Z^2\Big[A_u(\cdot)$$

$$\cdot\frac{\partial^2(U+V)}{\partial x^2}+F_u(\cdot)\Big]_{t=\bar t}dx+\int_{l_1(\bar t)}^{l_1(0)} Z^2\Big[A_u(\cdot)\frac{\partial^2(U+V)}{\partial x^2}$$

$$+F_u(\cdot)\Big]\Big]_{t=l_1^{-1}(x)}dx-\int_{Q_t} Z^2\frac{\partial}{\partial t}\Big[A_u(\cdot)\frac{\partial^2(U+V)}{\partial x^2}+F_u(\cdot)\Big]dQ$$

其中， $A_u(\cdot)=A_u(U+V+\theta_1 Z)$， $F_u(\cdot)=F_u(U+V+\theta_3 Z)$， $A=A(U+V+Z)$

记 $A_{u'}(\cdot)=A_{u'}(U+V+\theta_2 Z)$， $F_{u'}(\cdot)=F_{u'}(U+V+\theta_4 Z)$

$$\therefore\ 2\int_{Q_t}\Big(\frac{\partial Z}{\partial t}\Big)^2 dQ+\int_{l_0(\bar t)}^{l_1(\bar t)} A\Big(\frac{\partial Z}{\partial x}\Big)^2\Big|_{t=\bar t}dx-\int_{l_0(\bar t)}^{l_1(\bar t)} Z^2F_u(\cdot)\Big|_{t=\bar t}dx$$

$$+2\int_{Q_t}\frac{\partial Z}{\partial x}\frac{\partial Z}{\partial t}\Big[\frac{\partial A}{\partial x}-A_{u'}(\cdot)\frac{\partial^2(U+V)}{\partial x^2}-F_{u'}(\cdot)\Big]dQ+\int_{Q_t} Z^2\frac{\partial}{\partial t}\Big[A_u(\cdot)$$

$$\cdot\frac{\partial^2(U+V)}{\partial x^2}+F_u(\cdot)\Big]dQ-\int_{Q_t}\frac{\partial A}{\partial t}\Big(\frac{\partial Z}{\partial x}\Big)^2 dQ-\int_{l_0(0)}^{l_0(\bar t)} Z^2\Big[A_u(\cdot)$$

$$\cdot\frac{\partial^2(U+V)}{\partial x^2}+F_u(\cdot)\Big]_{t=l_0^{-1}(x)}dx-\int_{l_1(\bar t)}^{l_1(0)} Z^2\Big[A_u(\cdot)\frac{\partial^2(U+V)}{\partial x^2}$$

$$+F_u(\cdot)\Big]_{t=l_1^{-1}(x)}dx-\int_{l_0(\bar t)}^{l_1(\bar t)} Z^2A_u(\cdot)\frac{\partial^2(U+V)}{\partial x^2}\Big|_{t=\bar t}dx$$

$$\leqslant 2\int_{Q_t}\frac{\partial Z}{\partial t}g(x,t,\varepsilon)dQ$$

$$\therefore\ 2\int_{Q_t}\Big(\frac{\partial Z}{\partial t}\Big)^2 dQ+\min(a_0,-c_0)\int_{l_0(\bar t)}^{l_1(\bar t)}\Big[Z^2+\Big(\frac{\partial Z}{\partial x}\Big)^2\Big]_{t=\bar t}dx$$

$$\leqslant M_1\int_{Q_t}\Big|\frac{\partial Z}{\partial x}\frac{\partial Z}{\partial t}\Big|dQ+M_2\int_{Q_t} Z^2 dQ$$

$$+M_3\int_{Q_t}\Big(\frac{\partial Z}{\partial x}\Big)^2 dQ+M_4\Big\{\int_{l_0(0)}^{l_0(\bar t)} Z^2\Big|_{t=l_0^{-1}(x)}dx$$

$$+\int_{l_0(\bar t)}^{l_1(\bar t)} Z^2\Big|_{t=\bar t}dx+\int_{l_1(\bar t)}^{l_0(0)} Z^2\Big|_{t=l_1^{-1}(x)}dx\Big\}+2\int_{Q_t}\frac{\partial Z}{\partial t}g(x,t,\varepsilon)dQ$$

其中由于 $Z(x,0)=0$， $\therefore M_4\{\cdots\}=M_4\int_{Q_t}\frac{\partial Z^2}{\partial t}dQ=2M_4\int_{Q_t} Z\frac{\partial Z}{\partial t}dQ$

我们可以选取λ，μ，ν 足够小，使 $(2-M_1\lambda-2M_4\mu-2\nu)>0$。

$$\therefore\ (2-M_1\lambda-2M_4-2\nu)\int_{Q_t}\Big(\frac{\partial Z}{\partial t}\Big)^2 dQ+\min(a_0,-c_0)\int_{l_0(\bar t)}^{l_1(\bar t)}\Big[Z^2+\Big(\frac{\partial Z}{\partial x}\Big)^2\Big]_{t=\bar t}dx$$

$$\leqslant\frac{M_1}{\lambda}\int_{Q_t}\Big(\frac{\partial Z}{\partial x}\Big)^2 dQ+M_2\int_{Q_t} Z^2 dQ+M_3\int_{Q_t}\Big(\frac{\partial Z}{\partial x}\Big)^2 dQ+\frac{2M_0}{\mu}\int_{Q_t} Z^2 dQ$$

$$+\frac{2}{\nu}\int_{Q_t} g^2(x,t,\varepsilon)dQ$$

记 $K=\min(a_0,-c_0)$， $M=\max\left(\frac{M_1}{\lambda}+M_3, M_2+\frac{2M_4}{\mu}\right)$, $N=\frac{2}{\nu}$

$$\therefore \quad K\int_{l_0(\bar t)}^{l_1(\bar t)}\left[Z^2+\left(\frac{\partial Z}{\partial x}\right)^2\right]\Big|_{t=\bar t}dx\leqslant M\int_{Q_t}\left[Z^2+\left(\frac{\partial Z}{\partial x}\right)^2\right]dQ+N\int_{Q_t}g^2(x,t,\varepsilon)dQ \tag{3.6}$$

记 $E(t)=\int_{Q_t}\left[Z^2+\left(\frac{\partial Z}{\partial x}\right)^2\right]dQ$, $F(t)=\int_{Q_t}g^2(x,t,\varepsilon)dQ$

上述不等式可以写为

$$K\frac{dE(t)}{dt}\leqslant ME(t)+NF(t)$$

$$\therefore \quad E(t)\leqslant\frac{Nt}{K}F(t)\exp\left[\frac{M}{K}t\right]$$

取 $C=\frac{NT}{K}\exp\left[\frac{M}{K}T\right]$

$$\therefore \quad E(T)=\int_Q\left[Z^2+\left(\frac{\partial Z}{\partial x}\right)^2\right]dQ\leqslant C\int_Q g^2(x,t,\varepsilon)dQ \tag{3.7}$$

记$\bar C=C+(MC+N)/(2-M_1\lambda-2M_4\mu-2\nu)$，不难推得

$$\int_Q\left[Z^2+\left(\frac{\partial Z}{\partial x}\right)^2+\left(\frac{\partial Z}{\partial t}\right)^2\right]dQ\leqslant\bar C\int_Q g^2(x,t,\varepsilon)dQ \tag{3.8}$$

$$\therefore \quad \|Z\|=\left\{\int_Q\left[Z^2+\left(\frac{\partial Z}{\partial x}\right)^2+\left(\frac{\partial Z}{\partial t}\right)^2\right]dQ\right\}^{1/2}=O(\varepsilon^2) \tag{3.9}$$

最后，我们得到如下的定理。

定理 对于拟线性双曲一抛物型方程，具有变动边界的初边值问题(2.1)～(2.4)，当假设(H1)～(H4)满足，且ε充分小时，则在存在充分光滑解的区域$Q=\{(x,t)\,|\,l_0(t)\leqslant x\leqslant l_1(t),\ 0\leqslant t\leqslant T\}$中，具有一致有效的广义渐近解

$$u_\varepsilon=u_0+\varepsilon u_1+\varepsilon^2u_2+\varepsilon v_1+\varepsilon^2v_2+Z \tag{3.10}$$

其中u_0为退化问题的充分光滑解，v_1，v_2为边界层校正项（在$t=0$的邻域中）。对于余项Z，则在区域Q上一致成立下面的估计式

$$\|Z\|=\left\{\int_Q\left[Z^2+\left(\frac{\partial Z}{\partial x}\right)^2+\left(\frac{\partial Z}{\partial t}\right)^2\right]dQ\right\}^{1/2}=O(\varepsilon^2) \tag{3.11}$$

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On Singular Perturbation for a Nonlinear Initial-Boundary Value Problem (Ⅱ)

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Abstract

In this paper, we consider a singularly perturbed problem of a kind of quasilinear hyperbolic-parabolic equations, subject to initial-boundary value conditions with moving boundary:

$$\varepsilon\frac{\partial^2 u_\varepsilon}{\partial t^2}+\frac{\partial u_\varepsilon}{\partial t}=A\left(x,t,u_\varepsilon,\frac{\partial u_\varepsilon}{\partial x}\right)\frac{\partial^2 u_\varepsilon}{\partial x^2}+F\left(x,t,u_\varepsilon,\frac{\partial u_\varepsilon}{\partial x}\right)$$

$$u_\varepsilon(x,0)=\varphi_0(x),\ \frac{\partial u_\varepsilon}{\partial t}(x,0)=\varphi_1(x);\ \frac{\partial u_\varepsilon}{\partial x}(l_i(t),t)=\psi_i(t)\qquad(i=0,1)$$

When certain assumptions are satisfied and ε is sufficiently small, the solution of this problem has a generalized asymptotic expansion (in the Van der Corput sense), which takes the sufficiently smooth solution of the reduced problem as the first term, and is uniformly valid in domain Q where the sufficiently smooth solution exists. The layer exists in the neighborhood of $t=0$. This paper is the development of references[3]～[5].

Key words singular perturbation, moving boundary, asymptotic expansion